Derivatives of rational Bézier curves

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Abstract. Two equations are presented which express the derivative of a rational Bézier curve in terms of its control points and weights. These equations are natural generalisations of the non-rational case and various properties are found from them. Bounds on the magnitude of the derivative and the direction of the derivative (the hodograph) are obtained. Expressions for the second derivative, curvature, and torsion of the curve are also calculated.

§1. Introduction

Rational Bézier and B-spline curves and surfaces are well established as a convenient way to represent Computer Aided Design geometry. Good introductions can be found in Faux & Pratt (1979) and Farin (1988). There are also articles covering the recursive algorithm for evaluating rational Bézier curves, Farin (1983), and recursive algorithms for rational Bézier surfaces, Piegl (1986). Various geometric constructions using rational B-splines are given by Piegl & Tiller (1987).

Many properties of non-rational curves have been naturally extended to equivalent properties of rational curves such as evaluation, subdivision, the convex hull property and degree elevation. Assuming the rational curve lies in $\mathbb{R}^3$, a typical method is as follows.
(i) Map the rational curve into $\mathbb{R}^4$ using homogeneous coordinates. The resulting curve is then non-rational.
(ii) Observe that the desired property holds for the non-rational curve in $\mathbb{R}^4$.
(iii) Show that when we map the curve back to $\mathbb{R}^3$ the property continues to be true.

Properties which do not extend to rational curves in this way are those which involve derivatives. This is due to the fact that the derivative of the $\mathbb{R}^3$ rational curve is very different to the projection of the non-rational $\mathbb{R}^4$ derivative. Indeed, this deficiency could be seen as a drawback of using rational curves and surfaces instead of non-rational ones in CAD. The evaluation of derivatives is less stable and more time consuming. In this paper we attempt to improve the situation.

There are two known formulas for the derivative of a (non-rational) Bézier curve $P(t)$ defined by (13). One of them expresses $P'(t)$ as a function (14) of the last two intermediate points in the de Casteljau algorithm $(P_{1,n-1}(t) - P_{0,n-1}(t))$. The other expresses $P'(t)$ as a function (15) of the control polygon segments $(P_{i+1} - P_i)$. In this paper we generalise both these formulas to the rational case (11) and (12). Equations (11) and (12) provide numerically stable ways of computing $P'(t)$. From them we obtain upper bounds on $P'(t)$ and the hodograph property. Formulas for the curvature and torsion of a rational curve in terms of the intermediate de Casteljau points are found from (11).

To keep things as simple as possible we only treat Bézier curves here though some of the results are being extended to NURBS in a forthcoming paper, Floater (1991). We start by making some basic definitions.

Definition.

Given the control points $P_0, P_1, \ldots, P_n \in \mathbb{R}^3$ and the associated nonnegative weights $w_0, w_1, \ldots, w_n$ we can define, for $0 \le t \le 1$, the rational Bézier curve P of degree n as

$$P(t) = \frac{\sum_{i=0}^n B_{i,n}(t) w_i P_i}{\sum_{i=0}^n B_{i,n}(t) w_i} \tag{1}$$

where $B_{0,n}, B_{1,n}, \ldots, B_{n,n}$ are the Bernstein polynomials,

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}. \tag{2}$$

Farin (1983) gave a recursive algorithm for calculating $P(t)$, analogous to the de Casteljau algorithm. If we define the intermediate weights $w_{i,k}(t)$ as

$$w_{i,k}(t) = \sum_{j=0}^k B_{j,k}(t) w_{i+j}, \tag{3}$$

we can compute them using the de Casteljau algorithm,

$$w_{i,k} = (1-t) w_{i,k-1} + t w_{i+1,k-1}. \tag{4}$$

Now define the intermediate points $P_{i,k}(t)$ as

$$P_{i,k}(t) = \frac{\sum_{j=0}^k B_{j,k}(t) w_{i+j} P_{i+j}}{\sum_{j=0}^k B_{j,k}(t) w_{i+j}}. \tag{5}$$

It follows that $P_{i,0}(t) = P_i$ and $P_{0,n}(t) = P(t)$. Farin showed that

$$w_{i,k} P_{i,k} = (1-t) w_{i,k-1} P_{i,k-1} + t w_{i+1,k-1} P_{i+1,k-1} \tag{6}$$

and hence that (4) and (6) together represent a recursive algorithm for computing $P(t)$, see figure 1.

Now consider the problem of finding the derivative $P'(t)$ of the curve $P(t)$. Assuming $P(t)$ is already known, an obvious solution is to rewrite (1) as

$$P(t) = \frac{a(t)}{b(t)}.$$

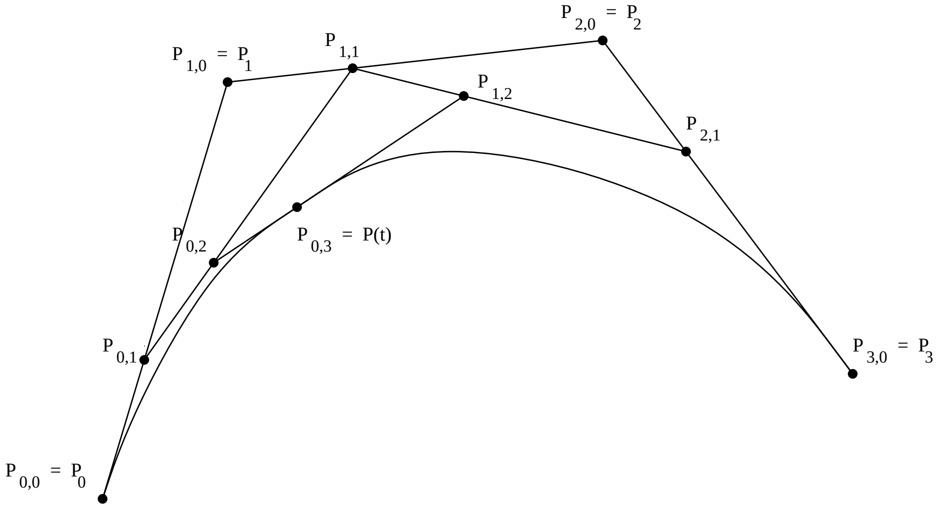

Figure 1. The recursive evaluation scheme. It depends on the points and their weights.

Now since $bP = a$ it follows that $b'P + bP' = a'$ and hence

$$P'(t) = \frac{a'(t) - b'(t)P(t)}{b(t)}. \tag{7}$$

Now the functions $a(t)$ and $b(t)$ are differentiated by using the identity

$$B'_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t)), \tag{8}$$

see Farin (1988). We then find

$$a'(t) = \sum_{i=0}^{n} B'_{i,n}(t) w_i P_i = n \sum_{i=0}^{n-1} B_{i,n-1}(t)(w_{i+1}P_{i+1} - w_i P_i) \tag{9}$$

and

$$b'(t) = n \sum_{i=0}^{n-1} B_{i,n-1}(t)(w_{i+1} - w_i). \tag{10}$$

This method enables us to determine $P'(t)$ but it is difficult to gain any geometrical insight into its behaviour. In this paper, we derive the formulas

$$P'(t) = n \frac{w_{0,n-1}(t) w_{1,n-1}(t)}{w_{0,n}^2(t)} (P_{1,n-1}(t) - P_{0,n-1}(t)) \tag{11}$$

and

$$P'(t) = \sum_{i=0}^{n-1} \lambda_i(t)(P_{i+1} - P_i) \tag{12}$$

where, for $i = 0, \ldots, n-1$,

$$\lambda_i(t) = \frac{n}{w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (B_{j,n-1}(t)B_{k-1,n-1}(t) - B_{j-1,n-1}(t)B_{k,n-1}(t))w_j w_k \quad (12a)$$

$$= \frac{1}{(1-t)tw_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (k-j)B_{j,n}(t)B_{k,n}(t)w_j w_k. \quad (12b)$$

These two expressions generalise those for the non-rational curve. By setting $w_i = 1$ for $i = 0, \ldots, n$, $P(t)$ becomes the non-rational Bézier curve

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t)P_i. \quad (13)$$

In this case $w_{i,k}(t) = 1$ for all t, i, k and so equation (11) reduces to the well known equation

$$P'(t) = n(P_{1,n-1}(t) - P_{0,n-1}(t)), \quad (14)$$

see e.g. Farin (1988). We can show that equation (12) also simplifies when $w_i = 1$, for all i. If this were the case, we could use the fact that

$$\sum_{j=0}^{i} \sum_{k=0}^{i} (B_{j,n-1}B_{k-1,n-1} - B_{j-1,n-1}B_{k,n-1}) = 0$$

to show that

$$\begin{aligned} \lambda_i &= n \sum_{j=0}^{i} \sum_{k=i+1}^{n} (B_{j,n-1}B_{k-1,n-1} - B_{j-1,n-1}B_{k,n-1}) \\ &= n \sum_{j=0}^{i} \sum_{k=0}^{n} (B_{j,n-1}B_{k-1,n-1} - B_{j-1,n-1}B_{k,n-1}) \\ &= n \sum_{j=0}^{i} (B_{j,n-1} - B_{j-1,n-1}) \\ &= nB_{i,n-1} \end{aligned}$$

and hence

$$P'(t) = n \sum_{i=0}^{n-1} B_{i,n-1}(t)(P_{i+1} - P_i) \quad (15)$$

as expected when P is non-rational.

After giving the proofs of (11) and (12) in Section 2 we derive various useful properties of $P'(t)$ in Section 3. First of all, from equation (11) it immediately follows that the curve P is tangential to $P_{1,n-1}(t) - P_{0,n-1}(t)$ at t. It is also simple to deduce that at the two ends of the curve we have $P'(0) = \frac{w_1}{w_0}(P_1 - P_0)$ and $P'(1) = \frac{w_{n-1}}{w_n}(P_n - P_{n-1})$. Moreover,

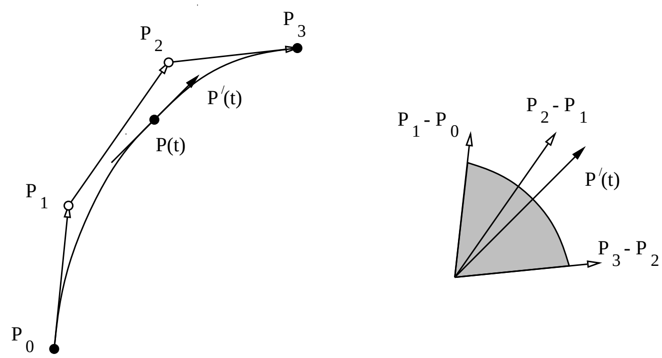

Figure 2. Every tangent to the curve lies between the control segments.

we show later that equation (11) also leads to the following upper bound on the magnitude of the derivative in terms of the weights and control points,

$$\|P'(t)\| \leq n\frac{W}{w} \max_{i,j=0,\ldots,n} \|P_i - P_j\| \tag{16}$$

where $W = \max_i\{w_i\}$ and $w = \min_i\{w_i\}$.

An immediate consequence of equation (12) combined with (12b) is that P has the hodograph property. That rational Bézier curves have this property was shown by Sederberg & Wang (1988) via a different approach. A curve is said to have the hodograph property if, at all points on the curve, the tangent (in the direction of increasing t) lies between the directions of the control polygon segments $P_{i+1} - P_i$, see figure 2. The existence of the hodograph has many implications. It means, for example, that one can often determine whether the curve is open or closed, whether two curves can intersect tangentially, and to compute a good bounding box for an offset curve. Further details and some applications of the hodograph property are given in Sederberg & Meyers (1988).

We further show in Section 3 that we can find another upper bound on $\|P'(t)\|$, namely

$$\|P'(t)\| \leq n\frac{W^2}{w^2} \max_{i=0,\ldots,n-1} \|P_{i+1} - P_i\|. \tag{17}$$

There are situations where the bound (17) is sharper than (16) and others where the opposite is true. If all the control points $P_0, \ldots, P_n$ are evenly spaced along a straight line and all the weights are set to 1 then (17) is the better bound. On the other hand, when $n = 2$ and P_0, P_1, and P_2 form an equilateral triangle and the weights are not all equal, then (16) is sharper. In applications one could calculate both bounds and use the minimum of the two.

Expressions for higher derivatives rapidly become complicated. Nevertheless, by demonstrating a simple recursive formula for the difference $P_{i+1,k} - P_{i,k}$ namely,

$$P_{i+1,k} - P_{i,k} = (1-t)\frac{w_{i,k-1}}{w_{i,k}}(P_{i+1,k-1} - P_{i,k-1}) + t\frac{w_{i+2,k-1}}{w_{i+1,k}}(P_{i+2,k-1} - P_{i+1,k-1}) \tag{18}$$

we are able to show that the second derivative is

$$\begin{aligned} P'' =& n\frac{w_{2,n-2}}{w_{0,n}^3}(2nw_{0,n-1}^2 - (n-1)w_{0,n-2}w_{0,n} - 2w_{0,n-1}w_{0,n})(P_{2,n-2} - P_{1,n-2}) \\ & - n\frac{w_{0,n-2}}{w_{0,n}^3}(2nw_{1,n-1}^2 - (n-1)w_{2,n-1}w_{0,n} - 2w_{1,n-1}w_{0,n})(P_{1,n-2} - P_{0,n-2}). \end{aligned} \tag{19}$$

Despite the complexity of higher derivatives, relatively neat expressions are obtained for the curvature and torsion of P in Section 4. We prove that the curvature is given by

$$\kappa = \frac{(n-1)}{n} R_1 \frac{\|(P_{1,n-2} - P_{0,n-2}) \times (P_{2,n-2} - P_{1,n-2})\|}{\|(P_{1,n-1} - P_{0,n-1})\|^3} \tag{20}$$

and the torsion is given by

$$\tau = \frac{(n-2)}{n} R_2 \frac{(P_{1,n-3} - P_{0,n-3}) \times (P_{2,n-3} - P_{1,n-3}).(P_{3,n-3} - P_{2,n-3})}{\|(P_{1,n-2} - P_{0,n-2}) \times (P_{2,n-2} - P_{1,n-2})\|^2} \tag{21}$$

where

$$R_1 = \frac{w_{0,n-2} w_{1,n-2} w_{2,n-2} w_{0,n}^3}{w_{0,n-1}^3 w_{1,n-1}^3} \quad \text{and} \quad R_2 = \frac{w_{0,n-3} w_{1,n-3} w_{2,n-3} w_{3,n-3} w_{0,n}^2}{w_{0,n-2}^2 w_{1,n-2}^2 w_{2,n-2}^2}.$$

These formulas offer a direct approach to calculating the curvature and torsion. A less direct method is to subdivide the curve at the given parameter value. One can then apply the known formulas for curvature and torsion at the end of a curve as described in Farin (1988).

§2. The derivative of a Bézier curve

Throughout this article we will only ever consider one parameter value t at a time so there is no ambiguity in dropping the t. Furthermore we will define $B_{i,k}(t) = 0$ for all t whenever $i < 0$ or $i > n$. First of all we derive equation (11).

Proposition 1. *Given the Bézier curve of degree n defined by equations (4) and (6), the derivative of P at t is*

$$P' = n \frac{w_{0,n-1} w_{1,n-1}}{w_{0,n}^2} (P_{1,n-1} - P_{0,n-1}).$$

Proof. Recalling equations (3) and (5) for $w_{i,k}$ and $P_{i,k}$, we find that from (9),

$$a' = n(w_{1,n-1} P_{1,n-1} - w_{0,n-1} P_{0,n-1})$$

and, from (10),

$$b' = n(w_{1,n-1} - w_{0,n-1}).$$

The substitution of these identities into (7) implies

$$w_{0,n} P' = n(w_{1,n-1} P_{1,n-1} - w_{0,n-1} P_{0,n-1}) - n(w_{1,n-1} - w_{0,n-1}) P_{0,n}.$$

Therefore, applying (4) and (6), we have

$$\begin{aligned} w_{0,n}^2 P' =& n(w_{1,n-1} w_{0,n} P_{1,n-1} - w_{0,n-1} w_{0,n} P_{0,n-1}) \\ & - n(w_{1,n-1} - w_{0,n-1})((1-t) w_{0,n-1} P_{0,n-1} + t w_{1,n-1} P_{1,n-1}) \\ =& n\{w_{1,n-1} w_{0,n} P_{1,n-1} - w_{0,n-1} w_{0,n} P_{0,n-1} - (1-t) w_{0,n-1} w_{1,n-1} P_{0,n-1} \\ & + (1-t) w_{0,n-1}^2 P_{0,n-1} - t w_{1,n-1}^2 P_{1,n-1} + t w_{0,n-1} w_{1,n-1} P_{1,n-1}\} \\ =& n\{w_{1,n-1}(w_{0,n} - t w_{1,n-1} + t w_{0,n-1}) P_{1,n-1} \\ & - w_{0,n-1}(w_{0,n} + (1-t) w_{1,n-1} - (1-t) w_{0,n-1}) P_{0,n-1}\} \\ =& n\{w_{1,n-1} w_{0,n-1} P_{1,n-1} - w_{0,n-1} w_{1,n-1} P_{0,n-1}\} \\ =& n w_{0,n-1} w_{1,n-1} (P_{1,n-1} - P_{0,n-1}) \end{aligned}$$

as claimed. ◁

Observe that we can find a similar expression for $P'_{i,k}(t)$ given any $k \in \{1, \ldots, n\}$ and $i \in \{0, \ldots, n-k\}$ by noticing that $P_{i,k}(t)$ is itself a Bézier curve with control points $P_i, \ldots, P_{i+k}$ and weights $w_i, \ldots, w_{i+k}$. Equation (11) then becomes

$$P'_{i,k} = k \frac{w_{i,k-1} w_{i+1,k-1}}{w_{i,k}^2} (P_{i+1,k-1} - P_{i,k-1}). \tag{22}$$

because $P(t) = P_{0,n}(t)$ and the computation is valid for every Bézier curve.

In order to prove equation (12) it is necessary to employ the following lemma concerning the basis functions.

Lemma 2. *For any $i, j \in \{0, \ldots, n\}$, $t \in [0, 1]$,*

$$B'_{i,n} B_{j,n} - B_{i,n} B'_{j,n} = n(B_{i-1,n-1} B_{j,n-1} - B_{i,n-1} B_{j-1,n-1}) \tag{23}$$

$$B'_{i,n} B_{j,n} - B_{i,n} B'_{j,n} = (i-j) \frac{B_{i,n} B_{j,n}}{(1-t)t} \tag{24}$$

Proof. To prove the first identity, recall the formula for $B'_{i,n}(t)$, equation (8), and the recursive formula for $B_{i,n}(t)$,

$$B_{i,n}(t) = (1-t) B_{i,n-1}(t) + t B_{i-1,n-1}(t).$$

By substituting these into the left hand side of (23) and simplifying the subsequent expression we prove (23). Now expand the right hand side of (23) using definition (2). After collecting together common terms, (24) follows from straightforward algebra. ◁

Note the right hand side of (24) is a removable singularity at $t = 0$ and $t = 1$ since $B_{i,n}(t) B_{j,n}(t)$ is always a multiple of $(1-t)t$ unless $i = j = 0$ or $i = j = n$. The identities (23) and (24) are used to prove the following.

Proposition 3. *Given the Bézier curve of degree n defined by equation (1), the derivative of P at t is*

$$P'(t) = \sum_{i=0}^{n-1} \lambda_i(t) (P_{i+1} - P_i)$$

where

$$\begin{aligned} \lambda_i(t) =& \frac{n}{w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (B_{j,n-1}(t) B_{k-1,n-1}(t) - B_{j-1,n-1}(t) B_{k,n-1}(t)) w_j w_k \\ =& \frac{1}{(1-t) t w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (k-j) B_{j,n}(t) B_{k,n}(t) w_j w_k. \end{aligned}$$

Proof. Rewrite (1) as

$$P(t) = \sum_{i=0}^{n} \mu_i(t) w_i P_i, \qquad \mu_i(t) = \frac{B_{i,n}(t)}{\sum_{k=0}^{n} B_{k,n}(t) w_k} = \frac{B_{i,n}(t)}{w_{0,n}(t)}.$$

Then

$$\mu_i' = \frac{1}{w_{0,n}^2}\left(\sum_{k=0}^{n} B_{k,n} w_k B_{i,n}' - \sum_{k=0}^{n} B_{k,n}' w_k B_{i,n}\right)$$
$$= \frac{1}{w_{0,n}^2}\sum_{k=0}^{n}(B_{i,n}' B_{k,n} - B_{i,n} B_{k,n}') w_k.$$

Now, at any t, P' is in the form

$$P' = \sum_{i=0}^{n} a_i P_i$$

where $a_i = \mu_i'(t) w_i$ and we wish to rewrite it in the form

$$P' = \sum_{i=0}^{n-1} \lambda_i (P_{i+1} - P_i). \tag{25}$$

If we set $\lambda_i = -\sum_{j=0}^{i} a_j$ we find

$$\sum_{i=0}^{n-1} \lambda_i (P_{i+1} - P_i) = \sum_{i=1}^{n} \lambda_{i-1} P_i - \sum_{i=0}^{n-1} \lambda_i P_i$$
$$= \lambda_{n-1} P_n + \sum_{i=1}^{n-1} (\lambda_{i-1} - \lambda_i) P_i - \lambda_0 P_0$$
$$= \left(-\sum_{j=0}^{n-1} a_j\right) P_n + \sum_{i=1}^{n-1} a_i P_i + a_0 P_0$$
$$= \left(-\sum_{j=0}^{n} a_j\right) P_n + \sum_{i=0}^{n} a_i P_i.$$

Thus we can write P' in the form of (25) if the condition

$$\sum_{i=0}^{n} a_i = 0$$

is satisfied. Well, the condition is indeed satisfied since, by symmetry,

$$\sum_{i=0}^{n} \mu_i'(t) w_i = \frac{1}{w_{0,n}^2(t)} \sum_{i=0}^{n}\sum_{k=0}^{n} (B_{i,n}'(t) B_{k,n}(t) - B_{i,n}(t) B_{k,n}'(t)) w_i w_k = 0.$$

Therefore we can express P' in the form of (25) where

$$\begin{aligned}\lambda_i(t) =& \frac{-1}{w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=0}^{n} (B'_{j,n}(t) B_{k,n}(t) - B_{j,n}(t) B'_{k,n}(t)) w_j w_k \\ =& \frac{1}{w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=0}^{n} (B'_{k,n}(t) B_{j,n}(t) - B_{k,n}(t) B'_{j,n}(t)) w_j w_k \\ =& \frac{1}{w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (B'_{k,n}(t) B_{j,n}(t) - B_{k,n}(t) B'_{j,n}(t)) w_j w_k.\end{aligned}$$

The last step comes from the fact that, again by symmetry,

$$\sum_{j=0}^{i} \sum_{k=0}^{i} (B'_{k,n}(t) B_{j,n}(t) - B_{k,n}(t) B'_{j,n}(t)) w_j w_k = 0.$$

Using Lemma 2, we can write λ_i in the two alternative forms

$$\lambda_i(t) = \frac{n}{w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (B_{j,n-1}(t) B_{k-1,n-1}(t) - B_{j-1,n-1}(t) B_{k,n-1}(t)) w_j w_k$$

and

$$\lambda_i(t) = \frac{1}{(1-t) t w_{0,n}^2(t)} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (k-j) B_{j,n}(t) B_{k,n}(t) w_j w_k$$

which completes the proof. $\triangleleft$

§3. Upper bounds and the hodograph

In some applications of rational Bézier curves and surfaces it is important to have a measure of the size of the derivatives. For example, in a paper by Filip, Magedson & Markot (1986) an efficient subdivision algorithm was presented for the intersection of surfaces. Testing for flatness can be eliminated and determining bounding boxes can be speeded up provided upper bounds on the magnitudes of the first and second derivatives of the surface can be calculated in advance. We are thus motivated to find a bound for $P'(t)$.

By applying equation (11) we can obtain information about the magnitude of the derivative of $P(t)$. Previously an upper bound has only been obtained for the derivative of a non-rational Bézier curve where

$$\|P'(t)\| \leq n \max_{i=0,\ldots,n-1} \|P_{i+1} - P_i\|$$

as described in Filip, Magedson & Markot (1986). In the rational case it follows from (11) that

$$\|P'(t)\| \leq n \frac{w_{0,n-1} w_{1,n-1}}{w_{0,n}^2} \|P_{1,n-1} - P_{0,n-1}\|.$$

Now since $P_{0,n-1}$ and $P_{1,n-1}$ are convex combinations of $P_0, \ldots, P_n$, we can write

$$P_{0,n-1} = \sum_{i=0}^{n} a_i P_i \qquad \text{and} \qquad P_{1,n-1} = \sum_{j=0}^{n} b_j P_j$$

where $\sum_{i=0}^{n} a_i = 1$, $a_i \geq 0$ and $\sum_{j=0}^{n} b_j = 1$, $b_j \geq 0$. It follows that

$$\begin{aligned} P_{1,n-1} - P_{0,n-1} &= \sum_{j=0}^{n} b_j P_j - \sum_{i=0}^{n} a_i P_i \\ &= \sum_{i=0}^{n} a_i \sum_{j=0}^{n} b_j P_j - \sum_{i=0}^{n} a_i P_i \sum_{j=0}^{n} b_j \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n} a_i b_j (P_i - P_j). \end{aligned}$$

By considering each of the two possibilities $w_{0,n-1} \leq w_{1,n-1}$ and $w_{0,n-1} \geq w_{1,n-1}$ it follows that either $w_{0,n} \geq w_{0,n-1}$ or $w_{0,n} \geq w_{1,n-1}$ and thus

$$\frac{w_{0,n-1} w_{1,n-1}}{w_{0,n}^2} \leq \max \left\{ \frac{w_{0,n-1}}{w_{0,n}}, \frac{w_{1,n-1}}{w_{0,n}} \right\} \leq \frac{W}{w}$$

for all t, where $W = \max_i \{w_i\}$ and $w = \min_i \{w_i\}$ (it is clear that $w \leq w_{i,k}(t) \leq W$ from the definition of $w_{i,k}$). Therefore

$$\|P'(t)\| \leq n \frac{W}{w} \max_{i,j=0,\ldots,n} \|P_i - P_j\|.$$

A different upper bound for $\|P'(t)\|$ can be computed from equation (12). Following a similar argument to that on page 4 and since $w_{i,k} \geq 0$, $B_{i,k} \geq 0$, and $w \leq w_{i,k} \leq W$ for all i, k, and t, we can deduce that

$$\begin{aligned}
\|P'(t)\| \leq& M \frac{W^2}{w^2} \frac{1}{(1-t)t} \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{k=i+1}^{n} (k-j) B_{j,n}(t) B_{k,n}(t) \\
=& nM \frac{W^2}{w^2} \sum_{i=0}^{n-1} \sum_{j=0}^{i} \sum_{k=0}^{n} (B_{j,n-1}(t) B_{k-1,n-1}(t) - B_{j-1,n-1}(t) B_{k,n-1}(t)) \\
=& nM \frac{W^2}{w^2} \sum_{i=0}^{n-1} B_{i,n-1}(t) \\
=& nM \frac{W^2}{w^2}
\end{aligned}$$

where $M = \max_i \|P_{i+1} - P_i\|$.

Finally, equation (12b) not only yields a bound on the magnitude of P' but on its direction as well. Since $k - j > 0$ for all $j \leq i$, $k \geq i+1$ it is clear that $\lambda_i \geq 0$. A consequence of this is that P has the hodograph property defined by Sederberg & Meyers (1988) i.e. P' can be expressed in the form

$$P'(t) = \sum_{i=0}^{n} \lambda_i(t)(P_{i+1} - P_i)$$

where $\lambda_i(t) \geq 0$ for all i and t. Thus we have demonstrated an alternative proof of the hodograph property for rational Bézier curves.

§4. Higher derivatives, curvature and torsion

Having found equation (11) for the first derivative of $P(t)$ it is natural to search for similar expressions for higher derivatives. These derivatives quickly become unwieldy and we were not able to obtain a general expression for the nth derivative. However we will demonstrate equation (19) for the second derivative. First we prove a lemma.

Lemma 4. *Given the Bézier curve of degree n defined by equations (4) and (6) and given any $k \in \{1, \dots, n-1\}$ and $i \in \{0, \dots, n-k\}$, the following expression holds:*

$$P_{i+1,k} - P_{i,k} = (1-t)\frac{w_{i,k-1}}{w_{i,k}}(P_{i+1,k-1} - P_{i,k-1}) + t\frac{w_{i+2,k-1}}{w_{i+1,k}}(P_{i+2,k-1} - P_{i+1,k-1}).$$

Proof. To ease notation we will only prove it for $i = 0$; the algebra is identical for any $i \in \{0, \dots, n-k\}$. From definition (6) we find

$$P_{1,k} - P_{0,k} = \frac{(1-t)w_{1,k-1}P_{1,k-1} + tw_{2,k-1}P_{2,k-1}}{w_{1,k}} - \frac{(1-t)w_{0,k-1}P_{0,k-1} + tw_{1,k-1}P_{1,k-1}}{w_{0,k}}.$$

Therefore

$$\begin{aligned} w_{0,k}w_{1,k}(P_{1,k} - P_{0,k}) & \\ =& w_{0,k}((1-t)w_{1,k-1}P_{1,k-1} + tw_{2,k-1}P_{2,k-1}) \\ & - w_{1,k}((1-t)w_{0,k-1}P_{0,k-1} + tw_{1,k-1}P_{1,k-1}) \\ =& tw_{0,k}w_{2,k-1}P_{2,k-1} - (1-t)w_{1,k}w_{0,k-1}P_{0,k-1} \\ & + ((1-t)w_{0,k}w_{1,k-1} - tw_{1,k}w_{1,k-1})P_{1,k-1}. \end{aligned}$$

But the coefficient of $P_{1,k-1}$ can be rewritten as

$$\begin{aligned} (1-t)w_{0,k}w_{1,k-1} - tw_{1,k}w_{1,k-1} =& w_{0,k}(w_{1,k} - tw_{2,k-1}) - w_{1,k}(w_{0,k} - (1-t)w_{0,k-1}) \\ =& -tw_{0,k}w_{2,k-1} + (1-t)w_{1,k}w_{0,k-1}. \end{aligned}$$

Hence

$$\begin{aligned} w_{0,k}w_{1,k}(P_{1,k} - P_{0,k}) =& \\ & tw_{0,k}w_{2,k-1}P_{2,k-1} - (1-t)w_{1,k}w_{0,k-1}P_{0,k-1} \\ & + (-tw_{0,k}w_{2,k-1} + (1-t)w_{1,k}w_{0,k-1})P_{1,k-1} \\ =& (1-t)w_{1,k}w_{0,k-1}(P_{1,k-1} - P_{0,k-1}) + tw_{0,k}w_{2,k-1}(P_{2,k-1} - P_{1,k-1}) \end{aligned}$$

as required. ◃

Note that Lemma 4 provides yet another way of computing $P'(t)$. One can begin by computing $P_{i+1} - P_i$ for each i. One can then recursively compute $P_{i+1,k}(t) - P_{i,k}(t)$ until $P_{1,n-1}(t) - P_{0,n-1}(t)$ has been found. This can then be substituted into (11).

Proposition 5. *Given the Bézier curve of degree $n \geq 2$ defined by equations (4) and (6), the second derivative of $P(t)$ is*

$$
\begin{aligned}
P'' =& n\frac{w_{2,n-2}}{w_{0,n}^3}(2nw_{0,n-1}^2 - (n-1)w_{0,n-2}w_{0,n} - 2w_{0,n-1}w_{0,n})(P_{2,n-2} - P_{1,n-2}) \\
&- n\frac{w_{0,n-2}}{w_{0,n}^3}(2nw_{1,n-1}^2 - (n-1)w_{2,n-2}w_{0,n} - 2w_{1,n-1}w_{0,n})(P_{1,n-2} - P_{0,n-2}).
\end{aligned}
$$

Proof. Rewrite (11) as

$$
w_{0,n}^2 P' = nw_{0,n-1}w_{1,n-1}(P_{1,n-1} - P_{0,n-1}).
$$

Then differentiating and applying Lemma 4 and (22) gives

$$
\begin{aligned}
&2w_{0,n}w_{0,n}'P' + w_{0,n}^2 P'' \\
&\quad= n(w_{0,n-1}'w_{1,n-1} + w_{0,n-1}w_{1,n-1}')(P_{1,n-1} - P_{0,n-1}) \\
&\qquad + nw_{0,n-1}w_{1,n-1}(P_{1,n-1}' - P_{0,n-1}') \\
&\quad= n(n-1)((w_{1,n-2} - w_{0,n-2})w_{1,n-1} + w_{0,n-1}(w_{2,n-2} - w_{1,n-2})) \\
&\qquad\qquad \{(1-t)\frac{w_{0,n-2}}{w_{0,n-1}}(P_{1,n-2} - P_{0,n-2}) + t\frac{w_{2,n-2}}{w_{1,n-1}}(P_{2,n-2} - P_{1,n-2})\} \\
&\qquad + n(n-1)w_{0,n-1}w_{1,n-1} \\
&\qquad\qquad \{\frac{w_{1,n-2}w_{2,n-2}}{w_{1,n-1}^2}(P_{2,n-2} - P_{1,n-2}) - \frac{w_{0,n-2}w_{1,n-2}}{w_{0,n-1}^2}(P_{1,n-2} - P_{0,n-2})\} \\
&\quad= n(n-1)a(P_{1,n-2} - P_{0,n-2}) + n(n-1)b(P_{2,n-2} - P_{1,n-2}) \qquad (26)
\end{aligned}
$$

where a and b have to be calculated. Well

$$
\begin{aligned}
w_{0,n-1}a =& (1-t)w_{0,n-2}(w_{1,n-2} - w_{0,n-2})w_{1,n-1} + (1-t)w_{0,n-2}w_{0,n-1}(w_{2,n-2} - w_{1,n-2}) \\
&- w_{0,n-2}w_{1,n-2}w_{1,n-1} \\
=& - w_{0,n-2}w_{0,n-1}w_{1,n-1} + (1-t)w_{0,n-2}w_{0,n-1}(w_{2,n-2} - w_{1,n-2}) \\
=& w_{0,n-2}w_{0,n-1}\{-w_{1,n-1} + (1-t)(w_{2,n-2} - w_{1,n-2})\} \\
=& w_{0,n-2}w_{0,n-1}(w_{2,n-2} - 2w_{1,n-1})
\end{aligned}
$$

so that

$$
a = w_{0,n-2}(w_{2,n-2} - 2w_{1,n-1})
$$

and similarly

$$
b = w_{2,n-2}(2w_{0,n-1} - w_{0,n-2}).
$$

Now by Lemma 4,

$$
\begin{aligned}
2w_{0,n}w_{0,n}'P' =& 2n^2(w_{1,n-1} - w_{0,n-1})\frac{w_{0,n-1}w_{1,n-1}}{w_{0,n}}(P_{1,n-1} - P_{0,n-1}) \\
=& 2\frac{n^2}{w_{0,n}}(w_{1,n-1} - w_{0,n-1})\{(1-t)w_{0,n-2}w_{1,n-1}(P_{1,n-2} - P_{0,n-2}) \\
&\qquad + tw_{2,n-2}w_{0,n-1}(P_{2,n-2} - P_{1,n-2})\} \\
=& 2\frac{n^2}{w_{0,n}}\{w_{0,n-2}w_{1,n-1}(w_{1,n-1} - w_{0,n})(P_{1,n-2} - P_{0,n-2}) \\
&\qquad + w_{2,n-2}w_{0,n-1}(w_{0,n} - w_{0,n-1})(P_{2,n-2} - P_{1,n-2})\}.
\end{aligned}
$$

Therefore, from (26),

$$w_{0,n}^3 P'' = A(P_{2,n-2} - P_{1,n-2}) - B(P_{1,n-2} - P_{0,n-2})$$

where

$$\begin{aligned}A =& n(n-1)w_{2,n-2}(2w_{0,n-1} - w_{0,n-2})w_{0,n} - 2n^2 w_{2,n-2}w_{0,n-1}(w_{0,n} - w_{0,n-1})\\ =& n w_{2,n-2}(2n w_{0,n-1}^2 - (n-1)w_{0,n-2}w_{0,n} - 2w_{0,n-1}w_{0,n})\end{aligned}$$

and

$$\begin{aligned}B =& -n(n-1)w_{0,n-2}(w_{2,n-2} - 2w_{1,n-1})w_{0,n} + 2n^2 w_{0,n-2}w_{1,n-1}(w_{1,n-1} - w_{0,n})\\ =& n w_{0,n-2}(2n w_{1,n-1}^2 - (n-1)w_{2,n-2}w_{0,n} - 2w_{1,n-1}w_{0,n})\end{aligned}$$

which finishes the proof. ◁

It is interesting to check that (19) really generalises the corresponding expression for non-rational Bézier curves. Indeed, setting $w_i = 1$, equation (19) becomes

$$P'' = n(n-1)(P_{0,n-2} - 2P_{1,n-2} + P_{2,n-2}).$$

As another application of (11) we combine it with (19) to obtain an expression for the curvature of $P(t)$ defined as

$$\kappa(t) = \frac{\|P'(t) \times P''(t)\|}{\|P'(t)\|^3}.$$

Corollary 6. *Given the Bézier curve of degree $n \geq 2$ defined by equations (4) and (6), the curvature of $P(t)$ is*

$$\kappa = \frac{(n-1)}{n} \frac{w_{0,n-2}w_{1,n-2}w_{2,n-2}w_{0,n}^3}{w_{0,n-1}^3 w_{1,n-1}^3} \frac{\|(P_{1,n-2} - P_{0,n-2}) \times (P_{2,n-2} - P_{1,n-2})\|}{\|(P_{1,n-1} - P_{0,n-1})\|^3}.$$

Proof. From (11) and Lemma 4 it follows that

$$\begin{aligned}P' =& n\frac{w_{0,n-1}w_{1,n-1}}{w_{0,n}^2}(P_{1,n-1} - P_{0,n-1})\\ =& n\frac{w_{0,n-1}w_{1,n-1}}{w_{0,n}^2}\left\{(1-t)\frac{w_{0,n-2}}{w_{0,n-1}}(P_{1,n-2} - P_{0,n-2}) + t\frac{w_{2,n-2}}{w_{1,n-1}}(P_{2,n-2} - P_{1,n-2})\right\}.\end{aligned}$$

Then from (19),

$$P' \times P'' = n^2 \frac{w_{0,n-2}w_{2,n-2}}{w_{0,n}^5} A(P_{1,n-2} - P_{0,n-2}) \times (P_{2,n-2} - P_{1,n-2})$$

where

$$
\begin{aligned}
A =&(1-t)w_{1,n-1}(2nw_{0,n-1}^2 - (n-1)w_{0,n-2}w_{0,n} - 2w_{0,n-1}w_{0,n}) \\
&+ tw_{0,n-1}(2nw_{1,n-1}^2 - (n-1)w_{2,n-2}w_{0,n} - 2w_{1,n-1}w_{0,n}) \\
=&2nw_{0,n-1}w_{1,n-1}w_{0,n} - (n-1)w_{0,n}((1-t)w_{0,n-2}w_{1,n-1} + tw_{2,n-2}w_{0,n-1}) \\
&- 2w_{0,n-1}w_{1,n-1}w_{0,n} \\
=&(n-1)w_{0,n}\{2w_{0,n-1}w_{1,n-1} - (1-t)w_{0,n-2}w_{1,n-1} - tw_{2,n-2}w_{0,n-1}\} \\
=&(n-1)w_{0,n}w_{1,n-2}w_{0,n}.
\end{aligned}
$$

Therefore

$$P' \times P'' = n^2(n-1)\frac{w_{0,n-2}w_{1,n-2}w_{2,n-2}}{w_{0,n}^3}(P_{1,n-2} - P_{0,n-2}) \times (P_{2,n-2} - P_{1,n-2}).$$

By substituting this and equation (11) into the definition of $\kappa(t)$ we have proved the corollary. ◁

Finally, despite even more algebra, we find the torsion $\tau(t)$ of P defined as

$$\tau(t) = \frac{P'(t) \times P''(t).P'''(t)}{\|P'(t) \times P''(t)\|^2}$$

in a similar way. We will not prove the details but mention that there exist scalar functions $A(t)$, $B(t)$, $C(t)$ (not necessarily positive) such that

$$P''' = A(P_{1,n-3} - P_{0,n-3}) + B(P_{2,n-3} - P_{1,n-3}) + C(P_{3,n-3} - P_{2,n-3}).$$

After calculating A, B, C it can be shown that

$$(P' \times P'').P''' = R(P_{1,n-3} - P_{0,n-3}) \times (P_{2,n-3} - P_{1,n-3}).(P_{3,n-3} - P_{2,n-3})$$

where

$$R = n^3(n-1)^2(n-2)\frac{w_{0,n-3}w_{1,n-3}w_{2,n-3}w_{3,n-3}}{w_{0,n}^4}.$$

Therefore, since

$$P' \times P'' = n^2(n-1)\frac{w_{0,n-2}w_{1,n-2}w_{2,n-2}}{w_{0,n}^3}(P_{1,n-2} - P_{0,n-2}) \times (P_{2,n-2} - P_{1,n-2})$$

equation (21) follows.

§5. References

G. Farin (1983) “ Algorithms for rational Bézier curves”, *Computer-Aided Design*, **15**, 73–77.

G. Farin (1988) “Curves and surfaces for computer aided geometric design”, *Academic Press, San Diego.*

I. Faux & M. Pratt (1979) “Computational geometry for design and manufacture”, *Ellis Horwood, England.*

D. Filip, R. Magedson & R. Markot (1986) “Surface algorithms using bounds on derivatives”, *Computer Aided Geometric Design*, **3**, 295–311.

M. S. Floater (1992) “Evaluation and properties of the derivative of a NURBS curve”, *Mathematical Methods in CAGD, T. Lyche & L. L. Schumaker (eds.), Academic Press, Boston*, **2**, 261–274.

L. Piegl (1986) “Representation of rational Bézier curves and surfaces by recursive algorithms”, *Computer-Aided Design*, **18**, 361–366.

L. Piegl & W. Tiller (1987) “Curve and surface constructions using rational B-splines”, *Computer-Aided Design*, **19**, 485–498.

T. W. Sederberg & R. J. Meyers (1988) “Loop detection in surface patch intersections”, *Computer Aided Geometric Design*, **5**, 161–171.

T. W. Sederberg & X. Wang (1987) “Rational hodographs”, *Computer Aided Geometric Design*, **4**, 333–335.