On the convergence of derivatives of Bernstein approximation

Michael S. Floater

Abstract: By differentiating a remainder formula of Stancu, we derive both an error bound and an asymptotic formula for the derivatives of Bernstein approximation.

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1. Introduction

The Bernstein approximation $B_n(f)$ to a function $f : [0,1] \to \mathbb{R}$ is the polynomial

$$B_n f(x) = \sum_{i=0}^{n} f\Big(\frac{i}{n}\Big) p_{n,i}(x), \tag{1.1}$$

where $p_{n,i}$ is the polynomial of degree n,

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n.$$

Bernstein [**1**] used this approximation to give the first constructive proof of the Weierstrass theorem. One of the many remarkable properties of Bernstein approximation is that derivatives of $B_n(f)$ of any order converge to corresponding derivatives of f; see Lorentz [**7**]. If $f \in C^k[0,1]$ for any $k \ge 0$, then

$$\lim_{n\to\infty} (B_n f)^{(k)} = f^{(k)} \qquad \text{uniformly on } [0,1].$$

Other remarkable properties are shape-preservation and variation-diminution [**5**].

These many properties can be viewed as compensation for the slow convergence of $B_n(f)$ to f. With $\|\cdot\|$ the max norm on $[0,1]$, the error bound

$$|B_n(f,x) - f(x)| \le \frac{1}{2n} x(1-x)\|f''\|, \tag{1.2}$$

given in Chapter 10 of [**4**], shows that the rate of convergence is at least $1/n$ for $f \in C^2[0,1]$. On the other hand, the asymptotic formula

$$\lim_{n\to\infty} n\big(B_n f(x) - f(x)\big) = \frac{1}{2} x(1-x) f''(x), \tag{1.3}$$

due to Voronovskaya [**9**], shows that for $x \in (0,1)$ with $f''(x) \ne 0$, the rate of convergence is precisely $1/n$.

In this note we show that all derivatives of the operator B_n converge at essentially the same rate by extending both the error bound (1.2) and the Voronovskaya formula (1.3). Firstly, the error bound generalizes to:

Theorem 1. *If $f \in C^{k+2}[0,1]$ for some $k \geq 0$ then*

$$|(B_n f)^{(k)}(x) - f^{(k)}(x)| \leq \frac{1}{2n}\left(k(k-1)\|f^{(k)}\| + k|1-2x|\|f^{(k+1)}\| + x(1-x)\|f^{(k+2)}\|\right).$$

Secondly, Voronovskaya's formula can be 'differentiated':

Theorem 2. *If $f \in C^{k+2}[0,1]$ for some $k \geq 0$, then*

$$\lim_{n\to\infty} n\big((B_n f)^{(k)}(x) - f^{(k)}(x)\big) = \frac{1}{2}\frac{d^k}{dx^k}\{x(1-x)f''(x)\},$$

uniformly for $x \in [0,1]$.

Thus the k-th derivative of $B_n(f)$ converges at the rate of $1/n$ when the k-th derivative of $x(1-x)f''(x)$ is non-zero.

We remark that after completion of this note, it was found that López-Moreno, Martínez-Moreno, and Muñoz-Delgado [**6**] very recently established Theorem 2 using a completely different approach.

2. Stancu's remainder formula

The traditional way to analyze the error $B_n(f) - f$ and indeed to derive both (1.2) and (1.3) is to substitute the Taylor expansion

$$f\Big(\frac{i}{n}\Big) = f(x) + \Big(\frac{i}{n} - x\Big)f'(x) + \dots$$

into equation (1.1). To deal with derivatives of B_n we will instead borrow an idea from numerical differentiation [**2**]. As is well known, error formulas for numerical differentiation can be obtained from differentiating Newton's remainder formula for polynomial interpolation. This suggests finding an analogous remainder formula for Bernstein approximation and subsequently differentiating it. A natural remainder formula for this purpose is

$$B_n f(x) - f(x) = \frac{1}{n}x(1-x)\sum_{i=0}^{n-1}\Big(\Big[\frac{i}{n}, \frac{i+1}{n}, x\Big]f\Big)p_{n-1,i}(x). \tag{2.1}$$

Here $[x_0, x_1, \dots, x_k]f$ denotes the k-th order divided difference of f at the points $x_0, \dots, x_k$, and we note that the right hand side of (2.1) is valid at least for f in $C^2[0,1]$.

A more general form of this formula for the remainder in tensor-product bivariate Bernstein approximation was derived by Stancu [**8**], but does not appear to be too well known. It therefore seems worth offering the following proof, especially as it is shorter than the orginal in [**8**]. If one recalls the identity

$$\frac{1}{n}x(1-x)p'_{n,i}(x) = \Big(\frac{i}{n} - x\Big)p_{n,i}(x),$$

given in Chapter 10 of [**4**], Stancu's formula follows simply from:

$$\begin{aligned} B_n f(x) - f(x) &= \sum_{i=0}^{n} \Big(f\Big(\frac{i}{n}\Big) - f(x) \Big) p_{n,i}(x) \\ &= \sum_{i=0}^{n} \Big[\frac{i}{n}, x\Big] f \Big(\frac{i}{n} - x\Big) p_{n,i}(x) = \frac{1}{n} x(1-x) \sum_{i=0}^{n} \Big(\Big[\frac{i}{n}, x\Big] f \Big) p'_{n,i}(x) \\ &= x(1-x) \sum_{i=0}^{n-1} \Big(\Big[\frac{i+1}{n}, x\Big] f - \Big[\frac{i}{n}, x\Big] f \Big) p_{n-1,i}(x). \end{aligned}$$

3. Error analysis

In what follows it will help to generalize the operator B_n to

$$B_{n,s,t} f(x) = \sum_{i=0}^{n-s} \Big(\Big[\frac{i}{n}, \dots, \frac{i+s}{n}, \underbrace{x, \dots, x}_{t}\Big] f \Big) p_{n-s,i}(x), \tag{3.1}$$

for any $s, t \geq 0$. We have $B_{n,0,0} = B_n$ and the remainder formula (2.1) can be written as

$$B_n f(x) - f(x) = \frac{1}{n} x(1-x) B_{n,1,1} f(x).$$

Differentiating this k times and using the Leibniz rule gives

$$\begin{aligned} (B_n f)^{(k)}(x) - f^{(k)}(x) = \frac{1}{n} \Big(&- k(k-1)(B_{n,1,1} f)^{(k-2)}(x) \\ &+ k(1-2x)(B_{n,1,1} f)^{(k-1)}(x) + x(1-x)(B_{n,1,1} f)^{(k)}(x) \Big). \end{aligned} \tag{3.2}$$

This leads us to study the derivatives of $B_{n,1,1} f$.

Lemma 1. *If $f \in C^{r+2}[0,1]$ for some $r \geq 0$ then*

$$(B_{n,1,1} f)^{(r)} = r! \sum_{j=1}^{r+1} j \frac{n-1}{n} \cdots \frac{n-j+1}{n} B_{n,j,r-j+2} f.$$

Proof: Using the formula (see Chapter 2 of [**2**])

$$\frac{d^r}{dx^r} \Big[\frac{i}{n}, \frac{i+1}{n}, x\Big] f = r! \Big[\frac{i}{n}, \frac{i+1}{n}, \underbrace{x, \dots, x}_{r+1}\Big] f,$$

differentiation of (3.1) with $s = t = 1$ implies

$$\begin{aligned} (B_{n,1,1} f)^{(r)}(x) &= \sum_{i=0}^{n-1} \sum_{j=0}^{r} \binom{r}{j} (r-j)! \Big(\Big[\frac{i}{n}, \frac{i+1}{n}, \underbrace{x, \dots, x}_{r-j+1}\Big] f \Big) p_{n-1,i}^{(j)}(x) \\ &= r! \sum_{j=0}^{r} \frac{(n-1)\dots(n-j)}{j!} \sum_{i=0}^{n-j-1} \Big(\Delta^j \Big[\frac{i}{n}, \frac{i+1}{n}, \underbrace{x, \dots, x}_{r-j+1}\Big] f \Big) p_{n-j-1,i}(x), \end{aligned} \tag{3.3}$$

where Δ is the forward difference operator w.r.t. i. Now notice that

$$\Delta\Big[\frac{i}{n},\frac{i+1}{n},x,\dots,x\Big]f = \Big[\frac{i+1}{n},\frac{i+2}{n},x,\dots,x\Big]f - \Big[\frac{i}{n},\frac{i+1}{n},x,\dots,x\Big]f$$
$$= \frac{2}{n}\Big[\frac{i}{n},\frac{i+1}{n},\frac{i+2}{n},x,\dots,x\Big]f,$$

and continuing to apply Δ implies

$$\Delta^j\Big[\frac{i}{n},\frac{i+1}{n},x,\dots,x\Big]f = \frac{2\,.\,3\dots(j+1)}{n^j}\Big[\frac{i}{n},\dots,\frac{i+j+1}{n},x,\dots,x\Big]f.$$

Substituting this identity into equation (3.3) and replacing j by $j-1$ gives the result. ■

Due to Lemma 1, we have for $f \in C^{r+2}[0,1]$ and $r \geq 0$,

$$\|(B_{n,1,1}f)^{(r)}\| \leq r!\sum_{j=1}^{r+1} j\,\frac{\|f^{(r+2)}\|}{(r+2)!} = \frac{1}{2}\|f^{(r+2)}\|.$$

Theorem 1 now follows from applying this bound to equation (3.2). To prove Theorem 2 we study the convergence of the operators $B_{n,s,t}$.

Lemma 2. *If $f \in C^{s+t}[0,1]$ for some $s,t \geq 0$ then*

$$\lim_{n\to\infty} B_{n,s,t}f = \frac{f^{(s+t)}}{(s+t)!} \qquad \textit{uniformly on } [0,1].$$

Proof: We extend Davis's proof of Bernstein's theorem, namely the proof of Theorem 6.2.2 of [**3**]. Let $q := s+t$. Then for each i, $0 \leq i \leq n-s$, there is some ξ_i in the smallest interval containing $x, i/n, \dots, (i+s)/n$ such that

$$\Big[\frac{i}{n},\dots,\frac{i+s}{n},\underbrace{x,\dots,x}_{t}\Big]f = \frac{f^{(q)}(\xi_i)}{q!},$$

and it is sufficient to show that

$$S_n := \sum_{i=0}^{n-s}(f^{(q)}(\xi_i) - f^{(q)}(x))p_{n-s,i}(x) \to 0.$$

Let $\epsilon > 0$. Since $f \in C^q[0,1]$, $\exists\delta > 0$ such that $|y-x| < \delta$ implies $|f^{(q)}(y) - f^{(q)}(x)| < \epsilon$. Let I_n be the set of all i, $0 \leq i \leq n-s$, for which $x-\delta < i/n < (i+s)/n < x+\delta$, and split S_n into the two terms

$$C_n = \sum_{i\in I_n}(f^{(q)}(\xi_i) - f^{(q)}(x))p_{n-s,i}(x), \qquad D_n = \sum_{i\notin I_n}(f^{(q)}(\xi_i) - f^{(q)}(x))p_{n-s,i}(x).$$

Now for $i \in I_n$ we clearly have $|\xi_i - x| < \delta$, and so

$$|C_n| \leq \sum_{i \in I_n} \epsilon p_{n-s,i}(x) \leq \epsilon.$$

Regarding D_n, notice that

$$\left|\frac{i}{n} - x\right| \leq \left|\frac{i}{n-s} - x\right| + \left|\frac{i}{n} - \frac{i}{n-s}\right| \leq \left|\frac{i}{n-s} - x\right| + \frac{s}{n},$$

and similarly

$$\left|\frac{i+s}{n} - x\right| \leq \left|\frac{i}{n-s} - x\right| + \left|\frac{i+s}{n} - \frac{i}{n-s}\right| \leq \left|\frac{i}{n-s} - x\right| + \frac{s}{n},$$

and therefore

$$\max\left\{\left|\frac{i}{n} - x\right|^2, \left|\frac{i+s}{n} - x\right|^2\right\} \leq \left(\frac{i}{n-s} - x\right)^2 + O(1/n),$$

uniformly for $x \in [0,1]$. It follows that

$$\begin{aligned}
|D_n| &\leq \frac{2}{\delta^2}\|f^{(q)}\| \sum_{i \notin I_n} \max\left\{\left|\frac{i}{n} - x\right|^2, \left|\frac{i+s}{n} - x\right|^2\right\} p_{n-s,i}(x) \\
&\leq \frac{2}{\delta^2}\|f^{(q)}\| \sum_{i=0}^{n-s} \left(\frac{i}{n-s} - x\right)^2 p_{n-s,i}(x) + O(1/n). \\
&= \frac{2}{(n-s)\delta^2}\|f^{(q)}\| x(1-x) + O(1/n).
\end{aligned}$$

Thus $\lim_{n\to\infty} |S_n| \leq \epsilon$ for any $\epsilon > 0$. ∎

Due to Lemmas 1 and 2, we have for $f \in C^{r+2}[0,1]$ and $r \geq 0$,

$$\lim_{n\to\infty} (B_{n,1,1}f)^{(r)} = r! \sum_{j=1}^{r+1} j \frac{f^{(r+2)}}{(r+2)!} = \frac{f^{(r+2)}}{2} \qquad \text{uniformly on } [0,1],$$

and Theorem 2 follows from multiplying equation (3.2) by n and letting $n \to \infty$.

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Michael S. Floater
Centre of Mathematics for Applications
Department of Informatics
University of Oslo
Postbox 1053, Blindern
0316 Oslo, NORWAY
`michaelf@ifi.uio.no`