On Systems of Functions Satisfying Hodograph Properties

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Abstract. We say that a system of functions is hodograph diminishing if the tangent directions of the curve are contained in the cone generated by the sides of the control polygon. We characterize such systems and conclude that Bézier curves, B-spline curves and, more generally, curves generated by totally positive bases are hodograph diminishing. A characterization is also found for rational curves. Hodograph diminution can be used to identify when two curves have at most one intersection. We also consider conditions for bounding the number of intersections by two.

§1. Introduction

The first derivative of a curve $P(t)$ is sometimes called its hodograph. Bounds on derivative directions are useful in detecting intersection between two curves (see [11]). The case of rational curves has been considered in several papers (see [10], [4], [5] and [9]). Given functions $u_0, \ldots, u_n$ defined on $[a, b]$ and $P_0, \ldots, P_n \in \mathbb{R}^d$, a curve $P(t)$ may be defined by

$$P(t) = \sum_{i=0}^{n} P_i u_i(t). \tag{1.1}$$

The points $P_0, \ldots, P_n$ are usually called *control points* and the polygon with vertices $P_0, \ldots, P_n$ is called the *control polygon* of $P(t)$. In Computer Aided Geometric Design the functions $u_0, \ldots, u_n$ are usually nonnegative and besides $\sum_{i=0}^{n} u_i(t) = 1 \; \forall\, t \in [a, b]$, and in this case we call them *blending functions*. In this paper we characterize the systems of functions which satisfy the following hodograph property: the tangent directions of the curve are contained in the cone generated by the sides of the control polygon. These systems of functions will be called hodograph diminishing.

A system of functions $(u_0, \ldots, u_n)$ is said to be monotonicity preserving if for any increasing sequence of coefficients $(\lambda_0, \ldots, \lambda_n)$ $(\lambda_0 \le \cdots \le \lambda_n)$ the function $u(t) = \sum_{i=0}^{n} \lambda_i u_i(t)$ is an increasing function. In [2] monotonicity preserving systems of functions were characterized and it was shown that the length of a curve generated by a monotonicity preserving system of blending functions is less than or equal to the length of the corresponding control polygon. In Section 2 we shall prove that the concepts of monotonicity preservation and hodograph diminution are equivalent.

Let us recall that a system of functions $(u_0, \ldots, u_n)$ defined on $[a, b]$ is called *totally positive* if all the minors of its collocation matrices

$$M \begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_m \end{pmatrix} = \big(u_j(t_i)\big)_{0 \le i \le m}^{0 \le j \le n} \tag{1.2}$$

$(a \le t_0 < \cdots < t_m \le b)$ are nonnegative. If all minors of order up to k of any collocation matrix are nonnegative then we say that $(u_0, \ldots, u_n)$ is $\mathrm{TP_k}$.

In Section 3 (in Theorem 3.1) it is proved that a system of nonnegative functions $(u_0, \ldots, u_n)$ is TP_2 if and only if all rational curves obtained with any positive weights satisfy the aforementioned hodograph property.

In Section 4 we describe an application of our hodograph property to bound the intersections of two curves generated by hodograph diminishing systems. In Section 5 a binormal diminishing property is introduced to bound the number of intersections between two curves by two. Binormal diminishing systems are characterized in terms of weak Tchebycheff systems. A system $(u_0, \ldots, u_n)$ is said to be weak Tchebycheff if all square collocation matrices (1.2) have nonnegative determinant. We also treat the rational case.

Many common systems of blending functions are totally positive. Some examples are the Bernstein and generalized Ball bases of polynomials (c.f. [7]) and the B-spline and β-spline bases. It is well known (see, for instance, [6] or [1]) that for totally positive systems many shape properties of the control polygon are inherited by the corresponding curve. Due to the results of this paper, totally positive systems of blending functions provide a source of examples of hodograph and binormal diminishing systems.

§2. Hodograph Diminishing Systems

The *hodograph* of a curve $P \in C^1([a, b], \mathbb{R}^d)$, $d \ge 1$, (see (1.1)) is its derivative $P'(t)$. The *hodograph* of a control polygon $P_0 \cdots P_n$, $P_i \in \mathbb{R}^d$ is the set of vectors $\{P_i - P_{i-1}; i = 1, \ldots, n\}$.

Let us recall that the convex cone generated by a set of vectors S is

$$\langle S \rangle_+ := \left\{ \sum_{i=1}^{n} \lambda_i c_i | \lambda_i \ge 0, c_i \in S, n \ge 1 \right\}.$$

When S is infinite, we shall sometimes use the notation $\overline{\langle S \rangle_+}$ to refer to the closure of $\langle S \rangle_+$.

Definition 2.2. *The hodographic hull of a curve* $P \in C^1([a, b], \mathbb{R}^d)$ *is the closure of the convex cone generated by the hodograph* $P'(t)$*, that is to say*

$$\overline{\langle P'(t) | t \in [a, b] \rangle_+}.$$

Similarly, the hodographic hull of the control polygon $P_0 \cdots P_n$ *of* P *is the convex cone generated by the* $P_i - P_{i-1}$.

In this paper we are interested in the following property.

Definition 2.3. *We say that a system of blending functions* $(u_0, \dots, u_n)$ *is hodograph diminishing if for any control polygon* $P_0 \cdots P_n$ *the hodographic hull of* $P_0 \cdots P_n$ *contains the hodographic hull of the curve (1.1).*

Proposition 2.4. *Let* $P \in C^1([a,b], \mathbb{R}^d)$*. Then one has*

$$\overline{\langle P'(t) | t \in [a,b] \rangle_+} = \overline{\langle P(s_2) - P(s_1) | s_1 < s_2 \in [a,b] \rangle_+}.$$

Proof: Let $C = \langle P(s_2) - P(s_1) | s_1 < s_2 \in [a,b] \rangle_+$ and then for all $s \neq t \in [a,b]$, $(P(t) - P(s))/(t-s) \in C$. Since $P'(t) = \lim_{t \to s}(P(t) - P(s))/(t-s)$, we see that $P'(t) \in \overline{C}$. On the other hand,

$$P(s_2) - P(s_1) = \int_{s_1}^{s_2} P'(t)\, dt = \lim_{n \to \infty} \sum_{i=1}^{n} P'(a + \frac{i}{n}(b-a))\frac{1}{n}(b-a),$$

and so $P(s_2) - P(s_1) \in \overline{\langle P'(t) | t \in [a,b] \rangle_+}$. ∎

Since the hodographic hull of a control polygon is closed, we may derive the following consequence of the previous result.

Corollary 2.5. *Let* $(u_0, \dots, u_n)$ *be a blending system of* C^1 *functions. Then* $(u_0, \dots, u_n)$ *is hodograph diminishing if and only if*

$$\langle P(s_2) - P(s_1) | s_1 < s_2 \in [a,b] \rangle_+ \subset \langle P_i - P_{i-1} | i = 1, \dots, n \rangle_+, \qquad (2.1)$$

for any $P_0, \cdots, P_n \in \mathbb{R}^d$, $P(t) = \sum P_i u_i(t)$, $t \in [a,b]$.

For nonsmooth functions we can therefore give an alternative definition of hodograph diminishing systems as follows:

Definition 2.3′. *An arbitrary blending system is hodograph diminishing if (2.1) holds.*

Definition 2.6. *A system of blending functions is monotonicity preserving if for any increasing sequence of coefficients* $c_0 \le \cdots \le c_n$ *the function* $\sum_{i=0}^{n} c_i u_i(t)$ *is increasing.*

Monotonicity preserving systems have been studied in the mathematical literature: see for instance [8]. Some recent characterizations of these systems have been presented in [2] and [3]. Let us see now how the concepts of hodograph diminution and monotonicity preservation are equivalent.

Theorem 2.7. *A blending system* $(u_0, \dots, u_n)$*,* $u_i : [a,b] \to \mathbb{R}$*, is hodograph diminishing if and only if it is monotonicity preserving.*

Proof: First, let us suppose that $(u_0, \dots, u_n)$ is hodograph diminishing. Then by considering the one-dimensional control polygon $\alpha_0, \dots, \alpha_n$, $\alpha_0 \le \cdots \le \alpha_n$, whose hodographic hull is $[0, +\infty)$, it follows that the function $\sum_{i=0}^{n} \alpha_i u_i$ satisfies

$$\sum_{i=0}^{n} \alpha_i u_i(s_2) - \sum_{i=0}^{n} \alpha_i u_i(s_1) \in [0, +\infty), \qquad s_1 < s_2,$$

which means that $\sum_{i=0}^{n} \alpha_i u_i$ is an increasing function. Therefore $(u_0, \ldots, u_n)$ is monotonicity preserving.

For the converse, assume that $(u_0, \ldots, u_n)$ is a monotonicity preserving system. Let us observe that the functions $v_i(t) = \sum_{j=i}^{n} u_j(t)$ are increasing because $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$ is an increasing sequence. Note also that $v_0(t) = 1$ for all $t \in [a, b]$. Now for $s < t$ we have

$$
\begin{aligned}
P(t) - P(s) =& \sum_{i=0}^{n} P_i(u_i(t) - u_i(s)) \\
=& P_0(v_0(t) - v_0(s)) + \sum_{i=1}^{n} (P_i - P_{i-1})(v_i(t) - v_i(s)) \\
=& \sum_{i=1}^{n} (P_i - P_{i-1})(v_i(t) - v_i(s)) \in \langle P_1 - P_0, \ldots, P_n - P_{n-1} \rangle_+
\end{aligned}
$$

and the result follows. ■

§3. Rational Hodograph Diminishing Systems

We now consider analogous properties for systems of rationals functions. The next result uses similar techniques to those found in Theorem 2.6 in [3].

Theorem 3.1. *Let $(u_0, \ldots, u_n)$ be a system of nonnegative functions such that $\sum_{i=0}^{n} u_i(t) > 0$. Then $(u_0, \ldots, u_n)$ is TP_2 if and only if for all positive weights $w_0, \ldots, w_n$, the system*

$$
\left(\frac{w_0 u_0}{\sum_{i=0}^{n} w_i u_i}, \ldots, \frac{w_n u_n}{\sum_{i=0}^{n} w_i u_i} \right) \tag{3.1}
$$

is hodograph diminishing.

Proof: Suppose $(u_0, \ldots, u_n)$ is TP_2. Let $P_0, \ldots, P_n \in \mathbb{R}^d$, $w_0, \ldots, w_n > 0$, and $r(s) = \sum_{i=0}^{n} w_i u_i(s) P_i / \big(\sum_{i=0}^{n} w_i u_i(s) \big)$. Then we see that, for $s_1 < s_2$,

$$
\begin{aligned}
r(s_2) - r(s_1) =& \\
& \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} w_j (u_j(s_1) u_i(s_2) - u_j(s_2) u_i(s_1)) w_i P_i}{\big(\sum_{i=0}^{n} w_i u_i(s_1) \big) \big(\sum_{j=0}^{n} w_j u_j(s_2) \big)} \\
=& \frac{\sum_{i=1}^{n} \sum_{k=i}^{n} \sum_{j=0}^{n} w_j (u_j(s_1) u_k(s_2) - u_j(s_2) u_k(s_1)) w_k (P_i - P_{i-1})}{\big(\sum_{i=0}^{n} w_i u_i(s_1) \big) \big(\sum_{j=0}^{n} w_j u_j(s_2) \big)} \\
=& \sum_{i=1}^{n} \frac{\sum_{k=i}^{n} \sum_{j=0}^{i-1} w_j (u_j(s_1) u_k(s_2) - u_j(s_2) u_k(s_1)) w_k}{\big(\sum_{i=0}^{n} w_i u_i(s_1) \big) \big(\sum_{j=0}^{n} w_j u_j(s_2) \big)} (P_i - P_{i-1}).
\end{aligned} \tag{3.2}
$$

Since $(u_0, \ldots, u_n)$ is TP_2, each coefficient of $P_i - P_{i-1}$ is positive. Therefore $r(s_2) - r(s_1)$ belongs to the hodographic hull of the control polygon $P_0 \cdots P_n$.

For the converse, suppose that the system in (3.1) is hodograph diminishing for all $w_0, \ldots, w_n > 0$. Choose s_1, s_2 such that $s_1 < s_2$ and p, q such that $0 \le p < q \le n$. Then define the following one-dimensional control polygon and weights:

$$P_i = \begin{cases} 0, & 0 \le i \le p, \\ 1, & p < i \le n, \end{cases} \qquad w_i = \begin{cases} 1, & i = p, q, \\ \epsilon, & \text{otherwise.} \end{cases}$$

Now, because the system in (3.1) is hodograph diminishing and since $P_i - P_{i-1} \ge 0$ for all i, we know that $r(s_2) - r(s_1) \ge 0$. Using equation (3.2) it follows, in this particular case, that

$$\sum_{k=p+1}^{n} \sum_{j=0}^{p} w_j w_k (u_j(s_1) u_k(s_2) - u_j(s_2) u_k(s_1)) \ge 0.$$

Finally, letting $\epsilon \to 0$, we find that

$$\begin{vmatrix} u_p(s_1) & u_q(s_1) \\ u_p(s_2) & u_q(s_2) \end{vmatrix} \ge 0$$

and so $(u_0, \ldots, u_n)$ is TP_2. ■

§4. Bounding Intersection Points

We now discuss the application of hodograph diminution to the analysis of intersections between two curves (c.f. [11]). The following proposition is illustrated in Figure 1, showing the control polygons of two curves P and Q and their respective hodographic hulls C and D.

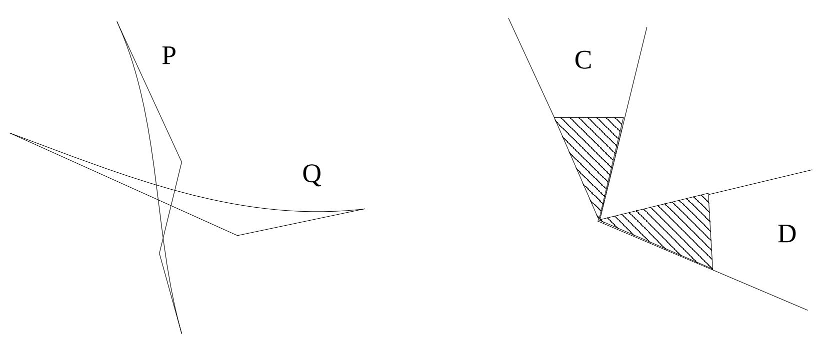

Fig. 1. Control polygons and hodographic hulls of two curves.

Proposition 4.1. *Let $P_0, \ldots, P_n$, $Q_0, \ldots, Q_m \in \mathbb{R}^d$. Let $P(t) = \sum P_i f_i(t)$, and $Q(s) = \sum Q_i g_i(s)$ be the curves generated by two hodograph diminishing systems $(f_0, \ldots, f_n)$ and $(g_0, \ldots, g_m)$. Let $C = \langle P_i - P_{i-1} | i = 1, \ldots, n \rangle_+$ and*

$D = \langle Q_i - Q_{i-1} | i = 1, \ldots, m \rangle_+$. *If* $C \cap D = \{0\}$ *and* $C \cap (-D) = \{0\}$ *then the curves* $P(t)$ *and* $Q(s)$ *cannot intersect at more than one point.*

Proof: Suppose, in order to get a contradiction, that $P(t_1) = Q(s_1)$, $P(t_2) = Q(s_2)$, and $P(t_1) \neq P(t_2)$. Without loss of generality we may assume that $t_1 < t_2$. If $s_1 < s_2$ then since

$$Q(s_2) - Q(s_1) = P(t_2) - P(t_1) \in C \cap D,$$

one deduces that $C \cap D \neq \{0\}$. Similarly, if $s_1 > s_2$, one obtains $C \cap (-D) \neq \{0\}$. In both cases there is a contradiction. ■

§5. Binormal Diminishing Systems

In this section we are concerned with properties of curves which provide further information about intersections between curves. We shall restrict our attention to the case of curves $P(t)$ embedded in $\mathbb{R}^3$. Here the binormal curve $P' \times P''$ plays a similar role to that of the tangent curve (hodograph) P'.

Let $P(t) = \sum_{i=0}^{n} P_i u_i(t)$ be as in (1.1) where $P_i \in \mathbb{R}^3$ and $(u_0, \ldots, u_n)$ is a system of blending functions defined on $[a, b]$. Then the binormal can be expressed in terms of the control polygon using auxiliary functions

$$v_i = \sum_{j=i}^{n} u_j, \qquad i = 1, \ldots, n, \tag{5.1}$$

so that

$$P(t) = P_0 + \sum_{i=1}^{n} (P_i - P_{i-1}) v_i(t). \tag{5.2}$$

Then

$$\begin{aligned} P'(t) \times P''(t) &= \sum_{i=1}^{n} \sum_{j=1}^{n} (P_i - P_{i-1}) \times (P_j - P_{j-1}) v_i'(t) v_j''(t) \\ &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (P_i - P_{i-1}) \times (P_j - P_{j-1}) (v_i'(t) v_j''(t) - v_j'(t) v_i''(t)). \end{aligned}$$

So we are interested in the following cone generated by the sides of the control polygon:

$$\langle (P_i - P_{i-1}) \times (P_j - P_{j-1}) | i < j \rangle_+. \tag{5.3}$$

This motivates the following definition valid even for nonsmooth functions.

Definition 5.1. *A blending system* $u_0, \ldots, u_n$ *is binormal diminishing if, for* $P(t)$ *as in (1.1),*

$$\begin{aligned} &\langle (P(s_2) - P(s_1)) \times (P(s_3) - P(s_2)) | s_1 < s_2 < s_3 \text{ in } [a, b] \rangle_+ \\ &\quad \subset \langle (P_i - P_{i-1}) \times (P_j - P_{j-1}) | 1 \leq i < j \leq n \rangle_+. \end{aligned}$$

The next result characterizes these systems.

Theorem 5.2. *Let* $(u_0,\ldots,u_n)$ *be a blending system and* v_i *be defined by (5.1). Then* $(u_0,\ldots,u_n)$ *is binormal diminishing if and only if the system* $(1,v_i,v_j)$ *is weak Tchebycheff for all* $i<j$ *in* $\{1,\ldots,n\}$*.*

Proof: Suppose that $(1,v_i,v_j)$ is weak Tchebycheff $\forall i<j\in\{1,\ldots,n\}$. From (5.2), we obtain

$$P(s_l)-P(s_r)=\sum_{i=1}^{n}(v_i(s_l)-v_i(s_r))(P_i-P_{i-1})$$

and therefore

$$\begin{aligned}
&(P(s_2)-P(s_1))\times(P(s_3)-P(s_2))\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}(v_i(s_2)-v_i(s_1))(v_j(s_3)-v_j(s_2))(P_i-P_{i-1})\times(P_j-P_{j-1})\\
&=\sum_{i<j}\big[(v_i(s_2)-v_i(s_1))(v_j(s_3)-v_j(s_2))-\\
&\qquad(v_j(s_2)-v_j(s_1))(v_i(s_3)-v_i(s_2))\big](P_i-P_{i-1})\times(P_j-P_{j-1})\\
&=\sum_{i<j}\det M\begin{pmatrix}1, & v_i, & v_j\\ s_1, & s_2, & s_3\end{pmatrix}(P_i-P_{i-1})\times(P_j-P_{j-1}),
\end{aligned}\tag{5.4}$$

which means that $(u_0,\ldots,u_n)$ is binormal diminishing.

Now suppose that $(u_0,\ldots,u_n)$ is binormal diminishing and let $k<l\in\{1,\ldots,n\}$. Now take $P_0=P_1=\cdots=P_{k-1}=(0,0,0)$, $P_k=P_{k+1}=\cdots=P_{l-1}=(1,0,0)$, and $P_l=P_{l+1}=\cdots=P_n=(1,1,0)$. Then from (5.4) we have that

$$(P(s_2)-P(s_1))\times(P(s_3)-P(s_2))=\det M\begin{pmatrix}1, & v_k, & v_l\\ s_1, & s_2, & s_3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix},$$

for all $s_1<s_2<s_3$. Since the cone (5.3) coincides with $\langle(0,0,1)^{\mathrm{T}}\rangle_+$, we have that $\det M\begin{pmatrix}1, & v_k, & v_l\\ s_1, & s_2, & s_3\end{pmatrix}\geq 0$ and $(1,v_k,v_l)$ is a weak Tchebycheff system. ∎

Under stronger conditions, all rational curves with positive weights obtained from $(u_0,\ldots,u_n)$ are binormal diminishing.

Theorem 5.3. *Let* $(u_0,\ldots,u_n)$ *be a system of nonnegative functions satisfying* $\sum_{i=0}^{n}u_i>0$*. Then all systems (3.1), for all* $w_0,\ldots,w_n>0$*, are binormal diminishing if and only if* (u_h,u_i,u_j) *is weak Tchebycheff for all* $h<i<j$ *in* $\{0,\ldots,n\}$*.*

Proof: From Theorem 5.2, the systems (3.1), with $w_0,\ldots,w_n>0$, are binormal diminishing if and only if the system

$$\left(1,\frac{\sum_{r=i}^{n}w_ru_r}{\sum_{r=0}^{n}w_ru_r},\frac{\sum_{r=j}^{n}w_ru_r}{\sum_{r=0}^{n}w_ru_r}\right)\tag{5.5}$$

is weak Tchebycheff for all $i < j$ in $\{1, \ldots, n\}$. Since the denominators in (5.5) are strictly positive, (5.5) is weak Tchebycheff if and only if the system

$$\left(\sum_{r=0}^{n} w_r u_r, \sum_{r=i}^{n} w_r u_r, \sum_{r=j}^{n} w_r u_r \right) \tag{5.6}$$

is also weak Tchebycheff and clearly this is equivalent to the system

$$\left(\sum_{r=0}^{i-1} w_r u_r, \sum_{r=i}^{j-1} w_r u_r, \sum_{r=j}^{n} w_r u_r \right) \tag{5.7}$$

being weak Tchebycheff.

Let us assume now that (u_h, u_k, u_l) is weak Tchebycheff for all $h < k < l$ in $\{0, \ldots, n\}$. Then

$$\det M \begin{pmatrix} \sum_{r=0}^{i-1} w_r u_r, & \sum_{r=i}^{j-1} w_r u_r, & \sum_{r=j}^{n} w_r u_r \\ s_1, & s_2, & s_3 \end{pmatrix}$$
$$= \sum_{h=0}^{i-1} \sum_{k=i}^{j-1} \sum_{l=j}^{n} w_h w_k w_l \det M \begin{pmatrix} u_h, & u_k, & u_l \\ s_1, & s_2, & s_3 \end{pmatrix} \geq 0,$$

and so (3.1) is binormal diminishing for all $w_0, \ldots, w_n > 0$.

Conversely, in (5.7) take $w_h = w_i = w_j = 1$ and $w_r = \epsilon \ \forall r \neq h, i, j$. Since the pointwise limit of a weak Tchebycheff system is weak Tchebycheff, we see, by letting $\epsilon \to 0$, that (u_h, u_i, u_j) is weak Tchebycheff. ■

We note that if a system $(u_0, \ldots, u_n)$ is totally positive, or merely TP_3, then (u_h, u_i, u_j) is weak Tchebycheff for all $h < i < j$ in $\{0, \ldots, n\}$. Therefore Bézier curves, B-spline curves and corresponding rational curves with positive weights satisfy the binormal diminishing property.

Combining Theorem 3.1 and Proposition 5.3 we obtain:

Corollary 5.4. *Let $(u_0, \ldots, u_n)$ be a system of nonnegative functions such that $\sum_{i=0}^{n} u_i > 0$. Then $(u_0, \ldots, u_n)$ is TP_3 if and only if all systems (3.1) are both hodograph and binormal diminishing for all $w_0, \ldots, w_n > 0$.*

Now we apply binormal diminution to bounding the number of intersections of two curves in $\mathbb{R}^3$.

Proposition 5.5. *Let $P_0, \ldots, P_n, Q_0, \ldots, Q_m \in \mathbb{R}^3$. Let $P(t) = \sum P_i f_i(t)$ and $Q(s) = \sum Q_i g_i(s)$ be two curves generated by two binormal diminishing systems $(f_0, \ldots, f_n)$ and $(g_0, \ldots, g_m)$. Let*

$$C = \langle (P_i - P_{i-1}) \times (P_j - P_{j-1}) | i < j \in \{1, \ldots, n\} \rangle_+,$$

and

$$D = \langle (Q_i - Q_{i-1}) \times (Q_j - Q_{j-1}) | i < j \in \{1, \ldots, m\} \rangle_+.$$

If $C \cap D = \{0\}$ and $C \cap (-D) = \{0\}$ then the curves $P(t)$ and $Q(s)$ cannot intersect at three non-collinear points.

Proof: Suppose in order to get a contradiction that $P(t_i) = Q(s_i)$, $i = 1, 2, 3$ and that $P(t_1)$, $P(t_2)$, $P(t_3)$ are not collinear. Without loss of generality we can assume that $t_1 < t_2 < t_3$ but the s_i can be in any order. Let $b \neq 0$ be one of the two unit normals to the plane spanned by $P(t_1)$, $P(t_2)$, $P(t_3)$. Then either b or $-b$ belongs to C, and similarly, either b or $-b$ belongs to D. Therefore in any case either $C \cap D \neq \{0\}$ or $C \cap (-D) \neq \{0\}$, which gives a contradiction. ■

It is not likely that two arbitrary curves in $\mathbb{R}^3$ intersect. However Proposition 5.5 also provides interesting information about intersections when the curves lie in the same plane.

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Acknowledgements. The authors were supported by the EU project CHRX CT94-0522. The first and third authors were supported by DGICYT PB93-0310 Research Grant.

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