Parameterization for curve interpolation

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Abstract

*A common task in geometric modelling is to interpolate a sequence of points or
derivatives, sampled from a curve, with a parametric polynomial or spline curve.
To do this we must first choose parameter values corresponding to the interpolation
points. The important issue of how this choice affects the accuracy of the approxi-
mation is the focus of this paper. The underlying principle is that full approximation
order can be achieved if the parameterization is an accurate enough approximation
to arc length. The theory is illustrated with numerical examples.*

Key words: interpolation, curves, arc length, approximation order

1. Introduction

One of the most basic tasks in geometric modelling is to fit a smooth curve
through a sequence of points $\mathbf{p}_0, \ldots \mathbf{p}_n$ in $\mathbb{R}^d$, with $d \geq 2$. The usual approach is
to fit a parametric curve. We choose parameter values $t_0 < \cdots < t_n$ and find a
parametric curve $\mathbf{c} : [t_0, t_n] \to \mathbb{R}^d$ such that

$$\mathbf{c}(t_i) = \mathbf{p}_i, \qquad i = 0, 1, \ldots, n. \tag{1}$$

Popular examples of $\mathbf{c}$ are polynomial curves (of degree at most n) for small n, and
spline curves for larger n.

In this paper we wish to focus on the role of the parameter values $t_0, \ldots, t_n$.
Empirical evidence has shown that the choice of parameterization has a significant
effect on the visual appearance of the interpolant $\mathbf{c}$. We will distinguish between
two different applications:

– *Design.* If a designer wants to use interpolation to design a curve, then we can simply view parameterization as a *design tool.* In addition to choosing the points $\mathbf{p}_i$, the designer is also free to choose the parameter values t_i in order to increase the degrees of freedom at his disposal.

– *Approximation.* If the points $\mathbf{p}_i$ have been sampled from a smooth curve, e.g.
  · a height contour,
  · a curve of intersection between two surfaces,
  · or a physical curve,

then the curve $\mathbf{c}$ will be an approximation to the curve the points $\mathbf{p}_i$ were sampled from. The important issue is now the *approximation error.* An interpolation method with higher approximation order will lead to faster convergence to the sampled curve as the sampling density increases.

While a lot of effort has been put into the effect of parameterization from the view point of *design*, little was known until recently about its effect on *approximation order*. The purpose of this paper is to summarize the recent results of [7,8] on approximation order and to illustrate the theory with numerical examples. Some comparisons with 'geometric interpolation' are made later in the paper.

2. Parameterization methods

In order to discuss different parameterization methods, we will use the example of C^2 cubic spline interpolation. Suppose $\boldsymbol{\sigma} : [t_0, t_n] \to \mathbb{R}^d$ is the C^2 cubic spline curve such that

$$\boldsymbol{\sigma}(t_i) = \mathbf{p}_i, \qquad i = 0, 1, \ldots, n,$$

and

$$\boldsymbol{\sigma}'(t_i) = \mathbf{m}_i, \qquad i = 0, n, \tag{2}$$

for some chosen vectors $\mathbf{m}_0$, $\mathbf{m}_n$. By 'cubic spline curve' we understand that $\boldsymbol{\sigma}$ is a cubic polynomial curve in each interval $[t_i, t_{i+1}]$, and that $\boldsymbol{\sigma}$ has C^2 continuity at the break points $t_1, \ldots, t_{n-1}$. As is well-known [1,2,6], $\boldsymbol{\sigma}$ is unique. The end conditions (2) are sometimes known as 'clamped' end conditions, and this kind of spline interpolation is sometimes called 'complete' spline interpolation.

The first thing to note about parameterization is that applying a linear transformation to $t_0, \ldots, t_n$ does not change the resulting spline curve. Provided we treat the end conditions (2) correctly, we simply get a reparameterization of $\boldsymbol{\sigma}$. To be precise, if we define

$$\hat{t}_i := \lambda t_i + \mu, \qquad i = 0, 1, \ldots, n,$$

for $\lambda > 0$ and we define the curve $\hat{\boldsymbol{\sigma}} : [\hat{t}_0, \hat{t}_n] \to \mathbb{R}^d$ by

$$\hat{\boldsymbol{\sigma}}(t) := \boldsymbol{\sigma}((t - \mu)/\lambda),$$

then $\hat{\boldsymbol{\sigma}}$ is clearly a C^2 cubic spline curve over the partition $\hat{t}_0 < \hat{t}_1 < \cdots < \hat{t}_n$, satisfying the conditions

$$\hat{\boldsymbol{\sigma}}(\hat{t}_i) = \mathbf{p}_i, \qquad i = 0, 1, \ldots, n,$$

and
$$\hat{\boldsymbol{\sigma}}'(\hat{t}_i) = \mathbf{m}_i/\lambda, \qquad i = 0, n.$$
For this reason, we may as well set $t_0 = 0$ and we typically specify a parameterization by recursively setting
$$t_{i+1} := t_i + d_i$$
for some chosen interval lengths $d_0, d_1, \ldots, d_{n-1} > 0$. Multiplying the interval lengths d_i by a common factor λ will not change the intrinsic geometry of the spline curve $\boldsymbol{\sigma}$ as long as we divide the vectors $\mathbf{m}_0$ and $\mathbf{m}_n$ by the same factor λ.

The simplest choice is the *uniform parameterization* defined by $d_i = 1$, where the values t_i are uniformly spaced. But as early as 1967, Ahlberg, Nilson, and Walsh [1] proposed using the *chordal parameterization* in which
$$d_i := |\mathbf{p}_{i+1} - \mathbf{p}_i|, \tag{3}$$
and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The motivation behind this was that the distance between two points on a curve is a reasonable approximation to the length of the associated curve segment. Thus the hope was that the 'speed' $|\boldsymbol{\sigma}'(t)|$ of the spline curve might be close to unity at all $t \in [t_0, t_n]$.

Epstein [5] showed that when using periodic boundary conditions, a chordal C^2 cubic spline interpolant is always regular, i.e., $\boldsymbol{\sigma}'(t) \neq \mathbf{0}$ at every $t \in [t_0, t_n]$.

Later, it was realized that the uniform and chordal parameterizations are the special cases $\mu = 0$ and $\mu = 1$ of the more general parameterization
$$d_i := |\mathbf{p}_{i+1} - \mathbf{p}_i|^{\mu},$$
with $0 \leq \mu \leq 1$ acting as a kind of blending parameter. The choice $\mu = 1/2$ was termed by Lee [11] the *centripetal parameterization*, and suggested that it sometimes leads to a spline curve which is closer to the polygon passing through the data points $\mathbf{p}_i$.

Later still, Foley and Nielson [9] observed that the chordal parameterization lacks affine invariance. If a non-uniform scaling is applied to the data points $\mathbf{p}_i$ and the new chord lengths are computed, then the resulting spline curve will not in general be a scaling of the original. This suggested replacing the Euclidean distance in (3) by a metric which is invariant to affine transformations of the data points. Nielson had earlier found such an affine-invariant metric, and they defined d_i with the aid of both the affine-invariant distance between $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ and the angles of the polygon passing through $\mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{p}_{i+2}$. Thus, unlike the previous methods, every value of d_i depends on all the data points $\mathbf{p}_0, \ldots, \mathbf{p}_n$.

Our goal in this paper is to study the effect of parameterization on *approximation* and with this in mind we have sampled points $\mathbf{p}_i$ from a fixed smooth curve, and computed the four spline curves $\boldsymbol{\sigma}$ resulting from the uniform, chordal, centripetal, and Foley-Nielson knot-vectors. Figure 1 shows the fixed curve in black and the four spline curves shown in grey. We chose the end vectors $\mathbf{m}_0$, $\mathbf{m}_n$ to have the same direction as the given curve and to have the magnitude
$$|\mathbf{m}_0| := |\mathbf{p}_1 - \mathbf{p}_0|/d_0, \qquad |\mathbf{m}_n| := |\mathbf{p}_n - \mathbf{p}_{n-1}|/d_{n-1},$$
the motivation being that $\boldsymbol{\sigma}'(t_0) \approx (\mathbf{p}_1 - \mathbf{p}_0)/d_0$.

(a) uniform   (b) chordal   (c) centripetal   (d) Foley-Nielson
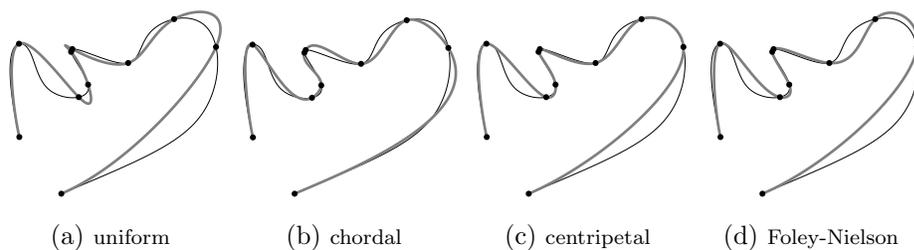
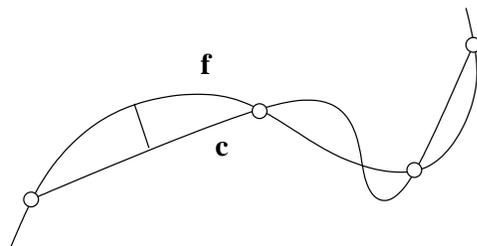Fig. 1. Choice of knot-vector for cubic spline interpolation.

Fig. 2. Distance between two curves.

It is quite evident from the figure that the chordal curve appears to be a particularly good approximation to the original curve. As we will see, this is not coincidence, and due to the fact that the length of a chord is a relatively good approximation to the length of an arc.

3. Approximation error

The example of Figure 1 clearly shows that parameterization affects approximation error. In order to study approximation, we will assume that the points $\mathbf{p}_i$ are samples, $\mathbf{p}_i = \mathbf{f}(s_i)$, from a parametric curve $\mathbf{f} : [a,b] \to \mathbb{R}^d$, $d \geq 2$, with $a \leq s_0 < \cdots < s_n \leq b$. We will assume that $\mathbf{f}$ is parameterized with respect to arc length, by which we mean that it is (at least) continuously differentiable and that $|\mathbf{f}'(s)| = 1$ for all $s \in [a,b]$. Then the interpolation problem becomes

$$\mathbf{c}(t_i) = \mathbf{f}(s_i), \qquad i = 0, 1, \ldots, n, \tag{4}$$

and the error is the distance between the curve piece $\mathbf{f}|_{[s_0,s_n]}$ and the curve $\mathbf{c}$, see Figure 2. One well-known way of measuring the distance between two such curves is the Hausdorff distance,

$$\mathrm{dist}_H(\mathbf{f}|_{[s_0,s_n]}, \mathbf{c}) = \max\{\max_{t_0 \leq t \leq t_n} \min_{s_0 \leq s \leq s_n} |\mathbf{f}(s) - \mathbf{c}(t)|, \max_{s_0 \leq s \leq s_n} \min_{t_0 \leq t \leq t_n} |\mathbf{f}(s) - \mathbf{c}(t)|\}.$$

Fig. 3. Cubic interpolation.

4. Approximation order

It seems difficult to derive bounds on the error between $\mathbf{f}$ and $\boldsymbol{\sigma}$. We will instead study the approximation *order*. A higher approximation order implies faster convergence of the interpolating curves to the given curve as the sampling density increases. Consider for simplicity polynomial interpolation, i.e., let $\mathbf{c}$ in (4) be the Lagrange polynomial interpolant $\mathbf{p} : [t_0, t_n] \to \mathbb{R}^d$ of degree at most n, and let h be the length of the curve piece $\mathbf{f}|_{[s_0, s_n]}$, i.e.,

$$h = L(\mathbf{f}|_{[s_0, s_n]}) = \int_{s_0}^{s_n} |\mathbf{f}'(s)| \, ds = s_n - s_0.$$

The following table gives numerical estimates of k s.t. $\operatorname{dist}_H(\mathbf{f}|_{[s_0, s_n]}, \mathbf{p}) = O(h^k)$ as $h \to 0$ for smooth curves $\mathbf{f}$:

Degree	1	2	3	4	5	$\cdots$
Uniform	2	2	2	2	2	$\cdots$
Chordal	2	3	4	4	4	$\cdots$

The table suggests that the uniform parameterization only provides an $O(h^2)$ approximation when applied to interpolation of any degree $n \geq 1$, while the chordal parameterization provides full approximation order $O(h^{n+1})$ for $n = 1, 2, 3$, and then stays at $O(h^4)$ for higher degrees $n > 3$. Arguably the most interesting of these numerical results is that of cubic interpolation ($n = 3$) with the chordal parameterization. The following theorem is a special case of a result proved in [7].

Theorem 1. Suppose $\mathbf{f} \in C^4[a, b]$ and for each sample $s_0 < s_1 < s_2 < s_3$ in $[a, b]$, let $t_0 = 0$ and $t_{i+1} - t_i = |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)|$ for $i = 0, 1, 2$, and let $\mathbf{p} : [t_0, t_3] \to \mathbb{R}^d$ be the cubic polynomial such that $\mathbf{p}(t_i) = \mathbf{f}(s_i)$, for $i = 0, 1, 2, 3$. Then $\operatorname{dist}_H(\mathbf{f}|_{[s_0, s_3]}, \mathbf{p}) = O(h^4)$ as $h \to 0$, where $h = s_3 - s_0$.

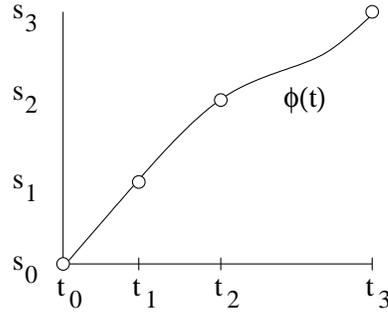The basic steps in the proof are:
(i) Show that
$$(t_{i+1} - t_i) - (s_{i+1} - s_i) = O\big((s_{i+1} - s_i)^3\big). \tag{5}$$
(ii) Use (5) to show that if $\phi : [t_0, t_3] \to \mathbb{R}$ is the cubic polynomial such that $\phi(t_i) = s_i$, $i = 0, 1, 2, 3$, then
$$\|\mathbf{f} \circ \phi - \mathbf{p}\|_\infty = O(h^4),$$
where $\|\mathbf{q}\|_\infty := \max_{t_0 \leq t \leq t_3} |\mathbf{q}(t)|$, see Figure 4.

Fig. 4. Reparameterization ϕ.

Consider now some of the details. Step (i) is to establish (5), that the error between the length of a chord and the length of an arc is of third order. This follows immediately from the following explicit error bound.

Lemma 1. If $\mathbf{f} \in C^2[a, b]$ then

$$0 \le (s_{i+1} - s_i) - |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)| \le \frac{1}{12}(s_{i+1} - s_i)^3 \max_{s_i \le s \le s_{i+1}} |\mathbf{f}''(s)|.$$

The proof follows from the use of Taylor series and the identities $\mathbf{f}'(s) \cdot \mathbf{f}'(s) = 1$ and $\mathbf{f}'(s) \cdot \mathbf{f}''(s) = 0$ [7].

In Step (ii) we use the reparameterization ϕ. If $\mathbf{g}(t) := \mathbf{f}(\phi(t))$, then

$$\mathbf{p}(t_i) = \mathbf{g}(t_i), \qquad i = 0, 1, 2, 3,$$

and so

$$\|\mathbf{g} - \mathbf{p}\|_\infty \le (t_3 - t_0)^4 \|\mathbf{g}^{(4)}\|_\infty/4! \le h^4 \|\mathbf{g}^{(4)}\|_\infty/4!.$$

So it is enough to show that $\phi' > 0$ for small enough h (so that $\mathbf{g}$ is well-defined) and that $\mathbf{g}^{(4)}$ is bounded as $h \to 0$. By the chain rule,

$$\mathbf{g}^{(4)} = (\phi')^4 \mathbf{f}^{(4)} + 6(\phi')^2 \phi'' \mathbf{f}''' + (3(\phi'')^2 + 4\phi'\phi''')\mathbf{f}'' + \phi^{(4)}\mathbf{f}',$$

and so $\mathbf{g}^{(4)}$ is indeed bounded if ϕ', ϕ'', ϕ''' are bounded as $h \to 0$. To see this, observe that (5) implies

$$[t_i, t_{i+1}]\phi - 1 = \frac{s_{i+1} - s_i}{t_{i+1} - t_i} - 1 = O\left(\frac{(s_{i+1} - s_i)^3}{t_{i+1} - t_i}\right) = O\big((s_{i+1} - s_i)^2\big), \qquad i = 0, 1, 2.$$

Therefore,

$$[t_i, t_{i+1}, t_{i+2}]\phi = O(s_{i+1} - s_i), \quad i = 0, 1, \qquad [t_0, t_1, t_2, t_3]\phi = O(1).$$

Thus all divided differences of ϕ are bounded and by expressing ϕ in its Newton form and differentiating, it follows that all derivatives of ϕ are indeed bounded, and that $\phi' > 0$ for small enough h.
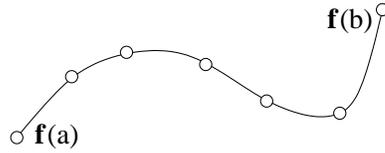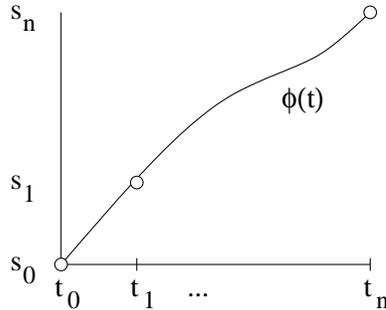
Fig. 5. Spline interpolation to **f**.

Fig. 6. Reparameterization ϕ.

5. Extension to cubic splines

The error analysis of chordal cubic polynomial interpolation extends to complete C^2 cubic spline interpolation. Again, chord lengths provide full fourth order approximation. The following was proved in [8]. We continue to assume that $|\mathbf{f}'(s)| = 1$ for all $s \in [a, b]$.

Theorem 2. Suppose $\mathbf{f} \in C^4[a, b]$ and for each sample $a = s_0 < \cdots < s_n = b$, let $t_0 = 0$ and $t_{i+1} - t_i = |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)|$, $0 \le i < n$, and let $\boldsymbol{\sigma} : [t_0, t_n] \to \mathbb{R}^d$ be the C^2 cubic spline curve such that

$$\boldsymbol{\sigma}(t_i) = \mathbf{f}(s_i), \quad i = 0, 1, \ldots, n,$$
$$\boldsymbol{\sigma}'(t_i) = \mathbf{f}'(s_i), \quad i = 0, n.$$

Then $\mathrm{dist}_H(\mathbf{f}, \boldsymbol{\sigma}) = O(h^4)$ as $h \to 0$, where $h = \max_{0 \le i < n}(s_{i+1} - s_i)$.

The basic ideas in the proof are similar to the polynomial case. But we now let $\phi : [t_0, t_n] \to \mathbb{R}$ be the C^2 cubic spline function satisfying

$$\phi(t_i) = s_i, \quad i = 0, 1, \ldots, n,$$
$$\phi'(t_i) = 1, \quad i = 0, n.$$

and use (5) to show that $\|\mathbf{f} \circ \phi - \boldsymbol{\sigma}\|_\infty = O(h^4)$, where $\|\mathbf{q}\|_\infty := \max_{t_0 \le t \le t_n} |\mathbf{q}(t)|$.

6. Parameter improvement for higher degree interpolation

Chord lengths do not give full approximation order $O(h^{n+1})$ for polynomial interpolation of degree $n > 3$. A solution is to use a parameterization which more accurately approximates the arc length of $\mathbf{f}$. A first improvement can be made by using the length of the cubic polynomial $\mathbf{p} : [t_0, t_3] \to \mathbb{R}^d$ in Theorem 1. It was shown in [7] that if $\mathbf{f} \in C^4[a, b]$ and $|\mathbf{f}'(s)| = 1$, for $s \in [a, b]$, then

$$L(\mathbf{p}|_{[t_0,t_1]}) - (s_1 - s_0) = O\Big((s_1 - s_0)^3 (s_2 - s_0)(s_3 - s_0)\Big),$$

$$L(\mathbf{p}|_{[t_1,t_2]}) - (s_2 - s_1) = O\Big((s_2 - s_0)(s_2 - s_1)^3 (s_3 - s_1)\Big),$$

$$L(\mathbf{p}|_{[t_2,t_3]}) - (s_3 - s_2) = O\Big((s_3 - s_0)(s_3 - s_1)(s_3 - s_2)^3\Big),$$

Thus, for example, the length of the cubic piece $\mathbf{p}|_{[t_1,t_2]}$ is a better approximation to the length of $\mathbf{f}|_{[s_1,s_2]}$ than the length of the chord $|\mathbf{f}(s_2) - \mathbf{f}(s_1)|$. Indeed the order of approximation has risen by two. Suppose now that $n = 4$ or $n = 5$ in (1). We start by letting $t_0, \ldots, t_n$ be chordal parameter values. Then we improve the parameterization as follows. For each $i = 0, \ldots, n-1$, we choose any sequence of four points $\mathbf{p}_j, \ldots, \mathbf{p}_{j+3}$ which includes $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ and we let $\mathbf{p} : [t_j, t_{j+3}] \to \mathbb{R}^d$ be the cubic interpolant $\mathbf{p}(t_k) = \mathbf{p}_k$, $j \le k \le j + 3$. Then we set

$$\hat{t}_{i+1} - \hat{t}_i := L(\mathbf{p}|_{[t_i,t_{i+1}]}).$$

For most i there is a choice of which cubic to use. Nevertheless, it can be shown, using a similar approach to the proof of Theorem 1, that if $\mathbf{q} : [\hat{t}_0, \hat{t}_n] \to \mathbb{R}^d$ is the interpolant of degree at most n with $\mathbf{q}(\hat{t}_i) = \mathbf{p}_i$, and $\mathbf{f}$ is in C^{n+1}, then

$$\mathrm{dist}_H(\mathbf{f}|_{[s_0,s_n]}, \mathbf{q}) = O(h^{n+1}), \qquad n = 4, 5.$$

Continuing this idea, one can write a recursive algorithm which generates a parameterization for any n which supports polynomial interpolation of degree n, see [7]. We can even view the uniform parameterization (or any other parameterization) as the start point for the iteration, because the length of the chord between two points is also the length of the linear interpolant to those points, with respect to any parameterization. Schematically, the parameter improvement looks as follows:

$$\text{Uniform} \;\to n = 1 \to O(h^2)$$
$$\nearrow$$
$$\text{Chordal} \;\to n = 3 \to O(h^4)$$
$$\nearrow$$
$$\text{Improved I} \to n = 5 \to O(h^6)$$
$$\nearrow$$
$$\text{Improved II} \to n = 7 \to O(h^8)$$

$\mathbf{f}(s_0)$
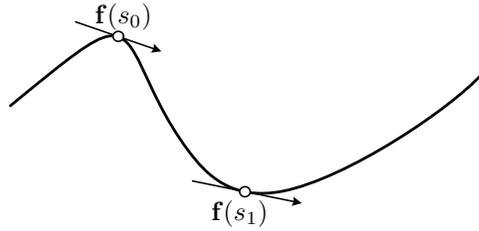
PSfrag replacements

$\mathbf{f}(s_1)$

Fig. 7. Two-point cubic Hermite interpolation.

7. Hermite interpolation

The theory of parameterization for polynomial interpolation extends to Hermite interpolation as long as we match arc length derivatives at every point.

7.1. *Cubic two-point Hermite*

Suppose we want to fit the Hermite cubic polynomial $\mathbf{p} : [t_0, t_1] \to \mathbb{R}^d$ to $\mathbf{f}$, i.e., such that

$$\mathbf{p}^{(k)}(t_i) = \mathbf{f}^{(k)}(s_i), \qquad i = 0, 1, \qquad k = 0, 1. \tag{6}$$

If we use chordal parameter values, i.e., $t_1 - t_0 = |\mathbf{f}(s_1) - \mathbf{f}(s_0)|$, then, noting that $|\mathbf{f}'(s_i)| = 1$, if $\mathbf{f} \in C^4[a, b]$ then

$$\mathrm{dist}_H(\mathbf{f}|_{[s_0, s_1]}, \mathbf{p}) = O(h^4) \quad \text{as } h \to 0.$$

7.2. *Quintic two-point Hermite*

How do we choose a suitable parameterization $\hat{t}_0 < \hat{t}_1$ when fitting a quintic polynomial $\mathbf{q} : [\hat{t}_0, \hat{t}_1] \to \mathbb{R}^d$ such that

$$\mathbf{q}^{(k)}(\hat{t}_i) = \mathbf{f}^{(k)}(s_i), \qquad i = 0, 1, \qquad k = 0, 1, 2, \tag{7}$$

where $|\mathbf{f}'(s_i)| = 1$, and $\mathbf{f}'(s_i) \cdot \mathbf{f}''(s_i) = 0$? One way of getting a sixth order approximation is to use the improved parameterization

$$\hat{t}_1 - \hat{t}_0 = (t_1 - t_0)(|\mathbf{p}'(\xi_-)| + |\mathbf{p}'(\xi_+)|)/2 \approx L(\mathbf{p}), \tag{8}$$

where $\xi_\pm = (t_0 + t_1)/2 \pm (t_1 - t_0)/(2\sqrt{3})$. Here $\mathbf{p}$ is the Hermite cubic interpolant in (6) using chordal values t_0, t_1, and we have used 2-point Gauss quadrature to estimate the integral

$$L(\mathbf{p}) = \int_{t_0}^{t_1} |\mathbf{p}'(t)| \, dt.$$

If $\mathbf{f} \in C^6[a, b]$, then

$$\mathrm{dist}_H(\mathbf{f}|_{[s_0, s_1]}, \mathbf{q}) = O(h^6) \quad \text{as } h \to 0.$$

Is the parameter improvement robust? Yes, in the sense that the new parameter intervals are always longer than the initial (chordal) parameter intervals. To see
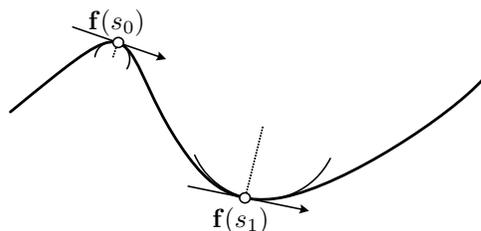
Fig. 8. Two-point quintic Hermite interpolation.

this, observe that since 2-point Gauss quadrature has quadratic (cubic) precision and positive weights,

$$
\begin{aligned}
\hat{t}_1 - \hat{t}_0 &= (t_1 - t_0)(|\mathbf{p}'(\xi_-)| + |\mathbf{p}'(\xi_+)|)/2 \\
&\geq |(t_1 - t_0)(\mathbf{p}'(\xi_-) + \mathbf{p}'(\xi_+))/2| \\
&= \left| \int_{t_0}^{t_1} \mathbf{p}'(t)\, dt \right| = |\mathbf{p}(t_1) - \mathbf{p}(t_0)| = |\mathbf{f}(s_1) - \mathbf{f}(s_0)| = t_1 - t_0.
\end{aligned}
\tag{9}
$$

Thus the new parameter intervals certainly do not shrink to zero. The above inequality is very natural because we know that since the straight line between two points is the shortest path between them,

$$
L(\mathbf{f}|_{[s_0, s_1]}) \geq t_1 - t_0.
$$

We could instead use Simpson's rule to get sixth-order accuracy:

$$
\begin{aligned}
\hat{t}_1 - \hat{t}_0 &= (t_1 - t_0)(|\mathbf{p}'(t_0)| + 4|\mathbf{p}'((t_0 + t_1)/2)| + |\mathbf{p}'(t_1)|)/6 \\
&= (t_1 - t_0)(1 + 2|\mathbf{p}'((t_0 + t_1)/2)|)/3 \\
&\approx L(\mathbf{p}).
\end{aligned}
$$

Again, $\hat{t}_1 - \hat{t}_0 \geq t_1 - t_0$. This same property holds for any quadrature method with positive weights and degree of precision at least two.

7.3. *Examples*

Figure 9 shows the C^1 cubic spline built from cubic 2-point Hermite interpolation, based on chordal parameter values. Figures 10 and 11 show the C^2 quintic splines built from quintic 2-point Hermite interpolation, based on chordal parameter values, and the improved parameter values. In all these figures, the black curve is the original and the grey one the approximation. Observe that the errors in Figures 10(b) and 11(b) are noticeably smaller than those in Figures 10(a) and 11(a).

Recall that from (9), the length $\hat{t}_1 - \hat{t}_0$ of the improved parameter interval is greater or equal to the original chord length $t_1 - t_0$ and it appears that lengthening the parameter interval has the effect of lengthening the quintic curve in Figure 10. We decided to explore this behaviour further and scaled the length of the chord by various factors: 1/2, 1, 2, and 5. The resulting curves are shown in Figure 12 which
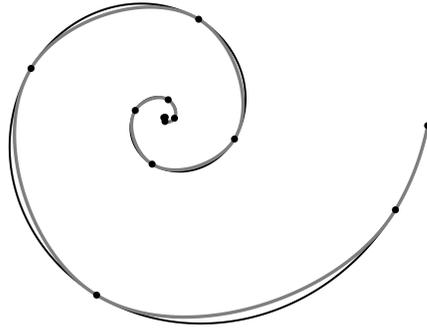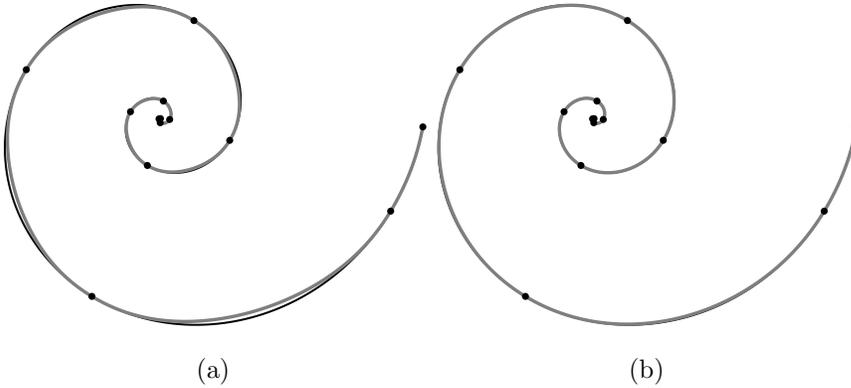
Fig. 9. Chordal Hermite cubic, C^1, $O(h^4)$.

(a)   (b)

Fig. 10. Chordal quintic Hermite, C^2, $O(h^4)$, (a) and improved quintic Hermite, C^2, $O(h^6)$, (b).

provides a striking illustration of what a dramatic effect parameterization can have on Hermite interpolation when the derivatives are fixed (not scaled as in Section 2). Clearly a shorter parameter interval leads to a tighter curve.

8. Geometric interpolation

An alternative approach to what we have discussed so far in this paper is so-called *geometric interpolation* as developed by [3,4,10,12–14]. These schemes, mainly Hermite, aim not only to retain full approximation order, but also to reduce the degree of the interpolant. The potential advantage of these schemes is that both the interpolant and parameterization are the simultaneous solutions to a set of equations. The disadvantage is that these equations are non-linear and only admit a solution under certain restrictions on the data points, and each scheme is dependent on the dimension d.

Since for planar data $(d = 2)$ the quintic Hermite $\mathbf{q}$ in (7) matches the tangents and curvatures of $\mathbf{f}$ at the two points, it is interesting to compare this quintic scheme
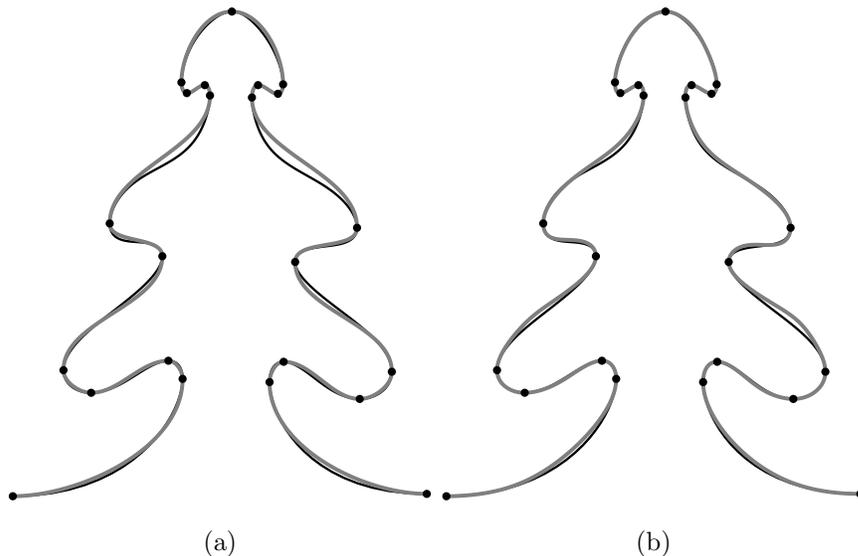
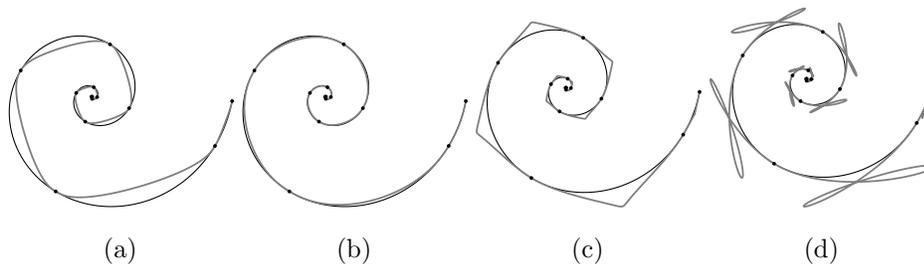Fig. 11. Chordal cubic, C^1, $O(h^4)$, (a) and improved quintic, C^2, $O(h^6)$, (b).

Fig. 12. Chord lengths multiplied by factors (a) $1/2$, (b) 1, (c) 2, and (d) 5.

with the scheme proposed for *planar* data by de Boor, Hollig, Sabin [3], which we will call the BHS scheme. When $\mathbf{f}$ is a curve in $\mathbb{R}^2$, the BHS scheme tries to fit a cubic to these tangents and curvatures. Thus the scheme attempts to satisfy

$$D_s^k \mathbf{p}(t_i) = \mathbf{f}^{(k)}(s_i), \qquad i = 0, 1, \quad k = 0, 1, 2,$$

where $\mathbf{p} : [t_0, t_1] \to \mathbb{R}^2$ is a cubic polynomial and D_s denotes differentiation with respect to arc length. The length $t_1 - t_0$ of the parameter interval is of no importance in this scheme and they set $t_0 = 0$, $t_1 = 1$. The approximation order is $O(h^6)$ around any point where the curvature is non-zero (a solution exists at such a point for small enough h). However there are examples of data sets for which there is no solution. Note that a method has been proposed in [15] for sampling points from a *given* curve $\mathbf{f}$ which supports the BHS scheme, in the sense that there is always a solution. For example, the BHS scheme has a solution for the data in Figure 13, generated by the method of [15], and as we can see in the figure, it is hard to distinguish the BHS (grey) curve from the original (black) curve, similar to the quintic in Figure 10(b).
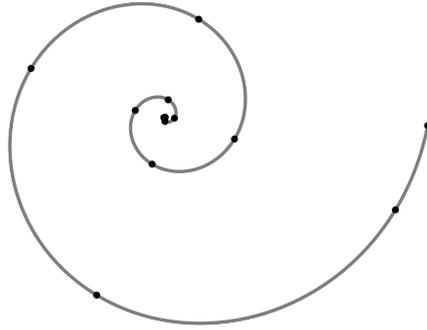
Fig. 13. BHS cubic, G^2.

Mørken and Scherer [12] view this scheme in a different way, in terms of reparameterization and degree reduction. First of all, instead of (7), we could find any quintic $\mathbf{q}$ which interpolates a reparameterization $\mathbf{g} = \mathbf{f} \circ \phi$ of the curve $\mathbf{f}$ where $\phi : [t_0, t_1] \to [s_0, s_1]$ is some increasing function with $\phi(t_i) = s_i$, $i = 0, 1$. Since

$$\mathbf{g}' = \phi' \mathbf{f}', \qquad \mathbf{g}'' = (\phi')^2 \mathbf{f}'' + \phi'' \mathbf{f}',$$

we can freely choose any values $\lambda_i, \mu_i \in \mathbb{R}$, $i = 0, 1$, and set

$$\begin{aligned}
\mathbf{q}(t_i) &= \mathbf{f}(s_i), & i = 0, 1, \\
\mathbf{q}'(t_i) &= \lambda_i \mathbf{f}'(s_i), & i = 0, 1, \\
\mathbf{q}''(t_i) &= \lambda_i^2 \mathbf{f}''(s_i) + \mu_i \mathbf{f}'(s_i), & i = 0, 1.
\end{aligned}$$

The case $\lambda_i = 1$ and $\mu_i = 0$ reduces to (7), whereas the BHS scheme attempts to find λ_i and μ_i to reduce the degree of the quintic $\mathbf{q}$ to three. Mørken and Scherer applied this approach to interpolation of other degrees.

Grandine and Hogan [10] have recently proposed raising the degree of the BHS scheme to four in order to guarantee a solution in all (planar) cases.

Clearly it is an advantage to interpolate with a polynomial with as low a degree as possible. On the other hand the attraction of the quintic scheme (7) is its simplicity: it is constructed in just a few steps without having to solve any non-linear equations. Moreover, unlike the BHS scheme and the scheme of Grandine and Hogan, the quintic scheme (7) applies to curves in *any* space dimension, including the important case $\mathbb{R}^3$. Moreover, the resulting curve has C^2 continuity rather than just G^2 which can be an advantage for certain post-processes such as surface lofting.

9. Conclusion

We have obtained precise information about how the choice of parameter values affects the approximation order of curve interpolation in *any* space dimension $d \geq 2$, based on polynomials and piecewise polynomials. Chordal parameter values give full approximation order for cubic interpolation, but more accurate approximations to
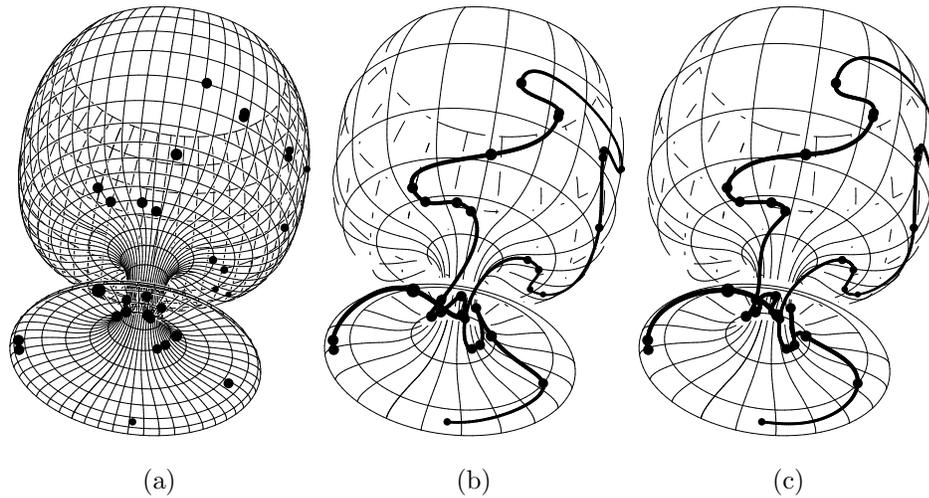
(a) (b) (c)

Fig. 14. Curve interpolation through points in $\mathbb{R}^3$.

arc length are required (and can be found) for higher degrees. Figure 14 illustrates an application where points in $\mathbb{R}^3$ are sampled from a curve on a glass-shaped surface (a). Two interpolating curves are shown: (b) a chordal C^2 cubic spline curve, and (c) a C^2 quintic Hermite spline curve, using the improved parameterization (8).

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