

On the injectivity of Wachspress and mean value mappings between convex polygons

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Abstract

Wachspress and mean value coordinates are two generalizations of triangular barycentric coordinates to convex polygons and have recently been used to construct mappings between polygons, with application to curve deformation and image warping. We show that Wachspress mappings between convex polygons are always injective but that mean value mappings can fail to be so in extreme cases.

Keywords: Barycentric coordinates, Wachspress coordinates, mean value coordinates, injectivity.

Math Subject Classification: 65D17, 26B10

1 Introduction

There has recently been considerable interest in using generalized barycentric coordinates to deform shapes, when modelling and processing geometry: shapes defined either by curves or surfaces [22, 14, 18]. For example, one way of deforming a curve is to first enclose it in a polygon, and then to use the vertices of the polygon as control points: moving the vertices will change the shape of the curve. Most of these methods focus on two kinds of barycentric coordinates: Wachspress coordinates [20, 21, 13] and mean value coordinates [7, 9, 14, 10, 5]. Further work on generalized barycentric coordinates can be

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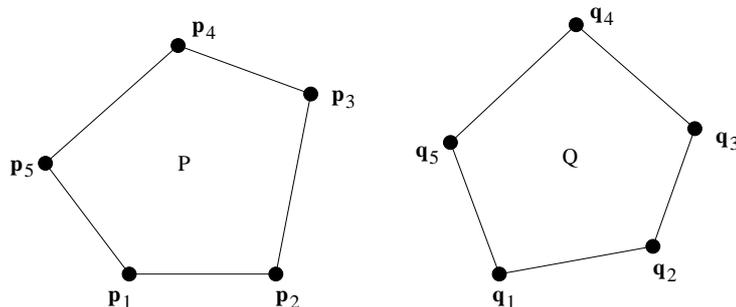


Figure 1: Two convex polygons.

found in [2, 8, 11, 15, 16, 17, 19]. In the planar case, such barycentric coordinates can be used to map one polygon into another. So far, however, nothing seems to be known about when such mappings are injective, even though this is clearly important for curve deformation: we would like a guarantee that when we deform a curve we do not introduce any new self-intersections.

In this paper we investigate the injectivity of Wachspress and mean value mappings between planar convex polygons. We show that Wachspress mappings are always injective. However, we have found examples of mean value mappings that are not injective when the polygons have five or more vertices.

2 Barycentric mappings

Let $P, Q \subset \mathbb{R}^2$ be strictly convex polygons with vertices $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ and $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ respectively, ordered anticlockwise, with $n \geq 3$. By strictly, we mean that no three vertices in the polygon are collinear. Figure 1 shows an example with $n = 5$. We view both polygons as open sets of \mathbb{R}^2 and denote their boundaries by ∂P and ∂Q and their closures by \overline{P} and \overline{Q} . We would like to define a smooth mapping $\mathbf{f} : \overline{P} \rightarrow \overline{Q}$. One way is to use barycentric coordinates. Let $\lambda_1, \lambda_2, \dots, \lambda_n : \overline{P} \rightarrow \mathbb{R}$ be a sequence of non-negative functions such that

$$\sum_{i=1}^n \lambda_i(\mathbf{x}) = 1, \quad \mathbf{x} \in \overline{P}, \quad (1)$$

$$\sum_{i=1}^n \lambda_i(\mathbf{x}) \mathbf{p}_i = \mathbf{x}, \quad \mathbf{x} \in \overline{P}. \quad (2)$$

We call $\lambda_1, \dots, \lambda_n$ a set of barycentric coordinates for P . If $n = 3$, \overline{P} is a triangle, and the equations (1-2) have, by Cramer's rule, the unique solution

$$\lambda_1(\mathbf{x}) = \frac{A(\mathbf{x}, \mathbf{p}_2, \mathbf{p}_3)}{A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}, \quad \lambda_2(\mathbf{x}) = \frac{A(\mathbf{p}_1, \mathbf{x}, \mathbf{p}_3)}{A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}, \quad \lambda_3(\mathbf{x}) = \frac{A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x})}{A(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}.$$

where $A(\mathbf{p}, \mathbf{q}, \mathbf{r})$ denotes the signed area of the triangle $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$,

$$A(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ p^1 & q^1 & r^1 \\ p^2 & q^2 & r^2 \end{vmatrix}, \quad (3)$$

and $\mathbf{p} = (p^1, p^2)$, $\mathbf{q} = (q^1, q^2)$, $\mathbf{r} = (r^1, r^2)$. For $n \geq 4$, there are several possible choices of barycentric coordinates. One is the construction that goes back to Wachspress. For $\mathbf{x} \in \overline{P}$ let

$$\lambda_i(\mathbf{x}) = w_i(\mathbf{x}) / \sum_{j=1}^n w_j(\mathbf{x}), \quad (4)$$

with

$$w_i(\mathbf{x}) = C_i \prod_{j \neq i-1, i} A_j(\mathbf{x}). \quad (5)$$

where $C_i = A(\mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1})$, and $A_j(\mathbf{x}) = A(\mathbf{x}, \mathbf{p}_j, \mathbf{p}_{j+1})$; see [20, 21, 13]. Here and throughout the paper we view the index i in \mathbf{p}_i , \mathbf{q}_i , A_i , w_i and so on cyclically with respect to n . If \mathbf{x} is not on the boundary of P we can replace (5) by

$$w_i(\mathbf{x}) = \frac{C_i}{A_{i-1}(\mathbf{x})A_i(\mathbf{x})}, \quad (6)$$

without altering λ_i . We will use this local expression later. A simple proof that the λ_i satisfy (2) is given in the appendix of [13].

One could instead consider mean value coordinates which, for $\mathbf{x} \in P$, are given by (4) with

$$w_i(\mathbf{x}) = \frac{1}{\|\mathbf{x} - \mathbf{p}_i\|} \left(\tan(\alpha_{i-1}(\mathbf{x})/2) + \tan(\alpha_i(\mathbf{x})/2) \right), \quad (7)$$

where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^2 and $\alpha_i(\mathbf{x})$ denotes the angle at \mathbf{x} in the triangle $[\mathbf{x}, \mathbf{p}_i, \mathbf{p}_{i+1}]$; see [7]. These mean value coordinates λ_i extend

continuously to the boundary ∂P in the same way as Wachspress and all other barycentric coordinates. Using (1-2), it can be shown (see [8]) that if

$$\mathbf{x} = (1 - \mu)\mathbf{p}_\ell + \mu\mathbf{p}_{\ell+1}, \quad (8)$$

for some ℓ and some $\mu \in [0, 1]$, with indexes treated cyclically, then

$$\lambda_\ell(\mathbf{x}) = 1 - \mu, \quad \lambda_{\ell+1}(\mathbf{x}) = \mu, \quad \text{and} \quad \lambda_i(\mathbf{x}) = 0 \quad \text{for } i \neq \ell, \ell + 1. \quad (9)$$

Consider next a barycentric mapping from P to Q . We define the mapping $\mathbf{f} : \overline{P} \rightarrow \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^n \lambda_i(\mathbf{x})\mathbf{q}_i, \quad \mathbf{x} \in \overline{P}. \quad (10)$$

Then (1) implies that $\mathbf{f}(\mathbf{x})$ is a convex combination of the points \mathbf{q}_i and so $\mathbf{f}(\overline{P}) \subset \overline{Q}$. Further, by (9), if \mathbf{x} is the boundary point (8) then

$$\mathbf{f}(\mathbf{x}) = (1 - \mu)\mathbf{q}_\ell + \mu\mathbf{q}_{\ell+1}.$$

Thus \mathbf{f} maps ∂P to ∂Q in a piecewise linear fashion, mapping vertices and edges of ∂P to corresponding vertices and edges of ∂Q .

Figure 2 shows examples of Wachspress and mean value mappings.

3 Conditions for injectivity

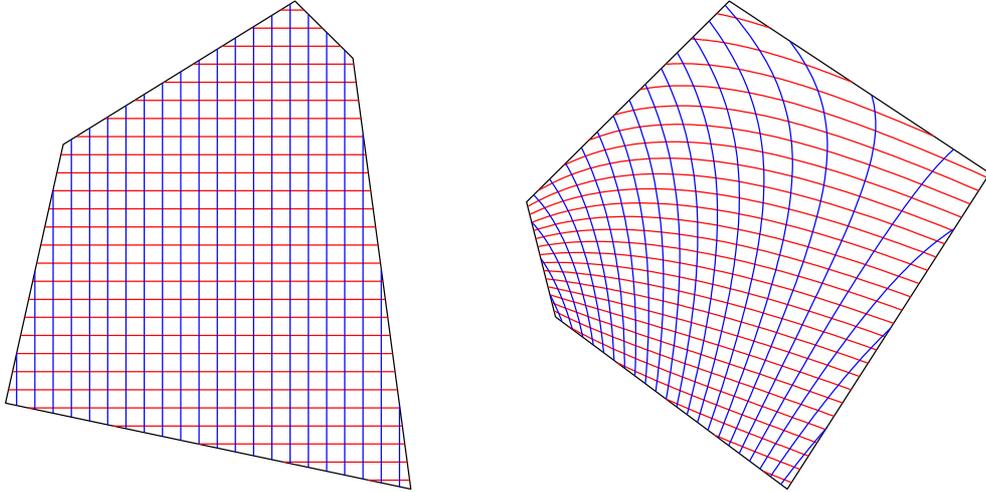
We start by deriving conditions on barycentric coordinates for injectivity: both sufficient conditions and necessary conditions.

Using basic results of real analysis, see for example [12], it can be shown that a sufficient condition for the injectivity of \mathbf{f} in (10) is that its Jacobian $J(\mathbf{f})$ is strictly positive in \overline{P} . This motivates looking for an expression for $J(\mathbf{f})$ in terms of signed areas of triangles formed by vertices of Q . If $\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, and $\partial_r h(\mathbf{x}) := \partial h / \partial x^r$, $r = 1, 2$, the Jacobian of \mathbf{f} is the determinant

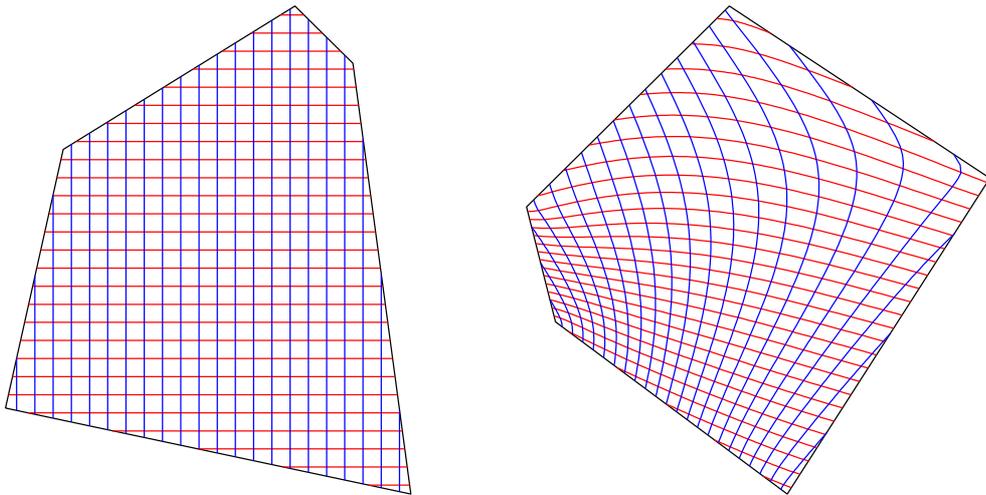
$$J(\mathbf{f}) = \begin{vmatrix} \partial_1 f & \partial_1 g \\ \partial_2 f & \partial_2 g \end{vmatrix}.$$

We define

$$\mathcal{D}(a, b, c) = \begin{vmatrix} a & b & c \\ \partial_1 a & \partial_1 b & \partial_1 c \\ \partial_2 a & \partial_2 b & \partial_2 c \end{vmatrix}. \quad (11)$$



a)



b)

Figure 2: Barycentric mappings between pentagons: a) Wachspress mapping, b) mean value mapping.

Lemma 1 For any set of differentiable functions $\lambda_1, \dots, \lambda_n$ satisfying (1),

$$J(\mathbf{f})(\mathbf{x}) = 2 \sum_{1 \leq i < j < k \leq n} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k). \quad (12)$$

Proof. Differentiating (1) and (10) gives the matrix identity

$$\begin{pmatrix} 1 & f & g \\ 0 & \partial_1 f & \partial_1 g \\ 0 & \partial_2 f & \partial_2 g \end{pmatrix} = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ \partial_1 \lambda_1 & \cdots & \partial_1 \lambda_n \\ \partial_2 \lambda_1 & \cdots & \partial_2 \lambda_n \end{pmatrix} \begin{pmatrix} 1 & q_1^1 & q_1^2 \\ \vdots & \vdots & \vdots \\ 1 & q_n^1 & q_n^2 \end{pmatrix}.$$

Applying the Cauchy-Binet theorem (see [1], formula (1.23)) to this equation and using the fact that the determinant of the matrix on the left equals $J(\mathbf{f})$, the result follows. \square

Since Q is strictly convex we have that $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) > 0$ whenever $1 \leq i < j < k \leq n$ because the triangle $[\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k]$ is positively oriented. Thus Lemma 1 leads to a sufficient condition for injectivity.

Theorem 1 If $\lambda_1, \dots, \lambda_n$ are differentiable barycentric coordinates such that for all $\mathbf{x} \in \bar{P}$, $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) \geq 0$ for all i, j, k with $1 \leq i < j < k \leq n$ and $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) > 0$ for some i, j, k with $1 \leq i < j < k \leq n$ then \mathbf{f} is injective.

Conversely, we derive necessary conditions for injectivity.

Theorem 2 If $\lambda_1, \dots, \lambda_n$ are differentiable barycentric coordinates and \mathbf{f} is injective then for all r, s, t with $1 \leq r < s < t \leq n$,

$$\sum_{r \leq i < s \leq j < t \leq k < n+r} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) \geq 0 \quad \text{in } P. \quad (13)$$

Proof. If \mathbf{f} is injective then $J(\mathbf{f}) \geq 0$ in P . Let Q be the polygon with vertices

$$\begin{aligned} \mathbf{q}_1 = \cdots = \mathbf{q}_{r-1} &= (0, 0), & \mathbf{q}_r = \cdots = \mathbf{q}_{s-1} &= (1, 0), \\ \mathbf{q}_s = \cdots = \mathbf{q}_{t-1} &= (0, 1), & \mathbf{q}_t = \cdots = \mathbf{q}_n &= (0, 0). \end{aligned}$$

Then $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = 1/2$ if $1 \leq i < r \leq j < s \leq k < t$ or $r \leq i < s \leq j < t \leq k \leq n$ and zero otherwise. Substituting these values into (12) gives

$$\sum_{1 \leq i < r \leq j < s \leq k < t} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) + \sum_{r \leq i < s \leq j < t \leq k \leq n} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) \geq 0. \quad (14)$$

Replacing i, j, k by k, i, j in the first sum and using the fact that

$$\mathcal{D}(\lambda_k, \lambda_i, \lambda_j) = \mathcal{D}(\lambda_i, \lambda_j, \lambda_k),$$

the first sum can be written as

$$\sum_{1 \leq k < r \leq i < s \leq j < t} \mathcal{D}(\lambda_i, \lambda_j, \lambda_k) = \sum_{\substack{r \leq i < s \leq j < t \\ n+1 \leq k < n+r}} \mathcal{D}(\lambda_i, \lambda_j, \lambda_{k-n}).$$

Replacing λ_{k-n} by λ_k , combining this second expression with the second sum in (14) gives (13). \square

We have thus derived sets of sufficient conditions and necessary conditions for injectivity, based on the signs of the determinants $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)$. We have not however been able to find a set of conditions that are both sufficient and necessary for general n . Interestingly, a similar difficulty arose in [3, 4, 6] when trying to characterize bases of functions that guarantee convexity preservation of curves: when the control polygon is convex, the curve should be too. So far, a characterization is not known. However, as we will see in the next section, the sufficient conditions of Theorem 1 are good enough to show that Wachspress mappings are injective and we use the necessary ones to help to derive a counterexample for mean value mappings.

For barycentric coordinates expressed in the form (4), the following identity helps to simplify the task of checking the signs of these determinants.

Lemma 2 *For differentiable functions a, b, c , and $\mu > 0$,*

$$\mathcal{D}(\mu a, \mu b, \mu c) = \mu^3 \mathcal{D}(a, b, c). \quad (15)$$

Proof. Recalling (11), expand the second and third rows of the determinant $\mathcal{D}(\mu a, \mu b, \mu c)$ using the product rule. Then, subtracting $\partial_1 \mu / \mu$ times the first row from the second row and subtracting $\partial_2 \mu / \mu$ times the first row from the third row, gives the result. \square

4 Wachspress mappings

We now use the sufficient conditions of Theorem 1 to show that Wachspress mappings are injective.

Theorem 3 *If $\lambda_1, \dots, \lambda_n$ are the Wachspress coordinates (4–5) then \mathbf{f} is injective.*

Proof. The proof uses Theorem 1 and consists of two steps dealing with $\mathbf{x} \in P$ and $\mathbf{x} \in \partial P$, respectively.

Step 1: First let $\mathbf{x} \in P$ and choose i, j, k such that $1 \leq i < j < k \leq n$. We will show that $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) > 0$. Since \mathbf{x} is not on the boundary of P we can define λ_i in (4) using the rational weight function w_i of (6), i.e., as $\lambda_i = \mu w_i$, where $\mu = 1/\sum_j w_j > 0$, and then by Lemma 2, it is sufficient to show that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0$. To this end, observe that

$$\mathcal{D}(w_i, w_j, w_k) = w_i \nabla w_j \times \nabla w_k + w_j \nabla w_k \times \nabla w_i + w_k \nabla w_i \times \nabla w_j, \quad (16)$$

where $\nabla w := (\partial_1 w, \partial_2 w)$ and $\mathbf{p} \times \mathbf{q} := p^1 q^2 - p^2 q^1$. Using (3), differentiation of $A_i(\mathbf{x})$ with respect to x^1 and x^2 gives

$$\nabla A_i = \frac{1}{2} \text{rot}(\mathbf{p}_{i+1} - \mathbf{p}_i), \quad (17)$$

where $\text{rot}(p^1, p^2) := (-p^2, p^1)$, a rotation through a positive angle of $\pi/2$. Therefore, from (6), we have

$$\nabla w_i = -w_i \left(\frac{\text{rot}(\mathbf{p}_i - \mathbf{p}_{i-1})}{2A_{i-1}(\mathbf{x})} + \frac{\text{rot}(\mathbf{p}_{i+1} - \mathbf{p}_i)}{2A_i(\mathbf{x})} \right),$$

and it follows that

$$\nabla w_i \times \nabla w_j = w_i w_j S_{ij}, \quad (18)$$

where

$$S_{ij} = R_{i-1, j-1} + R_{i-1, j} + R_{i, j-1} + R_{ij},$$

and

$$R_{ij} = \frac{\text{rot}(\mathbf{p}_{i+1} - \mathbf{p}_i) \times \text{rot}(\mathbf{p}_{j+1} - \mathbf{p}_j)}{4A_i(\mathbf{x})A_j(\mathbf{x})} = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_i) \times (\mathbf{p}_{j+1} - \mathbf{p}_j)}{4A_i(\mathbf{x})A_j(\mathbf{x})}. \quad (19)$$

Substituting (18) into (16) gives

$$\mathcal{D}(w_i, w_j, w_k) = w_i w_j w_k (S_{ij} + S_{jk} + S_{ki}).$$

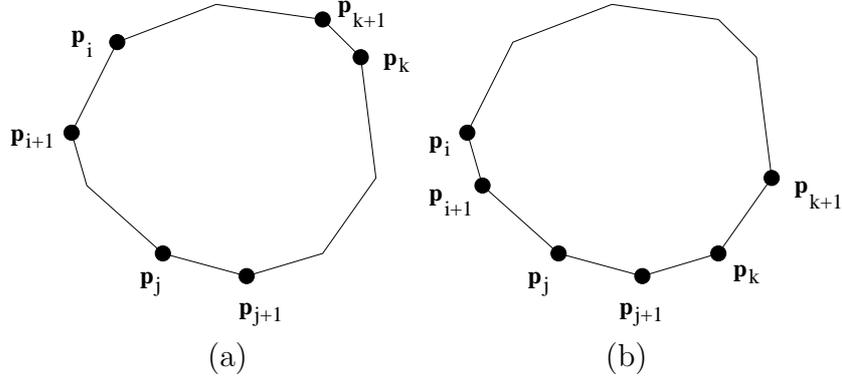


Figure 3: At least two cross products are positive.

Now, each of the three S terms expands into four R terms, giving twelve R terms altogether. We can group these twelve terms into four groups of three such that each group has the form

$$T_{pqr} := R_{pq} + R_{qr} + R_{rp}.$$

There are two ways of doing this. One is

$$\mathcal{D}(w_i, w_j, w_k) = w_i w_j w_k (T_{i,j,k} + T_{i,j-1,k-1} + T_{i-1,j,k-1} + T_{i-1,j-1,k}).$$

Now, since $R_{ii} = 0$, we see that $T_{pqr} = 0$ whenever $p = q$ or $q = r$. Therefore, we see that $\mathcal{D}(w_i, w_j, w_k) > 0$ if

$$T_{ijk} > 0, \quad 1 \leq i < j < k \leq n.$$

To show this, observe from the cross product in (19) and from the convexity of P that at least two of the terms R_{ij} , R_{jk} , R_{ki} must be strictly positive. We may assume without loss of generality that these are R_{ij} and R_{jk} . If the third term R_{ki} is non-negative, as illustrated in Figure 3(a), then $T_{ijk} > 0$ as claimed. Thus it remains to treat the case that $R_{ki} < 0$, as illustrated in Figure 3(b). To this end, with L_α denoting the infinite line that extends the line segment $[\mathbf{p}_\alpha, \mathbf{p}_{\alpha+1}]$, consider the three lines L_i , L_j , L_k . They intersect in three points: \mathbf{r}_1 the intersection of L_i and L_j ; \mathbf{r}_2 the intersection of L_i and L_k ; and \mathbf{r}_3 the intersection of L_j and L_k ; see Figure 4. Then, recalling the ‘half the base times the height’ rule for the area of a triangle, it is easy to see that

$$\frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{A_i(\mathbf{x})} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_2)},$$

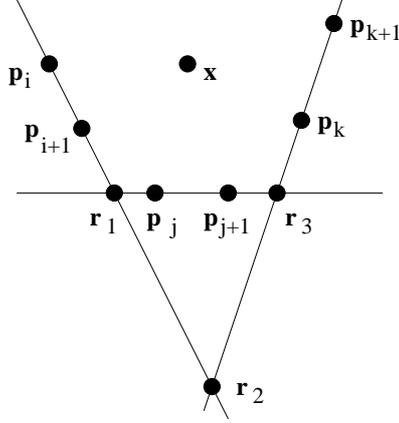


Figure 4: Intersections of three lines.

$$\frac{\mathbf{p}_{j+1} - \mathbf{p}_j}{A_j(\mathbf{x})} = \frac{\mathbf{r}_3 - \mathbf{r}_1}{A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_3)},$$

$$\frac{\mathbf{p}_{k+1} - \mathbf{p}_k}{A_k(\mathbf{x})} = \frac{\mathbf{r}_3 - \mathbf{r}_2}{A(\mathbf{x}, \mathbf{r}_2, \mathbf{r}_3)},$$

and it follows from (19) that

$$R_{ij} = \frac{A(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)}{2A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_2)A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_3)},$$

$$R_{jk} = \frac{A(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)}{2A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_3)A(\mathbf{x}, \mathbf{r}_2, \mathbf{r}_3)},$$

$$R_{ki} = \frac{-A(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)}{2A(\mathbf{x}, \mathbf{r}_2, \mathbf{r}_3)A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_2)}.$$

So

$$T_{ijk} = \frac{(A(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3))^2}{2A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_2)A(\mathbf{x}, \mathbf{r}_1, \mathbf{r}_3)A(\mathbf{x}, \mathbf{r}_2, \mathbf{r}_3)} > 0.$$

This shows that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0$ for $\mathbf{x} \in P$.

Step 2: It remains to deal with $\mathbf{x} \in \partial P$, which we may assume to be as in (8) for some ℓ and $\mu \in [0, 1)$. Since the λ_i are continuous on the whole of \overline{P} and $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) > 0$ on P , we know that $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) \geq 0$ by Step 1. However, for general n , it is possible that there are some triples i, j, k with $1 \leq i < j < k \leq n$ such that $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x}) = 0$. In fact, recalling (11),

this is the case when none of i, j, k equal either ℓ or $\ell + 1$, for then $\lambda_i(\mathbf{x}) = \lambda_j(\mathbf{x}) = \lambda_k(\mathbf{x}) = 0$. We will, however, show that $\mathcal{D}(\lambda_{\ell-1}, \lambda_\ell, \lambda_{\ell+1})(\mathbf{x}) > 0$ which is sufficient for injectivity by Theorem 1. To simplify this task we again appeal to Lemma 2 but this time must use the polynomial w_i . We will show that $\mathcal{D}(w_{\ell-1}, w_\ell, w_{\ell+1})(\mathbf{x}) > 0$ with w_i as in (5). From (16), we can write the determinant as

$$\mathcal{D}(w_{\ell-1}, w_\ell, w_{\ell+1}) = w_{\ell-1} \nabla w_\ell \times \nabla w_{\ell+1} + w_\ell^2 \nabla \left(\frac{w_{\ell+1}}{w_\ell} \right) \times \nabla w_{\ell-1}, \quad (20)$$

valid as long as $w_\ell \neq 0$. This expression is convenient because

$$\frac{w_{\ell+1}}{w_\ell} = \frac{C_{\ell+1} A_{\ell-1}}{C_\ell A_{\ell+1}},$$

and, using (17),

$$\nabla \left(\frac{A_{\ell-1}}{A_{\ell+1}} \right) = \frac{\text{rot}(\mathbf{p}_\ell - \mathbf{p}_{\ell-1})}{2A_{\ell+1}} - \frac{A_{\ell-1} \text{rot}(\mathbf{p}_{\ell+2} - \mathbf{p}_{\ell+1})}{2A_{\ell+1}^2}.$$

Using these observations, we now evaluate (20) at $\mathbf{x} = (1 - \mu)\mathbf{p}_\ell + \mu\mathbf{p}_{\ell+1}$. Since $w_{\ell-1}(\mathbf{x}) = 0$ and

$$\nabla w_{\ell-1}(\mathbf{x}) = K \text{rot}(\mathbf{p}_{\ell+1} - \mathbf{p}_\ell),$$

where

$$K = \frac{1}{2} C_{\ell-1} \prod_{j \neq \ell-2, \ell-1, \ell} A_j(\mathbf{x}) > 0,$$

we find

$$\mathcal{D}(w_{\ell-1}, w_\ell, w_{\ell+1})(\mathbf{x}) = K_1 A(\mathbf{p}_{\ell-1}, \mathbf{p}_\ell, \mathbf{p}_{\ell+1}) + K_2 A(\mathbf{p}_\ell, \mathbf{p}_{\ell+1}, \mathbf{p}_{\ell+2}),$$

where

$$K_1 = K \frac{C_{\ell+1}}{C_\ell} \frac{w_\ell^2(\mathbf{x})}{A_{\ell+1}(\mathbf{x})} > 0,$$

and

$$K_2 = K \frac{C_{\ell+1}}{C_\ell} \frac{w_\ell^2(\mathbf{x}) A_{\ell-1}(\mathbf{x})}{A_{\ell+1}^2(\mathbf{x})} > 0.$$

□

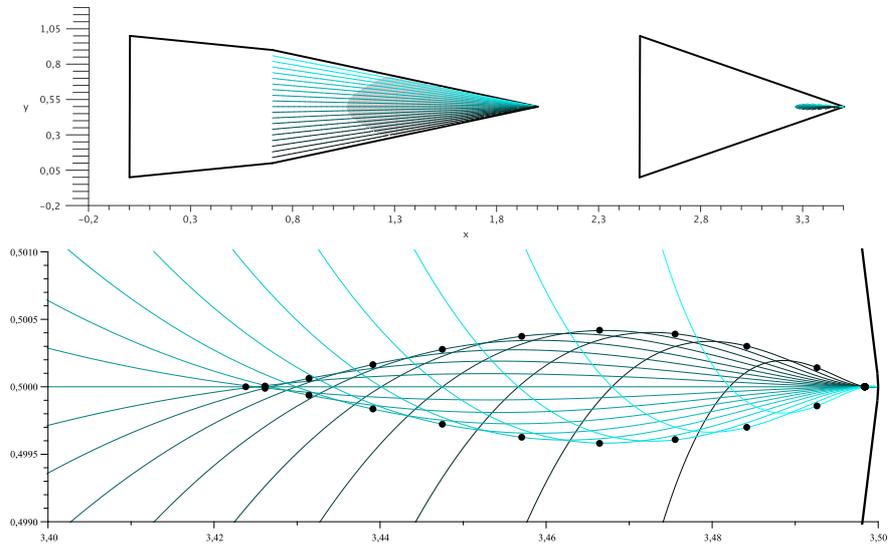


Figure 5: Counterexample 1: Foldover of a mean value mapping between two strictly convex polygons. Top: The gray region G corresponds to negative Jacobian in P . Bottom: The image of G , the foldover area, is bounded by the black dots in Q . Only the part of Q near its vertex \mathbf{q}_3 is shown.

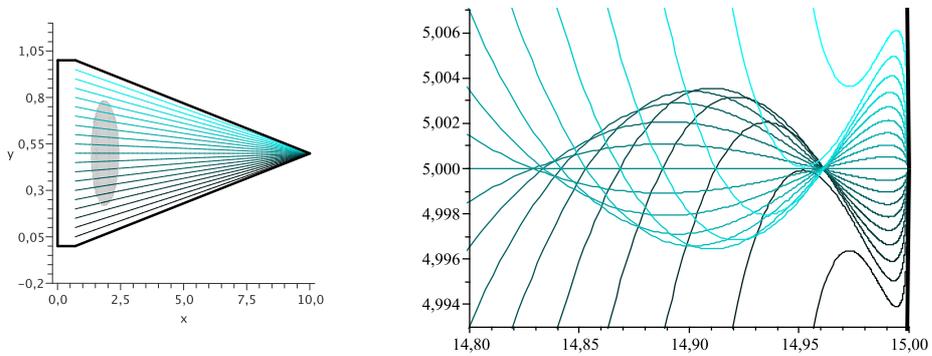


Figure 6: Counterexample 2: Another foldover of a mean value mapping over two strictly convex polygons. Only a part of Q is shown in order to emphasize the foldover.

5 Mean value mappings

Mean value mappings are defined by (10), (4) and (7). Following numerous numerical tests, we conjecture that they are always injective when $n = 4$. However, we have discovered counterexamples when $n = 5$. These were found by exploiting Theorem 2.

5.1 Counterexample 1

Let us suppose that P is given by $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (0.7, 0.1)$, $\mathbf{p}_3 = (2, 0.5)$, $\mathbf{p}_4 = (0.7, 0.9)$ and $\mathbf{p}_5 = (0, 1)$. Moreover, let $\mathbf{x} = (1.2, 0.5) \in P$. By evaluating $\mathcal{D}(\lambda_i, \lambda_j, \lambda_k)(\mathbf{x})$ for the mean value coordinates of P one obtains that

$$\begin{aligned}\mathcal{D}(\lambda_1, \lambda_2, \lambda_5)(\mathbf{x}) &= -0.00011838, \\ \mathcal{D}(\lambda_1, \lambda_3, \lambda_5)(\mathbf{x}) &= -0.00040841, \\ \mathcal{D}(\lambda_1, \lambda_4, \lambda_5)(\mathbf{x}) &= -0.00011838.\end{aligned}\tag{21}$$

Therefore,

$$\mathcal{D}(\lambda_1, \lambda_2, \lambda_5)(\mathbf{x}) + \mathcal{D}(\lambda_1, \lambda_3, \lambda_5)(\mathbf{x}) + \mathcal{D}(\lambda_1, \lambda_4, \lambda_5)(\mathbf{x}) = -0.00064517.$$

As this value is negative, it violates the necessary conditions of Theorem 2 for $n = 5$, $r = 1$, $s = 2$ and $t = 5$. Consequently, by choosing Q to be, for example, given by $\mathbf{q}_1 = (2.5, 0)$, $\mathbf{q}_2 = (3.4999, 0.4999)$, $\mathbf{q}_3 = (3.5, 0.5)$, $\mathbf{q}_4 = (3.4999, 0.5001)$ and $\mathbf{q}_5 = (2.5, 1)$, we arrive at an example of a mean value mapping over strictly convex polygons where the Jacobian is negative at some interior points. Indeed, $J(\mathbf{f})(\mathbf{x}) = -0.0006069$.

Figure 5 depicts the situation and shows the region of P where the corresponding Jacobian is negative. Moreover, it shows a star of line segments in P . One can see the intersections of their images in Figure 5 – a fact that contradicts the injectivity of mean value mappings over convex polygons.

5.2 Counterexample 2

In this second example we assume that P has vertices $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (0.7, 0)$, $\mathbf{p}_3 = (10, 0.5)$, $\mathbf{p}_4 = (0.7, 1)$, $\mathbf{p}_5 = (0, 1)$ and Q is given by $\mathbf{q}_1 = (13, 0)$, $\mathbf{q}_2 = (14.999, 4.99)$, $\mathbf{q}_3 = (15, 5)$, $\mathbf{q}_4 = (14.999, 5.01)$ and $\mathbf{q}_5 = (13, 10)$. Figure 6 shows that the corresponding mean value mapping is not injective.

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