

A divided difference identity for a class of multiple integrals

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Abstract

We derive an identity that relates a class of multiple integrals involving Vandermonde polynomials to divided differences. Alternatively the identity can be viewed as an integral formula for divided differences. As part of the derivation we show that both sums of pure partial derivatives and mixed partial derivatives of Vandermonde polynomials are zero, which might be of independent interest.

Keywords: multiple integral, Vandermonde determinant, Vandermonde polynomial, divided difference.

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1 Introduction

In this paper we show that there is a surprisingly simple formula for a certain type of multiple integral involving Vandermonde polynomials. This formula arose while working on bounds for the determinant of an exponential matrix, and will be used for that purpose in [5]. Here, we will focus purely on a proof of the formula (Theorem 1). The formula can alternatively be viewed as an integral formula for a divided difference (at distinct points) which is different to the Genocchi-Hermite formula and to the B-spline integral

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formula of Curry and Schoenberg [2]. For a survey of divided differences and their role in interpolation theory, see de Boor [3], and for some basic theory, see Isaacson and Keller [6, Section 6.1] and Steffensen [8].

For some $n \geq 1$, let $\mathbf{x} = (x_1, x_2, \dots, x_{n+1})$ be a real increasing sequence and consider the rectangle in $\mathbb{R}^n$,

$$R(\mathbf{x}) := [x_1, x_2] \times [x_2, x_3] \times \cdots \times [x_n, x_{n+1}]. \quad (1)$$

Let $\phi(\mathbf{t}) = \phi(t_1, t_2, \dots, t_n)$ be a continuous function $\phi : R(\mathbf{x}) \rightarrow \mathbb{R}$. Then its integral over $R(\mathbf{x})$ is

$$\int_{R(\mathbf{x})} \phi(\mathbf{t}) \, d\mathbf{t} := \int_{x_n}^{x_{n+1}} \cdots \int_{x_2}^{x_3} \int_{x_1}^{x_2} \phi(t_1, t_2, \dots, t_n) \, dt_1 \, dt_2 \cdots dt_n.$$

Let us denote by $V(\mathbf{t})$ the Vandermonde polynomial

$$V(\mathbf{t}) = V(t_1, t_2, \dots, t_n) := \prod_{1 \leq i < j \leq n} (t_j - t_i). \quad (2)$$

We note that Vandermonde polynomials appear in the study of generalized Vandermonde determinants and Schur polynomials [7, 4, 1]. Let

$$y_i := x_1 + \cdots + x_{n+1-i} + x_{n+3-i} + \cdots + x_{n+1}, \quad i = 1, 2, \dots, n+1. \quad (3)$$

Thus, y_i is the sum of the x_j except the ' i -th to last', x_{n+2-i} , and we can alternatively write

$$y_i = \left(\sum_{j=1}^{n+1} x_j \right) - x_{n+2-i}. \quad (4)$$

Thus,

$$\begin{aligned} y_1 &= x_1 + \cdots + x_n, \\ y_{n+1} &= x_2 + \cdots + x_{n+1}, \end{aligned} \quad (5)$$

and by (4),

$$y_{i+1} - y_i = x_{n+2-i} - x_{n+1-i} > 0$$

and so $y_1, \dots, y_{n+1}$ are increasing. It also follows from (5) that for $\mathbf{t} \in R(\mathbf{x})$,

$$y_1 \leq t_1 + t_2 + \cdots + t_n \leq y_{n+1}. \quad (6)$$

We will show

Theorem 1 For $f \in C^n[y_1, y_{n+1}]$,

$$\int_{R(\mathbf{x})} V(\mathbf{t}) f^{(n)}(t_1 + t_2 + \cdots + t_n) d\mathbf{t} = V(\mathbf{x})[y_1, y_2, \dots, y_{n+1}]f, \quad (7)$$

where $[y_1, y_2, \dots, y_{n+1}]f$ denotes the divided difference of f at the points $y_1, y_2, \dots, y_{n+1}$.

For example, for $n = 1$,

$$\int_{x_1}^{x_2} f'(t_1) dt_1 = (x_2 - x_1)[y_1, y_2]f = (x_2 - x_1)[x_1, x_2]f,$$

and for $n = 2$,

$$\int_{x_2}^{x_3} \int_{x_1}^{x_2} (t_2 - t_1) f''(t_1 + t_2) dt_1 dt_2 = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)[y_1, y_2, y_3]f,$$

where

$$y_1 := x_1 + x_2, \quad y_2 := x_1 + x_3, \quad y_3 := x_2 + x_3.$$

In [5], we will use both Theorem 1 and the following corollary. Letting $f(\alpha) := \alpha^n/n!$ shows:

Corollary 1

$$\int_{R(\mathbf{x})} V(\mathbf{t}) d\mathbf{t} = \frac{1}{n!} V(\mathbf{x}).$$

For example, in the case $n = 2$,

$$\int_{x_2}^{x_3} \int_{x_1}^{x_2} (t_2 - t_1) dt_1 dt_2 = \frac{1}{2} (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

2 A divided difference viewpoint

The identity of Theorem 1 can be expressed purely as a formula for the divided difference $[y_1, y_2, \dots, y_{n+1}]f$ for given $y_1 < y_2 < \cdots < y_{n+1}$. To see this observe first that from (4),

$$y_j - y_i = x_{n+2-i} - x_{n+2-j},$$

and so $V(\mathbf{x}) = V(\mathbf{y})$, where $\mathbf{y} := (y_1, y_2, \dots, y_{n+1})$. Next observe that treating $\mathbf{x}$ and $\mathbf{y}$ as column vectors, they are related by the matrix equation $\mathbf{y} = M\mathbf{x}$, where M is the $(n+1) \times (n+1)$ matrix

$$M := \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

It is easy to check that M has the inverse

$$M^{-1} = \frac{1}{n} \begin{bmatrix} 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & \cdots & 1 & 1 \\ 1-n & 1 & \cdots & 1 & 1 \end{bmatrix},$$

and therefore,

$$x_i = \frac{1}{n} \left(\sum_{j=1}^{n+1} y_j - n y_{n+2-i} \right), \quad i = 1, 2, \dots, n+1. \quad (8)$$

So, by Theorem 1,

$$[y_1, y_2, \dots, y_{n+1}]f = \frac{1}{V(\mathbf{y})} \int_{R(\mathbf{x})} V(\mathbf{t}) f^{(n)}(t_1 + t_2 + \cdots + t_n) d\mathbf{t},$$

with $\mathbf{x}$ given by (8). For example, in the case $n = 2$,

$$[y_0, y_1, y_2]f = \frac{1}{V(\mathbf{y})} \int_{x_2}^{x_3} \int_{x_1}^{x_2} (t_2 - t_1) f''(t_1 + t_2) dt_1 dt_2,$$

where

$$V(\mathbf{y}) = (y_2 - y_1)(y_3 - y_1)(y_3 - y_2),$$

and

$$x_1 := \frac{y_1 + y_2 - y_3}{2}, \quad x_2 := \frac{y_1 - y_2 + y_3}{2}, \quad x_3 := \frac{-y_1 + y_2 + y_3}{2}.$$

3 Partial derivatives of $V(\mathbf{t})$

The rest of this paper is devoted to proving Theorem 1. A key element is the fact that the sums of mixed partial derivatives of any fixed order of $V(\mathbf{t})$ are zero: Lemma 5. As the first step towards that, we consider a polynomial of the form

$$\omega(t) := (t - t_1)(t - t_2) \cdots (t - t_m), \quad (9)$$

where $t, t_1, \dots, t_m \in \mathbb{R}$. We want to express its derivatives in terms of products of the differences $t - t_i$.

3.1 Derivatives of ω

To compute derivatives of ω in (9), consider the elementary symmetric polynomials. These are defined, for $t_1, t_2, \dots, t_m \in \mathbb{R}$, as

$$e_0(t_1, \dots, t_m) := 1,$$

and for $k = 1, \dots, m$,

$$e_k(t_1, t_2, \dots, t_m) := \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq m} t_{i_1} t_{i_2} \cdots t_{i_k}. \quad (10)$$

Lemma 1 For $k = 1, \dots, m$,

$$\frac{d}{dt} e_k(t - t_1, \dots, t - t_m) = (m - k + 1) e_{k-1}(t - t_1, \dots, t - t_m).$$

Proof. By (10),

$$e_k(t - t_1, \dots, t - t_m) = \sum_{1 \leq i_1 < \dots < i_k \leq m} (t - t_{i_1}) \cdots (t - t_{i_k}),$$

and its derivative with respect to t is

$$\sum_{1 \leq i_1 < \dots < i_k \leq m} \sum_{r=1}^k (t - t_{i_1}) \cdots (t - t_{i_{r-1}})(t - t_{i_{r+1}}) \cdots (t - t_{i_k}). \quad (11)$$

We can rewrite this as a sum over products of the form

$$(t - t_{j_1}) \cdots (t - t_{j_{k-1}}) \quad (12)$$

where

$$1 \leq j_1 < \cdots < j_{k-1} \leq m.$$

Since there are $m - k + 1$ elements in the complement of $\{j_1, \dots, j_{k-1}\}$ in $\{1, \dots, m\}$, the product (12) appears $m - k + 1$ times in the sum in (11) and so

$$\frac{d}{dt} e_k(t - t_1, \dots, t - t_m) = (m - k + 1) \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq m} (t - t_{j_1}) \cdots (t - t_{j_{k-1}}).$$

□

This lemma enables us to find the derivatives of ω in (9).

Lemma 2 *Let ω be the polynomial in (9). Then for $k = 0, 1, \dots, m$,*

$$\omega^{(k)}(t) = k! e_{m-k}(t - t_1, t - t_2, \dots, t - t_m).$$

Proof. Since $\omega(t) = e_m(t - t_1, t - t_2, \dots, t - t_m)$, applying Lemma 1 iteratively gives

$$\begin{aligned} \omega'(t) &= e_{m-1}(t - t_1, t - t_2, \dots, t - t_m), \\ \omega''(t) &= 2! e_{m-2}(t - t_1, t - t_2, \dots, t - t_m), \\ \omega'''(t) &= 3! e_{m-3}(t - t_1, t - t_2, \dots, t - t_m), \end{aligned}$$

and so on.

□

3.2 Pure partial derivatives of V

Using Lemma 2, we next derive a formula for the ‘pure’ (not mixed) derivatives of the Vandermonde polynomial $V(\mathbf{t})$ in (2) when $t_1, \dots, t_n$ are distinct.

Lemma 3 *Let $V(\mathbf{t})$ be the Vandermonde polynomial of (2) with $t_1, \dots, t_n$ distinct. For each $i = 1, \dots, n$ and $k = 1, \dots, n - 1$,*

$$\frac{\partial^k}{\partial t_i^k} V(\mathbf{t}) = k! V(\mathbf{t}) e_k\left(\frac{1}{t_i - t_1}, \dots, \frac{1}{t_i - t_{i-1}}, \frac{1}{t_i - t_{i+1}}, \dots, \frac{1}{t_i - t_n}\right).$$

Proof. We can write $V(\mathbf{t}) = c\omega(t_i)$, where

$$c := (-1)^{n-i} V(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n),$$

which is independent of t_i , and

$$\omega(t) := (t - t_1) \cdots (t - t_{i-1})(t - t_{i+1}) \cdots (t - t_n).$$

Then

$$\frac{\partial^k}{\partial t_i^k} V(\mathbf{t}) = c \omega^{(k)}(t_i) = V(\mathbf{t}) \frac{\omega^{(k)}(t_i)}{\omega(t_i)}.$$

By Lemma 2,

$$\omega^{(k)}(t) = k! e_{n-1-k}(t - t_1, \dots, t - t_{i-1}, t - t_{i+1}, \dots, t - t_n),$$

and therefore,

$$\frac{\omega^{(k)}(t_i)}{\omega(t_i)} = k! e_k\left(\frac{1}{t_i - t_1}, \dots, \frac{1}{t_i - t_{i-1}}, \frac{1}{t_i - t_{i+1}}, \dots, \frac{1}{t_i - t_n}\right).$$

□

The lemma covers the cases $k = 1, \dots, n - 1$. Since $V(\mathbf{t})$ is a polynomial of degree $n - 1$ with respect to t_i , it follows that

$$\frac{\partial^n}{\partial t_i^n} V(\mathbf{t}) = 0. \tag{13}$$

3.3 Sums of pure derivatives

Next we show that the sums of pure derivatives of $V(\mathbf{t})$ are zero.

Lemma 4 *Let $V(\mathbf{t}) := V(t_1, \dots, t_n)$ for any $t_1, \dots, t_n \in \mathbb{R}$. For each $i = 1, \dots, n$ and $k = 1, \dots, n$,*

$$\sum_{i=1}^n \frac{\partial^k}{\partial t_i^k} V(\mathbf{t}) = 0.$$

Proof. For $k = n$ this is implied by (13). For $1 \leq k \leq n - 1$ it is sufficient to prove it for distinct $t_1, \dots, t_n$. Then, by Lemma 3,

$$\sum_{i=1}^n \frac{\partial^k}{\partial t_i^k} V(\mathbf{t}) = k! V(\mathbf{t}) S,$$

where

$$S := \sum_{i=1}^n \sum_{\substack{1 \leq i_1 < \dots < i_k \leq m \\ i \notin \{i_1, \dots, i_k\}}} \frac{1}{\prod_{j=1}^k (t_i - t_{i_j})}.$$

By considering the sequence of length $k+1$ formed by i and $i_1, \dots, i_k$, we can rewrite S as

$$S = \sum_{1 \leq j_1 < \dots < j_{k+1} \leq m} S(j_1, j_2, \dots, j_{k+1}),$$

wher

$$S(j_1, j_2, \dots, j_{k+1}) := \sum_{r=1}^{k+1} \frac{1}{\prod_{s=1, s \neq r}^{k+1} (t_{j_r} - t_{j_s})}.$$

We recognize $S(j_1, j_2, \dots, j_{k+1})$ as the divided difference of order k of the constant function 1 over the $k+1$ points $t_{j_1}, \dots, t_{j_{k+1}}$, which is zero. $\square$

3.4 Sums of mixed derivatives

We now show that the sums of mixed derivatives of $V(\mathbf{t})$ are also zero. For $k = 1, \dots, n$, let E_k be the differential operator

$$E_k := \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{\partial^k}{\partial t_{i_1} \dots \partial t_{i_k}}. \quad (14)$$

Lemma 5 *Let $V(\mathbf{t}) = V(t_1, \dots, t_n)$ for $t_1, \dots, t_n \in \mathbb{R}$. Then $E_k V(\mathbf{t}) = 0$ for all $k = 1, \dots, n$.*

Proof. Let P_k denote the differential operator

$$P_k := \sum_{i=1}^n \frac{\partial^k}{\partial t_i^k}.$$

We have already shown that $P_k V(\mathbf{t}) = 0$ for all $k = 1, \dots, n$. The operators P_k and E_k are analogous to the power sum and elementary symmetric

polynomials and thus obey Newton's identities:

$$\begin{aligned}
E_1 &= P_1, \\
2E_2 &= E_1P_1 - P_2, \\
3E_3 &= E_2P_1 - E_1P_2 + P_3, \\
&\dots \\
kE_k &= \sum_{i=1}^{k-1} (-1)^{i-1} E_{k-i}P_i + (-1)^{k-1}P_k.
\end{aligned}$$

Thus, since $P_1V(\mathbf{t}) = \dots = P_kV(\mathbf{t})$, it follows that $E_kV(\mathbf{t}) = 0$ as well. $\square$

4 The sum function

For $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$, let $s : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the sum

$$s(\mathbf{t}) := t_1 + \dots + t_n. \quad (15)$$

We note that s has the property that

$$\frac{\partial s}{\partial t_i} = 1, \quad i = 1, \dots, n.$$

It then follows from the chain rule that if $f \in C^1(s(\Omega))$ for some domain $\Omega \subseteq \mathbb{R}^n$, then

$$\frac{\partial}{\partial t_i}(f(s(\mathbf{t}))) = f'(s(\mathbf{t})), \quad i = 1, \dots, n.$$

Building on this fact, we next compute the highest order mixed derivative of the product of $f \circ s$ with some other function ψ . Recall the notation of (14) and define $E_0 := \text{id}$.

Lemma 6 *Suppose that $\psi \in C^n(\Omega)$ and $f \in C^n(s(\Omega))$ for some domain $\Omega \subseteq \mathbb{R}^n$, and define the product*

$$\phi(\mathbf{t}) := \psi(\mathbf{t})f(s(\mathbf{t})), \quad \mathbf{t} \in \Omega.$$

Then,

$$\frac{\partial^n}{\partial t_1 \dots \partial t_n} \phi(\mathbf{t}) = E_n \phi(\mathbf{t}) = \sum_{k=0}^n E_k \psi(\mathbf{t}) f^{(n-k)}(s(\mathbf{t})), \quad \mathbf{t} \in \Omega. \quad (16)$$

Proof. We show more generally that

$$\frac{\partial^m}{\partial t_1 \cdots \partial t_m} \phi(\mathbf{t}) = \sum_{k=0}^m \sum_{1 \leq i_1 < \cdots < i_k \leq m} \frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}} \psi(\mathbf{t}) f^{(m-k)}(s(\mathbf{t})) \quad (17)$$

for all $m = 0, 1, \dots, n$ by induction on m . This is true for $m = 0$ by the definition of ϕ . Suppose it holds for some m , $0 \leq m \leq n-1$. Differentiation with respect to t_{m+1} gives

$$\frac{\partial^{m+1}}{\partial t_1 \cdots \partial t_{m+1}} \phi(\mathbf{t}) = a + b,$$

where

$$\begin{aligned} a &:= \sum_{k=0}^m \sum_{1 \leq i_1 < \cdots < i_k \leq m} \frac{\partial^{k+1}}{\partial t_{i_1} \cdots \partial t_{i_k} \partial t_{m+1}} \psi(\mathbf{t}) f^{(m-k)}(s(\mathbf{t})), \\ b &:= \sum_{k=0}^m \sum_{1 \leq i_1 < \cdots < i_k \leq m} \frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}} \psi(\mathbf{t}) f^{(m+1-k)}(s(\mathbf{t})). \end{aligned}$$

We can write a as

$$\begin{aligned} a &= \sum_{k=1}^{m+1} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq m} \frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_{k-1}} \partial t_{m+1}} \psi(\mathbf{t}) f^{(m+1-k)}(s(\mathbf{t})) \\ &= \sum_{k=1}^{m+1} \sum_{1 \leq i_1 < \cdots < i_k = m+1} \frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}} \psi(\mathbf{t}) f^{(m+1-k)}(s(\mathbf{t})), \end{aligned}$$

and adding this to b yields the case $m+1$ of (17). $\square$

Now let $V(\mathbf{t})$ be the Vandermonde polynomial in (2). By Lemma 5, $E_k V(\mathbf{t}) = 0$ for all $k = 1, \dots, n$, while $E_0 V(\mathbf{t}) = V(\mathbf{t})$. Therefore, letting $\psi = V$ in Lemma 6 implies

Lemma 7 *Suppose that $f \in C^n(s(\Omega))$ for some domain $\Omega \subseteq \mathbb{R}^n$. Then,*

$$V(\mathbf{t}) f^{(n)}(s(\mathbf{t})) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} (V(\mathbf{t}) f(s(\mathbf{t}))), \quad \mathbf{t} \in \Omega.$$

This observation will simplify the integration in Theorem 1.

5 Integration on a rectangle

Let us now look at integration on a rectangle in $\mathbb{R}^n$. Such a rectangle has the form

$$R := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

and has 2^n vertices which we can denote by

$$\mathbf{v}_\epsilon := ((1 - \epsilon_1)a_1 + \epsilon_1 b_1, (1 - \epsilon_2)a_2 + \epsilon_2 b_2, \dots, (1 - \epsilon_n)a_n + \epsilon_n b_n),$$

for $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$. It is easy to integrate the highest order mixed derivative of a function of n variables on R . Similar to (15) we define

$$s(\epsilon) := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n.$$

Lemma 8 *For any $\phi \in C^n(R)$,*

$$\int_R \frac{\partial^n \phi}{\partial t_1 \cdots \partial t_n} dt_1 \cdots dt_n = \sum_{\epsilon \in \{0,1\}^n} (-1)^{n-s(\epsilon)} \phi(\mathbf{v}_\epsilon). \quad (18)$$

Proof. We show more generally that the integral can be written as

$$\sum_{\epsilon \in \{0,1\}^k} (-1)^{k-s(\epsilon)} \int_{a_n}^{b_n} \cdots \int_{a_{k+1}}^{b_{k+1}} \frac{\partial^{n-k}}{\partial t_{k+1} \cdots \partial t_n} \phi(\mathbf{v}_{\epsilon,k}) dt_{k+1} \cdots dt_n \quad (19)$$

for any $k = 0, 1, \dots, n$, where

$$\mathbf{v}_{\epsilon,k} := ((1 - \epsilon_1)a_1 + \epsilon_1 b_1, \dots, (1 - \epsilon_k)a_k + \epsilon_k b_k, t_{k+1}, \dots, t_n).$$

The case $k = 0$ of (19) is the left hand side of (18). Suppose (19) holds for some k , $0 \leq k \leq n-1$. Then its inner integral is

$$\begin{aligned} & \frac{\partial^{n-k-1}}{\partial t_{k+2} \cdots \partial t_n} \phi(\mathbf{v}_{(\epsilon,1),k+1}) - \frac{\partial^{n-k-1}}{\partial t_{k+2} \cdots \partial t_n} \phi(\mathbf{v}_{(\epsilon,0),k+1}) \\ &= \sum_{\epsilon_{k+1}=0}^1 (-1)^{1-\epsilon_{k+1}} \frac{\partial^{n-k-1}}{\partial t_{k+2} \cdots \partial t_n} \phi(\mathbf{v}_{(\epsilon,\epsilon_{k+1}),k+1}), \end{aligned}$$

and moving this summation out of the remaining integrals yields the case $k+1$ of (19). Thus (19) also holds for $k = n$, which is the right hand side of (18). $\square$

Suppose next that R is the sequential rectangle $R(\mathbf{x})$ of (1) and that ϕ has the zero property

$$\phi(\mathbf{t}) = 0 \quad \text{whenever } t_i = t_{i+1} \text{ for any } i \in \{1, 2, \dots, n-1\}. \quad (20)$$

Then the sum in (18) is reduced from 2^n terms to $n+1$.

Lemma 9 *If $\phi \in C^n(R(\mathbf{x}))$ and ϕ has the zero property (20) then*

$$\int_{R(\mathbf{x})} \frac{\partial^n \phi}{\partial t_1 \cdots \partial t_n} dt_1 \cdots dt_n = \sum_{i=1}^{n+1} (-1)^{n+1-i} \phi(\mathbf{v}_i), \quad (21)$$

where

$$\mathbf{v}_i := (x_1, \dots, x_{n+1-i}, x_{n+3-i}, \dots, x_{n+1}). \quad (22)$$

Proof. We apply Lemma 8 to the sequential rectangle $R = R(\mathbf{x})$, whose vertices are

$$\mathbf{v}_\epsilon := ((1 - \epsilon_1)x_1 + \epsilon_1 x_2, (1 - \epsilon_2)x_2 + \epsilon_2 x_3, \dots, (1 - \epsilon_n)x_n + \epsilon_n x_{n+1})$$

for $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$. By the zero property (20), $\phi(\mathbf{v}_\epsilon) = 0$ whenever $\epsilon_i = 1$ and $\epsilon_{i+1} = 0$ for some $i \in \{1, 2, \dots, n-1\}$. Thus $\phi(\mathbf{v}_\epsilon)$ can only be non-zero if the sequence ϵ is non-decreasing, $\epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_n$. There are just $n+1$ such sequences:

$$\epsilon_i := (\underbrace{0, \dots, 0}_{n+1-i}, \underbrace{1, \dots, 1}_{i-1}), \quad i = 1, 2, \dots, n+1, \quad (23)$$

and the corresponding $n+1$ vertices are $\mathbf{v}_i := \mathbf{v}_{(\epsilon_i)}$ which are as in (22). Furthermore, $s(\epsilon_i) = i-1$. Therefore, the right hand side of (18) reduces to the right hand side of (21). $\square$

6 Proof of Theorem 1

We now complete the proof of Theorem 1. Let $\Omega := R(\mathbf{x})$. Then, by (5) and (6), $s(\Omega) = [y_1, y_{n+1}]$, and by Lemma 7,

$$\int_{R(\mathbf{x})} V(\mathbf{t}) f^{(n)}(s(\mathbf{t})) d\mathbf{t} = \int_{R(\mathbf{x})} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} (V(\mathbf{t}) f(s(\mathbf{t}))) d\mathbf{t},$$

and by Lemma 9,

$$\int_{R(\mathbf{x})} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} (V(\mathbf{t})f(s(\mathbf{t}))) \, d\mathbf{t} = \sum_{i=1}^{n+1} (-1)^{n+1-i} V(\mathbf{v}_i) f(s(\mathbf{v}_i)), \quad (24)$$

where $\mathbf{v}_i$ is as in (22). We see that $s(\mathbf{v}_i) = y_i$ by the definition of y_i in (3). Further, by the definition of the $\mathbf{v}_i$ in (22), the factors in $V(\mathbf{v}_i)$ are those of $V(\mathbf{x})$ except those involving x_{n+2-i} , and we can write

$$(-1)^{n+1-i} V(\mathbf{v}_i) = \frac{V(\mathbf{x})}{\prod_{j=1, j \neq n+2-i}^{n+1} (x_j - x_{n+2-i})}.$$

Recalling (4), we see that

$$y_i - y_{n+2-j} = x_j - x_{n+2-i},$$

and so

$$\prod_{\substack{j=1 \\ j \neq n+2-i}}^{n+1} (x_j - x_{n+2-i}) = \prod_{\substack{j=1 \\ j \neq n+2-i}}^{n+1} (y_i - y_{n+2-j}) = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} (y_i - y_j).$$

Thus the right hand side of (24) can be written as

$$V(\mathbf{x}) \sum_{i=1}^{n+1} \frac{f(y_i)}{\prod_{j=1, j \neq i}^{n+1} (y_i - y_j)} = V(\mathbf{x})[y_1, y_2, \dots, y_{n+1}]f.$$

References

- [1] R. Ait-Haddou and M.-L. Mazure, *The fundamental blossoming inequality in Chebyshev spaces — I: applications to Schur functions*, Found. Comp. Math. **18** (2018), 135–158.
- [2] H. B. Curry and I. J. Schoenberg, *On Pólya frequency functions IV: the fundamental spline functions and their limits*, J. Analyse Math. **17** (1966), 71–107.
- [3] C. de Boor, *Divided differences*, Surveys in Approx. Theory **1** (2005), 46–69.

- [4] S. de Marchi, *Polynomials arising in factoring generalized Vandermonde determinants: an algorithm for computing their coefficients*, Math. Comp. Modelling **34** (2001), 271–281.
- [5] M. S. Floater, *Bounds on the determinant of an exponential matrix*, preprint, 2026.
- [6] E. Isaacson and H. B. Keller, *Analysis of numerical methods*, Dover, New York, 1994.
- [7] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1995.
- [8] J. F. Steffensen, *Interpolation*, Williams and Wilkins, Baltimore, 1927.