Title: Bézier Curves and Surfaces

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Bézier Curves and Surfaces

Definition

Computer-aided geometric design (CAGD) is the design of geometrical shapes using computer technology, and is used extensively in many applications, such as the automotive, shipbuilding, and aerospace industries, architectural design, and computer animation. A popular way of modelling geometry in CAGD is to represent the outer surface, or curve, of the object as a patchwork of parametric polynomial pieces. Bézier curves and surfaces are a representation of such polynomial pieces that makes their interactive design easier and more intuitive than with other representations. They were developed in the 1960's and 1970's by Paul de Casteljau and Pierre Bézier, for use in the automotive industry.

Curves

A Bézier curve, of degree n, on some interval $[a, b]$, is a parametric polynomial $\mathbf{p} : [a, b] \to \mathbb{R}^d$ given by the formula

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{c}_i B_i^n(u), \qquad t \in [a, b],$$

where: u is the local variable, $u = (t-a)/(b-a)$; the points $\mathbf{c}_i \in \mathbb{R}^d$ are the *control points* of $\mathbf{p}$; and B_i^n is the Bernstein (basis) polynomial

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}, \qquad u \in [0, 1].$$

The Euclidean space will often be $\mathbb{R}^2$ or $\mathbb{R}^3$. The polygon formed by connecting the sequence of control points $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n$ is known as the *control polygon* of $\mathbf{p}$. The shape of $\mathbf{p}$ tends to mimic the shape of the polygon, making it a popular choice for designing geometry in an interactive graphical environment. Figure 1 shows a cubic Bézier curve

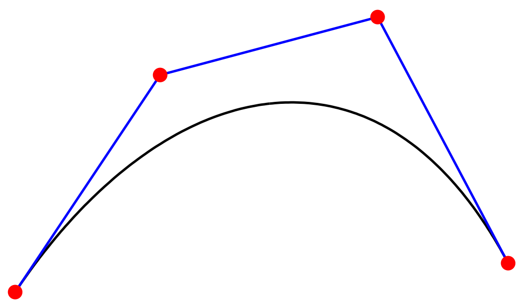

Fig. 1. A cubic Bézier curve

and its control polygon. The cubic Bernstein polynomials are

$$B_0^3(u) = (1-u)^3, \quad B_1^3(u) = 3u(1-u)^2, \quad B_2^3(u) = 3u^2(1-u), \quad B_3^3(u) = u^3,$$

shown in Figure 2.

Various properties of Bézier curves follow from properties of the Bernstein polynomials. One is the *endpoint property*: $\mathbf{p}(a) = \mathbf{c}_0$ and $\mathbf{p}(b) = \mathbf{c}_n$. Another is that since the B_i^n are non-negative and sum to one, every point $\mathbf{p}(t)$ is a *convex combination* of the control points and $\mathbf{p}$ lies in the *convex hull* of the control points. Similarly, $\mathbf{p}$ lies in the *bounding box*

$$[x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_d, y_d],$$

where, if the point $\mathbf{c}_i$ has coordinates $c_i^1, \ldots, c_i^d$,

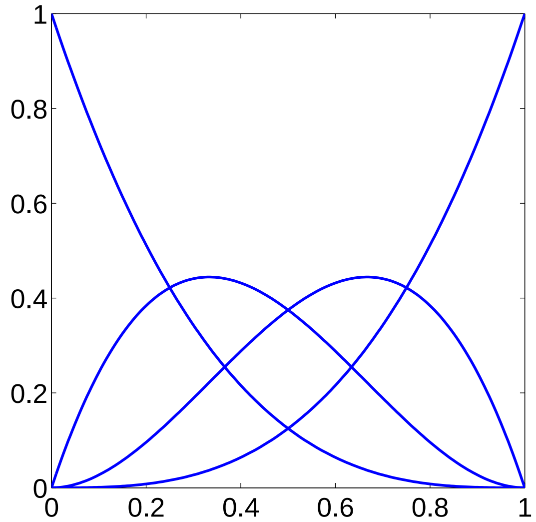

Fig. 2. The cubic Bernstein polynomials

$$x_k = \min_{0 \le i \le n} c_i^k \quad \text{and} \quad y_k = \max_{0 \le i \le n} c_i^k, \qquad k = 1, \ldots, d.$$

Bounding boxes are useful for visualization, and for detecting intersections between pairs of objects and self-intersections.

Due to the *recursion formula*,

$$B_i^n(u) = uB_{i-1}^{n-1}(u) + (1-u)B_i^{n-1}(u),$$

one can *evaluate* (compute) $\mathbf{p}(t)$ for some $t \in [a, b]$ using de Casteljau's algorithm. After the initialization $\mathbf{c}_i^0 = \mathbf{c}_i$, $i = 0, 1, \ldots, n$, we compute

$$\mathbf{c}_i^r = (1-u)\mathbf{c}_i^{r-1} + u\mathbf{c}_{i+1}^{r-1},$$

for $r = 1, \ldots, n$, and $i = 0, 1, \ldots, n-r$, the last point being the point on the curve: $\mathbf{p}(t) = \mathbf{c}_0^n$. This can be viewed as the following triangular scheme, here arranged row-wise, with each row being computed from the row above:

$$\begin{array}{ccccccccc} \mathbf{c}_0^0 & & \mathbf{c}_1^0 & & \mathbf{c}_2^0 & & \cdots & & \mathbf{c}_n^0 \\ & \mathbf{c}_0^1 & & \mathbf{c}_1^1 & & \cdots & & \mathbf{c}_{n-1}^1 & \\ & & \ddots & & & & \iddots & & \\ & & & \mathbf{c}_0^{n-1} & & \mathbf{c}_1^{n-1} & & & \\ & & & & \mathbf{c}_0^n & & & & \end{array}$$

Derivatives of $\mathbf{p}$ can be computed by expressing them as Bézier curves of lower degree:

$$\mathbf{p}'(t) = \frac{d\mathbf{p}}{dt} = \frac{n}{(b-a)} \sum_{i=0}^{n-1} \Delta \mathbf{c}_i B_i^{n-1}(u),$$

where Δ is the forward difference, $\Delta \mathbf{c}_i = \mathbf{c}_{i+1} - \mathbf{c}_i$, and more generally,

$$\mathbf{p}^{(r)}(t) = \frac{d^r\mathbf{p}}{dt^r} = \frac{n(n-1)\cdots(n-r+1)}{(b-a)^r} \sum_{i=0}^{n-r} \Delta^r \mathbf{c}_i B_i^{n-r}(u).$$

Complex curves are often modelled by joining several Bézier curves together. If $\mathbf{q} : [b,c] \to \mathbb{R}^d$ is another Bézier curve,

$$\mathbf{q}(t) = \sum_{i=0}^{n} \mathbf{d}_i B_i^n(v), \qquad t \in [b,c], \qquad v = \frac{t-b}{c-b},$$

then $\mathbf{p}$ and $\mathbf{q}$ join with C^k continuity at $t=b$, i.e., $\mathbf{q}^{(r)}(b) = \mathbf{p}^{(r)}(b)$ for all $r = 0, 1, \ldots, k$, if and only if

$$\frac{\Delta^r \mathbf{d}_0}{(c-b)^r} = \frac{\Delta^r \mathbf{c}_{n-r}}{(b-a)^r}, \qquad r = 0, 1, \ldots, k.$$

Tensor-product surfaces

A tensor-product Bézier surface in $\mathbb{R}^d$ is a parametric polynomial $\mathbf{p} : D \to \mathbb{R}^d$ of degree $m \times n$, given by the formula

$$\mathbf{p}(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{c}_{i,j} B_i^m(u) B_j^n(v), \qquad (s,t) \in D,$$

where D is a rectangle, $D = [a_1, b_1] \times [a_2, b_2]$, and

$$u = \frac{s-a_1}{b_1-a_1}, \qquad v = \frac{t-a_2}{b_2-a_2}.$$

The Euclidean space is usually $\mathbb{R}^3$. The *control net* of $\mathbf{p}$ is the network of *control points* $\mathbf{c}_{i,j} \in \mathbb{R}^d$ and all line segments of the form $[\mathbf{c}_{i,j}, \mathbf{c}_{i+1,j}]$ and $[\mathbf{c}_{i,j}, \mathbf{c}_{i,j+1}]$.

On each of the four sides of D, the surface $\mathbf{p}$ is a Bézier curve whose control polygon is one of the four boundaries of the control net of $\mathbf{p}$. At the four corners of D, the surface $\mathbf{p}$ equals one of the corners of the control net. Like Bézier curves, these surfaces have the convex hull and bounding box properties. The point $\mathbf{p}(s,t)$ can be evaluated by applying de Casteljau's algorithm to the rows of points in the control net, in each of the two directions, using m steps with respect to u and n steps with respect to v. These $m+n$ steps can be applied in any order.

Triangular surfaces

A triangular Bézier surface, of degree n, is a polynomial $\mathbf{p}: T \to \mathbb{R}^d$, in the form

$$\mathbf{p}(\mathbf{t}) = \sum_{|\mathbf{i}|=n} \mathbf{c}_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u}), \qquad \mathbf{t} \in T,$$

where: $T \subset \mathbb{R}^2$ is a triangle, with vertices $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$; $\mathbf{i} = (i,j,k)$; $|\mathbf{i}| = i+j+k$; $\mathbf{u} = (u,v,w)$, and the values $u,v,w \geq 0$ are the *barycentric coordinates* of the point $\mathbf{t}$ with respect to T, i.e., the three values such that

$$u + v + w = 1,$$

$$u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3 = \mathbf{t},$$

and $B_{\mathbf{i}}^n$ is the Bernstein polynomial

$$B_{\mathbf{i}}^n(\mathbf{u}) = \frac{n!}{i!j!k!} u^i v^j w^k.$$

For example, with $n=3$ there are 10 such polynomials,

$$\begin{array}{ccccccc} & & & B_{003}^3 & & & \\ & & B_{102}^3 & & B_{012}^3 & & \\ & B_{201}^3 & & B_{111}^3 & & B_{021}^3 & \\ B_{300}^3 & & B_{210}^3 & & B_{120}^3 & & B_{030}^3 \end{array}$$

given by the formulas

$$
\begin{array}{ccccccc}
 & & & w^3 & & & \\
 & & 3uw^2 & & 3vw^2 & & \\
 & 3u^2w & & 6uvw & & 3v^2w & \\
u^3 & & 3u^2v & & 3uv^2 & & v^3
\end{array}
$$

The points $\mathbf{c_i} \in \mathbb{R}^d$ are the *control points* of $\mathbf{p}$, which, together with all line segments that connect neighbouring points, form the *control net* of $\mathbf{p}$. Two control points are neighbours if they have one index in common and the other two indices differ by one. A point that is not on the boundary has six neighbours. On each of the three sides of T, the surface $\mathbf{p}$ is a Bézier curve whose control polygon is the corresponding boundary polygon of the control net of $\mathbf{p}$. At the corners of T, the surface equals one of the corner control points. Triangular Bézier surfaces have the convex hull and bounding box properties. There is a de Casteljau algorithm for evaluating $\mathbf{p}$ and there are formulas for derivatives and conditions for joining pairs of such triangular surfaces together with a certain order of continuity.

Figure 3 shows a biquadratic surface, where $m = n = 2$, and a quadratic surface, where $n = 2$, with their control nets.

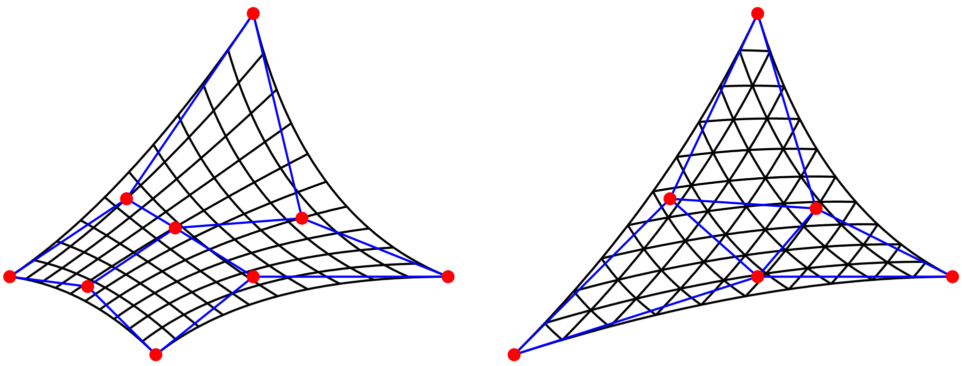

Fig. 3. Biquadratic and quadratic surfaces

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