Asymptotic Convergence of Degree-Raising

Michael S. Floater [a] and Tom Lyche [b]

[a] *SINTEF, Postbox 124 Blindern, 0314 Oslo, NORWAY*
E-mail: mif@math.sintef.no

[b] *Institutt for Informatikk, Postbox 1080 Blindern, 0316 Oslo, NORWAY*
E-mail: tom@ifi.uio.no

It is well known that the degree-raised Bernstein-Bézier coefficients of degree n of a polynomial g converge to g at the rate $1/n$. In this paper we consider the polynomial $A_n(g)$ of degree $\le n$ interpolating the coefficients. We show how A_n can be viewed as an inverse to the Bernstein polynomial operator and that the derivatives $A_n(g)^{(r)}$ converge uniformly to $g^{(r)}$ at the rate $1/n$ for all r. We also give an asymptotic expansion of Voronovskaya type for $A_n(g)$ and discuss some shape preserving properties of this polynomial.

Keywords: degree-raising, Bernstein polynomials, Voronovskaya estimates

AMS Subject classification: Primary 65B17, 41A10, 41A25 ; Secondary 65D05

1. Introduction

Suppose g is a polynomial, of degree say d, $g \in \pi_d$, and let $n \ge d$. In terms of the Bernstein basis for polynomials of degree n given by

$$p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \quad \nu = 0, \ldots, n, \tag{1.1}$$

we can represent g in *degree-raised Bernstein form* as

$$g = \sum_{\nu=0}^{n} a_{n,\nu} p_{n,\nu}, \tag{1.2}$$

where the $a_{n,\nu}$ are called the Bernstein/Bézier coefficients (BB-coefficients) of g of degree n.

This form of *degree-raising* has been considered by many authors, see [3,5,9–11,15] and references therein. Degree-raising is used in computer aided geometric design to model a curve by its *control polygon* $P_n(g)$, i.e. the piecewise linear function connecting the *control points* $(\nu/n, a_{n,\nu})$ for $\nu = 0, \ldots, n$. The curve inherits the shape of $P_n(g)$ ([6]) and it has been shown in [11] that the degree-raised BB-coefficients $a_{n,\nu}$ of g converge to g at the rate $1/n$,

$$a_{n,\nu} - g\Big(\frac{\nu}{n}\Big) = O(\frac{1}{n}), \quad n \to \infty. \tag{1.3}$$

This implies that the control polygon $P_n(g)$ converges uniformly to g as n tends to infinity.

The control polygon is only one of many curves which interpolates or approximates the control points and convergence can perhaps be better analyzed by considering a different approximation. In this paper we consider the polynomial $A_n(g)$ of degree $\leq n$ passing through the control points. In [15] this approach was used to show $1/n$ convergence of $A_n(g)$ to g. Since $A_n(g)$ is differentiable we can study the convergence of the derivatives of $A_n(g)$ to the corresponding derivatives of g. We do this in Section 2, where we show that all the derivatives of $A_n(g)$ converge to the derivatives of g at the same rate $1/n$. Using A_n we are also able to give a refined convergence analysis of degree-raising. In fact we derive in Section 3 an asymptotic expansion of Voronovskaya type ([1,16]) for A_n. The main tool is a simple three term recurrence relation for certain coefficients of A_n. Finally in Section 4 we study shape preserving properties of A_n.

After this paper was completed we discovered that the recurrence relation for A_n has recently been shown in [7] and [14].

2. Convergence of Derivatives

Degree-raising and approximation by Bernstein polynomials are closely related. In fact, g is the Bernstein polynomial of the control polygon $f = P_n(g)$ of g. In symbols $g = B_n(P_n(g))$, where for any continuous function f on $[0, 1]$

$$B_n(f) = \sum_{\nu=0}^{n} f(\frac{\nu}{n}) p_{n,\nu}. \tag{2.1}$$

Thus the operator P_n is a one-sided inverse of the Bernstein operator B_n. Suppose that $g \in \pi_n$ is given by (1.2) and let $A_n(g)$ be the polynomial of degree $\leq n$ interpolating the degree n control points $(\nu/n, a_{n,\nu})_{\nu=0}^{n}$ of g. Since $B_n(f)$ only depends on the values of f at the points $(\nu/n)_{\nu=0}^{n}$ it follows that also A_n is a one-sided inverse of B_n. Moreover, A_n is the unique inverse of B_n as an operator on π_n. The Lagrange form of $A_n(g)$ is

$$A_n(g) = \sum_{\nu=0}^{n} a_{n,\nu}\ell_{n,\nu}, \tag{2.2}$$

where $\ell_{n,\nu} \in \pi_n$ is the Lagrange polynomial defined by $\ell_{n,\nu}(\mu/n) = \delta_{\mu\nu}$ for $\mu = 0, \ldots, n$.

It is well known that if $f \in \pi_d$ for some $d \le n$, then also $B_n(f) \in \pi_d$. This follows from the formula

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0) x^k,$$

which can be found for example in Chapter 10 of [4]. Here Δ^k denotes powers of the usual forward difference with spacing $1/n$ given by $\Delta f(0) = f(1/n) - f(0)$. It follows that if $g \in \pi_d$ then $A_n(g) \in \pi_d$.

Let $g \in \pi_d$ be given. To give a representation for $A_n(g)$ which shows more clearly that it is a polynomial of degree $\le d$ it is convenient to introduce factorial notation. For any x and any nonnegative integer n we let $x^{(0)_n} = x^{(0)} = 1$ and for $m \ge 1$

$$x^{(m)_n} = x^{(m)} := x(x - \frac{1}{n}) \cdots (x - \frac{m-1}{n}) \tag{2.3}$$

be the mth degree factorial with spacing $1/n$. The factorial is related to a binomial coefficient by the formula

$$x^{(m)_n} = n^{-m} m! \binom{nx}{m}, \tag{2.4}$$

where for all $y \in \mathbb{R}$, $\binom{y}{m} = y(y-1)\ldots(y-m+1)/m!$ for $m \ge 0$ and we define $\binom{y}{m} = 0$ for $m < 0$.

Using this notation, we next define for integers $n \ge d \ge 0$,

$$p_{d,\nu,n}(x) = \binom{d}{\nu} x^{(\nu)} (1-x)^{(d-\nu)} / 1^{(d)}, \quad \nu = 0, \ldots, d, \quad x \in \mathbb{R}, \tag{2.5}$$

where the factorials $x^{(\nu)}$, $(1-x)^{(d-\nu)}$, and $1^{(d)}$ have a spacing of $1/n$ as in (2.3). Observe that $p_{d,\nu,n}(x)$ is a polynomial of degree d in x, with the d roots $0, 1/n, \ldots, (\nu-1)/n, 1-(d-\nu-1)/n, \ldots, 1-1/n, 1$, and it is non-negative in the interval $[(\nu-1)/n, 1-(d-\nu-1)/n]$. This implies that any non-negative linear combination of the $p_{d,\nu,n}$ is non-negative in the interval $[(d-1)/n, 1-(d-1)/n]$, for $n \geq 2d-2$. We call the function $p_{d,\nu,n}$ a *discrete Bernstein polynomial*. The first specific study of such functions was carried out in [9,10] and was later considered in [13].

¿From (2.4) we immediately obtain the alternative representation

$$p_{d,\nu,n}(x) = \binom{nx}{\nu}\binom{n(1-x)}{d-\nu}\Big/\binom{n}{d}. \tag{2.6}$$

We have the following representation for $A_n(g)$ in terms of the polynomials $p_{d,\nu,n}$.

Proposition 2.1. Let $n \geq d \geq 0$ and $g \in \pi_d$. Then

$$A_n(g) = \sum_{\nu=0}^{d} a_{d,\nu} p_{d,\nu,n}, \tag{2.7}$$

where the $a_{d,\nu}$ are the BB-coefficients of g of degree d.

Proof. The degree n BB-coefficients $(a_{n,\nu})_{\nu=0}^{n}$ can be expressed in terms of the degree d BB-coefficients as follows ([5])

$$a_{n,k} = \sum_{\nu=0}^{d} a_{d,\nu} \frac{\binom{d}{\nu}\binom{n-d}{k-\nu}}{\binom{n}{k}}, \quad k = 0, \ldots, n. \tag{2.8}$$

But using (2.6) and rearranging terms we find

$$\sum_{\nu=0}^{d} a_{d,\nu} p_{d,\nu,n}(\frac{k}{n}) = \sum_{\nu=0}^{d} a_{d,\nu} \frac{\binom{k}{\nu}\binom{n-k}{d-\nu}}{\binom{n}{d}} = \sum_{\nu=0}^{d} a_{d,\nu} \frac{\binom{d}{\nu}\binom{n-d}{k-\nu}}{\binom{n}{k}} = a_{n,k}. \tag{2.9}$$

Substituting this into (2.2) we obtain

$$A_n(g) = \sum_{k=0}^{n} \sum_{\nu=0}^{d} a_{d,\nu} p_{d,\nu,n}(k/n) \ell_{n,k}.$$

Switching the order of summation and observing that

$$p_{d,\nu,n} = \sum_{k=0}^{n} p_{d,\nu,n}(k/n) \ell_{n,k}$$

we obtain (2.7). □

Consider now convergence of $A_n(g)$ to g. We first show that the derivatives $D^k p_{d,\nu}(x)$ and $D^k p_{d,\nu,n}(x)$ are $O(1/n)$ perturbations of each other.

Lemma 2.2. Let d and n be integers with n positive and $0 \le d \le n$. There is a constant K depending only on d such that

$$|D^k p_{d,\nu,n}(x) - D^k p_{d,\nu}(x)| \le \frac{K}{n}, \quad x \in [0,1],\ 0 \le \nu, k \le d. \tag{2.10}$$

Proof. By (1.1) and (2.5) we can write

$$\begin{aligned}
&D^k p_{d,\nu,n}(x) - D^k p_{d,\nu}(x) \\
&\quad = \binom{d}{\nu} \left[D^k x^{(\nu)}(1-x)^{(d-\nu)} - 1^{(d)} D^k x^\nu (1-x)^{d-\nu} \right] / 1^{(d)} \\
&\quad = \binom{d}{\nu} \Big[D^k (x^{(\nu)} - x^\nu)(1-x)^{(d-\nu)} \\
&\quad + D^k x^\nu \big((1-x)^{(d-\nu)} - (1-x)^{d-\nu}\big) + D^k x^\nu (1-x)^{d-\nu}(1 - 1^{(d)})\Big] / 1^{(d)}.
\end{aligned} \tag{2.11}$$

For a positive integer r, any $x_1, \ldots, x_r$ with $0 \le x_i \le d/n$, and any $x \in [0,1]$ we have

$$\begin{aligned}
|(x-x_1)\cdots(x-x_r) - x^r| &\le \frac{rd}{n}, \\
|(x-x_1)\cdots(x-x_r)| &\le 1, \\
1^{(d)_n} \ge 1^{(d)_d} &= (d-1)!/d^{d-1}.
\end{aligned} \tag{2.12}$$

The last inequality in (2.12) is obvious, while the second follows since each factor is bounded by one. Finally, the first inequality clearly holds for $r = 1$ and if it holds for $r-1$ then for $x \in [0,1]$,

$$\begin{aligned}
|(x-x_1)\cdots(x-x_r) - x^r| &= |(x-x_r)\big((x-x_1)\cdots(x-x_{r-1}) - x^{r-1}\big) - x_r x^r| \\
&\le |(x-x_1)\cdots(x-x_{r-1}) - x^{r-1}| + x_r \\
&\le \frac{(r-1)d}{n} + \frac{d}{n} = \frac{rd}{n}.
\end{aligned}$$

To complete the proof we differentiate the products in (2.11) and invoke the triangle inequality. By using the inequalities (2.12) on each term the result follows. □

We can now show that derivatives of $A_n(g)$ converge to corresponding derivatives of g.

Theorem 2.3. For each nonnegative integer d and $g \in \pi_d$ there is a constant C such that for all $n \geq d$

$$|D^k A_n(g,x) - D^k g(x)| \leq \frac{C}{n}, \quad x \in [0,1],\ 0 \leq k \leq d. \tag{2.13}$$

Proof. By (1.2) with $n = d$, (2.7), and Lemma 2.2

$$|D^k A_n(g,x) - D^k g(x)| \leq \sum_{\nu=0}^{d} |a_{d,\nu}||D^k p_{d,\nu,n}(x) - D^k p_{d,\nu}(x)| \leq \frac{K}{n} \sum_{\nu=0}^{d} |a_{d,\nu}|,$$

where K is the constant in Lemma 2.2. The result now follows. □

3. Asymptotic Convergence of Degree-Raising

It was shown by Voronovskaya [16] that if $f \in C[0,1]$, then

$$\lim_{n\to\infty} n\big(B_n(f,x) - f(x)\big) = \frac{1}{2} x(1-x) f''(x),$$

where B_n is the Bernstein operator defined in (2.1). Bernstein [1] (see also [8]) later generalized this to an asymptotic formula involving higher derivatives of f, provided f is smooth enough. In particular when $f \in \pi_d$, $d \leq n$, the formula reduces to

$$\lim_{n\to\infty} n^q \left[B_n(f,x) - f(x) - \sum_{\mu=2}^{2q-1} \frac{1}{\mu!\ n^\mu} T_{n,\mu}(x) f^{(\mu)}(x) \right] = \left(\tfrac{1}{2} x(1-x)\right)^q \frac{1}{q!} f^{(2q)}(x), \tag{3.1}$$

for $q \leq (d+1)/2$, where

$$T_{n,\mu}(x) = \sum_{\nu=0}^{n} (\nu - nx)^\mu p_{n,\nu}(x).$$

Equation (3.1) is derived in [8] from the well-known recurrence formula,

$$T_{n,\mu+1}(x) - x(1-x)(T'_{n,\mu}(x) - \mu n T_{x,\mu-1}(x)) = 0,$$

and the identities

$$T_{n,0} = 1, \qquad T_{n,1} = 0.$$

In this section we wish to obtain an asymptotic formula similar to (3.1) for the inverse Bernstein operator A_n and as a consequence degree-raising.

Proposition 3.1. Let $g \in \pi_d$ and $n \geq d$. Then

$$A_n(g)(x) = \sum_{\mu=0}^{d} \frac{(n-\mu)!}{n!} S_{n,\mu}(x) \frac{g^{(\mu)}(x)}{\mu!}, \quad x \in \mathbb{R}, \tag{3.2}$$

where

$$S_{n,\mu}(x) = \sum_{k=0}^{\mu} s_{n,\mu,k}(x) \tag{3.3}$$

and

$$s_{n,\mu,k}(x) = (-1)^{\mu-k} \binom{\mu}{k} x^{\mu-k} \frac{(n-k)!}{(n-\mu)!} nx(nx-1)\dots(nx-k+1). \tag{3.4}$$

Proof. It is well known that the $p_{n,\nu}$ are the B-splines of degree n over the knot vector $t_0 = \cdots = t_n = 0$, $t_{n+1} = \cdots = t_{2n+1} = 1$. Specializing the de Boor-Fix formula (see [2], p. 159) to this case one finds that the nth degree BB-coefficients $a_{n,\nu}$ of g satisfy

$$a_{n,\nu} = \sum_{\mu=0}^{d} (-1)^{\mu} \psi_{\nu,n}^{(n-\mu)}(\nu/n) g^{(\mu)}(\nu/n), \tag{3.5}$$

where

$$\psi_{\nu,n}(x) = x^{n-\nu}(x-1)^{\nu}/n!.$$

Expanding $\psi_{\nu,n}(x)$ in powers of x we obtain

$$\psi_{\nu,n}(x) = \frac{1}{n!} \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} x^{n-\nu+k},$$

and differentiating $n-\mu$ times we find

$$\psi_{\nu,n}^{(n-\mu)}(x) = \frac{1}{n!} \sum_{k=\max(0,\nu-\mu)}^{\nu} (-1)^k \binom{\nu}{k} \frac{(n-\nu+k)!}{(\mu-\nu+k)!} x^{\mu-\nu+k}.$$

Now changing the summation index k to $\nu-k$ and removing a factor of $(n-\mu)!/\mu!$ one obtains

$$\begin{aligned} (-1)^{\mu} \psi_{\nu,n}^{(n-\mu)}(\nu/n) &= \frac{1}{n!} \sum_{k=0}^{\min(\nu,\mu)} (-1)^{\mu-k} \binom{\nu}{\nu-k} \frac{(n-k)!}{(\mu-k)!} (\nu/n)^{\mu-k} \\ &= \frac{(n-\mu)!}{\mu!\, n!} \sum_{k=0}^{\min(\nu,\mu)} (-1)^{\mu-k} \binom{\mu}{k} \frac{\nu!}{(\nu-k)!} \frac{(n-k)!}{(n-\mu)!} (\nu/n)^{\mu-k} \end{aligned}$$

$$= \frac{(n-\mu)!}{\mu!\ n!} S_{n,\mu}(\nu/n).$$

Combining this with (3.5), we find

$$B_n\Big(\sum_{\mu=0}^{d} \frac{(n-\mu)!}{\mu!\ n!} S_{n,\mu} g^{(\mu)}\Big) = \sum_{\nu=0}^{d} a_{n,\nu} p_{n,\nu} = g,$$

and (3.4) follows from applying A_n. □

It was noted in [12] that $A_n(g)$ can be expressed as a linear combination of derivatives of g and this observation was used as the start point for studying certain quasi-interpolant operators [12,17].

Now we take a closer look at the functions $S_{n,\mu}$. From (3.3) the first three polynomials in the sequence are $S_{n,0}(x) = 1$, $S_{n,1}(x) = 0$, and $S_{n,2}(x) = -nx(1-x)$. In order to compute further members of the sequence and to derive further facts about $S_{n,\mu}$ a three-term recurrence formula will now be derived.

Lemma 3.2. For all x, all n and $\mu = 1, 2, 3, \ldots, n-1$,

$$S_{n,\mu+1}(x) + \mu(1-2x)S_{n,\mu}(x) + \mu(n-\mu+1)x(1-x)S_{n,\mu-1}(x) = 0. \tag{3.6}$$

Proof. From (3.4) we observe that

$$x s_{n,\mu,k} = -\frac{(\mu-k+1)}{(n-\mu)(\mu+1)} s_{n,\mu+1,k} \tag{3.7}$$

and

$$(nx-k)s_{n,\mu,k} = \frac{(n-k)(k+1)}{(n-\mu)(\mu+1)} s_{n,\mu+1,k+1}. \tag{3.8}$$

Define $s_{n,\mu,k} = 0$ for $k > \mu$. Then, firstly applying (3.7) with μ replaced by $\mu-1$ and secondly applying both (3.7) and (3.8) we obtain

$$\begin{aligned}
S_{n,\mu+1} &+ \mu(1-2x)S_{n,\mu} + \mu(n-\mu+1)x(1-x)S_{n,\mu-1} \\
&= \sum_{k=0}^{\mu+1} \big(s_{n,\mu+1,k} + \mu(1-2x)s_{n,\mu,k} + \mu(n-\mu+1)x(1-x)s_{n,\mu-1,k}\big) \\
&= \sum_{k=0}^{\mu+1} \big(s_{n,\mu+1,k} + \mu(1-2x)s_{n,\mu,k} - (\mu-k)(1-x)s_{n,\mu,k}\big) \\
&= \sum_{k=0}^{\mu+1} \big(s_{n,\mu+1,k} + (n-k-\mu)x s_{n,\mu,k} - (nx-k)s_{n,\mu,k}\big)
\end{aligned}$$

$$= \frac{1}{(n-\mu)(\mu+1)} \sum_{k=0}^{\mu+1} \big(k(n-k+1)s_{n,\mu+1,k} - (k+1)(n-k)s_{n,\mu+1,k+1}\big)$$
$$= 0.$$

□

Using (3.6) it is straightforward to compute further polynomials in the sequence. Let $X = x(1-x)$ and $Y = 1-2x$. Then noting that $Y^2 = 1-4X$ we find

$$S_{n,0} = 1, \quad S_{n,1} = 0, \quad S_{n,2} = -nX, \quad S_{n,3} = 2nYX,$$
$$S_{n,4} = 3nX[(n+6)X-2], \quad S_{n,5} = -4nYX[(5n+12)X-6],$$
$$S_{n,6} = -5nX[(3n^2+86n+120)X^2-(26n+120)X+24].$$

Equation (3.4) shows that $S_{n,\mu}$ is a polynomial in n as well as x but (3.6) gives us the following more precise statement.

Lemma 3.3. The polynomials $S_{n,2\mu}$ and $S_{n,2\mu+1}$ can be expressed as

$$S_{n,2\mu}(x) = \sum_{j=0}^{\mu} b_{j,2\mu}(X)n^j, \qquad S_{n,2\mu+1}(x) = Y\sum_{j=0}^{\mu} b_{j,2\mu+1}(X)n^j, \tag{3.9}$$

where $b_{j,2\mu}$ and $b_{j,2\mu+1}$ are polynomials of degree $\le \mu$ and

$$b_{\mu,2\mu}(X) = (-1)^\mu 1\cdot 3\cdot 5\cdots(2\mu-1)X^\mu = (-1)^\mu \frac{(2\mu)!}{2^\mu \mu!}X^\mu. \tag{3.10}$$

Proof. By induction on μ using (3.6) and the identities $S_{n,0} = 1$ and $S_{n,1} = 0$, the expressions (3.9) are easily established once one sees the identity $Y^2 = 1-4X$.

To obtain an expression for the highest order coefficient $b_{\mu,2\mu}(X)$ in $S_{n,2\mu}$, we observe from (3.6) that

$$S_{n,2\mu}(x) = -(2\mu-1)YS_{n,2\mu-1}(x) - (2\mu-1)(n-2\mu+2)XS_{n,2\mu-2}(x),$$

and it follows that

$$b_{\mu,2\mu}(X) = -(2\mu-1)Xb_{\mu-1,2\mu-2}(X).$$

Then, because $b_{1,2}(X) = -X$, equation (3.10) follows. □

We now apply Lemma 3.3 to obtain an expansion similar to (3.1).

Theorem 3.4. Let $g \in \pi_d$. Then for $q = 1, 2, 3, \ldots, [d/2]$,

$$\lim_{n\to\infty} n^q \left[A_n(g,x) - g(x) - \sum_{\mu=2}^{2q-1} \frac{(n-\mu)!}{\mu!\ n!} S_{n,\mu}(x) g^{(\mu)}(x) \right] = (-1)^q \left(\tfrac{1}{2}X\right)^q \frac{1}{q!} g^{(2q)}(x), \tag{3.11}$$

and the limit is uniform for $x \in [0,1]$.

Proof. Noting that

$$(n-\mu)!/n! = 1/\bigl(n(n-1)\ldots(n-\mu+1)\bigr) = O(n^{-\mu})$$

and applying the expression for $b_{\mu,2\mu}$ in (3.10), we deduce that

$$n^q \sum_{\mu=2q}^{d} \frac{(n-\mu)!}{\mu!\ n!} S_{n,\mu} g^{(\mu)} = \frac{1}{(2q)!} b_{q,2q}(X) g^{(2q)} + O(\frac{1}{n}) = (-1)^q \left(\tfrac{1}{2}X\right)^q \frac{1}{q!} g^{(2q)} + O(\frac{1}{n}).$$

□

As an example, letting $q = 1$ in (3.11), we obtain

$$\lim_{n\to\infty} n[A_n(g,x) - g(x)] = -\frac{1}{2} x(1-x) g''(x).$$

In order to apply (3.11) to the convergence of the degree-raised BB-coefficients, we only have to observe from (2.3) that $A_n(g, \nu/n) = a_{\nu,n}$. Therefore since the right hand side of (3.11) is bounded for $x \in [0,1]$, we finally arrive at a generalization of equation (1.3).

Corollary 3.5. Under the hypothesis of Theorem 3.4

$$a_{n,\nu} - g\Bigl(\frac{\nu}{n}\Bigr) - \sum_{\mu=2}^{2q-1} \frac{(n-\mu)!}{\mu!\ n!} S_{n,\mu}\Bigl(\frac{\nu}{n}\Bigr) g^{(\mu)}\Bigl(\frac{\nu}{n}\Bigr) = O\Bigl(\frac{1}{n^q}\Bigr), \quad n \to \infty,$$

and the bound is uniform.

4. Shape Preservation

It was shown in [6] that the sequence $\{P_n(g)\}_n$ is *variation diminishing* in the sense that $P_{n+1}(g)$ has no more zeros in the interval $[0,1]$ than $P_n(g)$.

The sequence $\{P_n(g)\}_n$ enjoys several other shape preserving properties such as positivity, monotonicity, and convexity. In this section we study to what extent the sequence $\{A_n(g)\}_n$ is also shape preserving.

Considering the case $n = d + 1$ in (2.8), we obtain the well known degree-raising identity

$$a_{d+1,\nu} = \frac{\nu}{d+1} a_{d,\nu-1} + \frac{d+1-\nu}{d+1} a_{d,\nu}. \tag{4.1}$$

It immediately follows that

$$A_{d+1}(g,x) = x A_d\Big(g, \frac{(d+1)x-1}{d}\Big) + (1-x) A_d\Big(g, \frac{(d+1)x}{d}\Big), \tag{4.2}$$

due to the evident fact that the two polynomials on either side of this equation have degree $\le d+1$ and agree at the points $x = 0, 1/(d+1), \ldots, 1$. It follows that there is partial positivity preservation in the sense that if $A_d(g,x) \ge 0$ for all $x \in [0,1]$, then $A_{d+1}(g,x) \ge 0$ for all $x \in [1/(d+1), d/(d+1)]$. By strengthening equation (4.2), we can obtain a more general positivity property.

Proposition 4.1. Let $g \in \pi_d$. Then for $n \ge d$,

$$A_n(g,x) = \sum_{\nu=0}^{n-d} A_d\Big(g, \frac{nx-\nu}{d}\Big) p_{n-d,\nu,n}(x) \tag{4.3}$$

Proof. ¿From equation (2.9), we have

$$a_{n,k} = \sum_{\nu=0}^{d} a_{d,\nu} \frac{\binom{k}{\nu}\binom{n-k}{d-\nu}}{\binom{n}{d}}.$$

Then, since $a_{m,\nu} = A_m(g, \nu/m)$ for all ν and m, and replacing ν by $k-\nu$, we have

$$A_n\Big(g, \frac{k}{n}\Big) = \sum_{\nu=0}^{n-d} A_d\Big(g, \frac{k-\nu}{d}\Big) \frac{\binom{k}{\nu}\binom{n-k}{n-d-\nu}}{\binom{n}{d}},$$

and so, from (2.4),

$$A_n\Big(g, \frac{k}{n}\Big) = \sum_{\nu=0}^{n-d} A_d\Big(g, \frac{k-\nu}{d}\Big) \binom{n-d}{\nu} \Big(\frac{k}{n}\Big)^{(\nu)} \Big(1 - \frac{k}{n}\Big)^{(n-d-\nu)} \Big/ 1^{(n-d)}.$$

It follows that equation (4.3) holds for $x = 0, 1/n, \ldots, 1$, and therefore for all x, since both sides of the equation are polynomials of degree $\le n$. □

Corollary 4.2. For $d \le n \le 2d$, if $A_d(g,x) \ge 0$ for all $x \in [0,1]$, then $A_n(g,x) \ge 0$ for all $x \in [1-d/n, d/n]$.

Proof. We know that the polynomials $p_{n-d,\nu,n}(x)$ are non-negative for x in the interval $[(n-d-1)/n, 1-(n-d-1)/n]$ which is contained in the interval $[1-d/n, d/n]$ in which $A_d(g,(nx-\nu)/d)$ is non-negative for all $\nu = 0,\dots,n-d$. The result now follows from (4.3). □

However non-negativity of $A_d(g)$ in $[0,1]$ does not guarantee non-negativity of $A_{d+1}(g)$ in the whole of $[0,1]$. For example, letting

$$g(x) = \frac{x + 16x^2 + 8x^3}{36},$$

one finds that

$$A_3(g,x) = x(x-1/6)^2,$$

which is nonnegative on $[0,1]$. ¿From equation (4.2) however, we have that

$$A_4(g,x) = \frac{-5x + 16x^2 + 64x^3}{108}$$

and so $A_4(g,1/8) = -1/432$.

Corollary 4.2 is only valid for $n \le 2d$. However from Proposition 2.1 we also have

Proposition 4.3. For $d \ge 2$ and $n \ge 2d-2$, if $A_d(g) \ge 0$ in $[0,1]$, then $A_n(g,x) \ge 0$ for all $x \in [(d-1)/n, 1-(d-1)/n]$.

Finally, there appears to be no partial monotonicity or convexity preservation in the spirit of Corollary 4.2. One can show that if

$$g(x) = \frac{481x - 900x^2 + 840x^3 - 360x^4 + 144x^5}{18750},$$

then

$$A_5(g,x) = \int_0^x (y-2/5)^2(y-3/5)^2\,dy,$$

so that $D\,A_5(g,x) \ge 0$ for all x in $[0,1]$ and $A_5(g)$ is thus monotonically non-decreasing there. However, computing $A_6(g)$ and differentiating, one finds that

$$D\,A_6(g,1/2) = -1/93750 < 0,$$

and so $A_6(g)$ is decreasing in the centre of the interval $[0,1]$.

Similarly, if one takes

$$g(x) = \frac{637x + 935x^2 - 1200x^3 + 900x^4 + 240x^6}{466560},$$

then

$$A_6(g, x) = \int_0^x \int_0^y (z - 1/3)^2 (z - 1/2)^2 \, dz, \, dy$$

and so $D^2 A_6(g, x) \geq 0$ for all x in $[0, 1]$ and thus $A_6(g)$ is convex on $[0, 1]$. However the same property does not hold for A_7 since by computing it and differentiating it twice one deduces that

$$D^2 A_7(g, 3/7) = -7/699840 < 0.$$

References

[1] Bernstein M., Complétement à l'article de E. Voronovskaya, C. R. Acad. Sci. URSS (1932), 86–92.

[2] de Boor C., *A Practical Guide to Splines*, Springer-Verlag, New York, 1978.

[3] Cohen E., L. L. Schumaker, Rates of convergence of control polygons, Computer-Aided Geom. Design **2** (1985), 229–235.

[4] DeVore R. A. and G. G. Lorentz, *Constructive approximation*, Springer Verlag, Berlin, 1993.

[5] Farin G., *Curves and surfaces for computer aided geometric design*, Academic Press, San Diego, 1988.

[6] Goodman T. N. T., Shape preserving representations, in *Mathematical methods in Computer Aided Geometric Design*, T. Lyche and L.L. Schumaker (eds.), Academic Press, New York, 1989, 333–357.

[7] Kageyama Y., Generalization of the left Bernstein quasi-interpolants, J. Approx. Theory **94** (1998), 306–329.

[8] Lorentz G. G., *Bernstein polynomials*, Toronto, Canada, 1953.

[9] Neamtu M., Subdividing multivariate polynomials over simplices in Bernstein-Bézier form without de Casteljau algorithm, in *Curves and Surfaces*, P. J. Laurent, A. Le Méhauté, and L. L. Schumaker (eds.), Academic Press, Boston, 1991, 359–362.

[10] Neamtu M., *Multivariate Splines*, Dissertation, University of Twente, 1991.

[11] Prautzsch H. and L. Kobbelt, Convergence of subdivision and degree elevation, Advances in Comp. Math. **2** (1994), 143–154.

[12] Sablonnière P., A family of Bernstein quasi-interpolants on $[0, 1]$, Approx. Theory Appl. **8** (1992), 62–76.

[13] Sablonnière P., Discrete Bézier curves and surfaces, in *Mathematical Methods in Computer Aided Geometric Design II*, T. Lyche & L. L. Schumaker (eds.), Academic Press, Boston (1992), 497–515.

[14] Sablonnière P., Representation of quasi-interpolants as differential operators and applications, Publication LANS 79, I.N.S.A. Rennes, 1998.

[15] Trump W. and H. Prautzsch, Arbitrarily high degree elevation of Bézier representations, Computer-Aided Geom. Design **13** (1996), 387–398.

[16] Voronovskaya E., Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Bernstein, Doklady Akademii Nauk SSSR (1932), 79–85.

[17] Zhengchang W., Norm of the Bernstein left quasi-interpolant operator, J. Approx. Theory **66** (1991), 35–43.