Blow-up at the Boundary for Degenerate Semilinear Parabolic Equations

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Abstract. This paper concerns a superlinear parabolic equation, degenerate in the time derivative. It is shown that the solution may blow up in finite time. Moreover it is proved that for a large class of initial data blow-up occurs at the boundary of the domain when the nonlinearity is no worse than quadratic. Various estimates are obtained which determine the asymptotic behaviour near the blow-up. The mathematical analysis is then extended to equations with other degeneracies.

§1. Introduction

Consider the following semilinear problem

$$xu_t = u_{xx} + u^p \qquad \text{in } \Omega \times (0, \infty) \tag{1.1}$$
$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{1.2}$$
$$u(x,0) = u_0(x) \geq 0 \qquad x \in \Omega \tag{1.3}$$

where Ω is the unit interval $(0,1)$ and $p > 1$. The initial data u_0 is assumed to be in $C^1(\bar{\Omega})$ with $u_0(0) = u_0(1) = 0$. Since the coefficient of u_t is zero at $x = 0$ we can regard (1.1) as degenerate.

In this paper we investigate the effect of the degeneracy on the blow-up behaviour of the solution of (1.1)–(1.3). It is shown that if u_0 is large enough, $\sup_{x\in\Omega} u$ tends to infinity in finite time. More significantly, if $1 < p \leq 2$ and u_0 is concave (this can be weakened to $\frac{d}{dx}(u_0(x)/x) \leq 0$) u blows up at the boundary $x = 0$. Numerical evidence, see Floater (1988) and Stuart & Floater (1989), indicates that blow-up is away from the boundary when $p > 2$. The analytical results are generalised to

$$x^q u_t = u_{xx} + u^p \tag{1.4}$$

for any $q > 0$. Blow-up at the boundary can occur when $1 < p \leq q + 1$.

The motivation for studying (1.1) comes from the paper by Ockendon (1979). A model is derived for the dynamics of a fluid whose viscosity is temperature dependent in a channel. By making a reasonable approximation the equation

$$xu_t = u_{xx} + e^u$$

is found where u represents the temperature of the fluid, see Ockendon (1979) and Lacey (1984). The boundary $x = 0$ corresponds to one boundary of the channel (the other boundary of the channel is disregarded). So that analytical results may be obtained we have approximated e^u by u^p.

We begin in Section 2 by constructing a unique classical solution to (1.1)–(1.3) in some time interval $(0, t_0)$. Further, due to a continuation theorem, u can be continued for all $t > 0$ or until blow-up.

In Section 3 we show that if u_0 is large enough, u blows up at some finite time T say. Now we call $x_0 \in \bar{\Omega}$ a blow-up point if there exists a sequence $(x_n, t_n) \subset \Omega \times (0, T)$ such that $t_n \to T$, $x_n \to x_0$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$. The blow-up set $S \subset \bar{\Omega}$ is defined as the set of all blow-up points. Theorem 3.5 states that if $1 < p \leq 2$ and $\frac{d}{dx}(u_0(x)/x) \leq 0$ then $S = \{0\}$. This contrasts with a result by Friedman & McLeod (1985). Using the reflection principle they showed that for the solution of

$$v_t = v_{xx} + v^p$$

with the same conditions (1.2)–(1.3), the blow-up set S_v is a compact subset of Ω. This implies that S_v is bounded away from the boundary $\partial\Omega$.

Under the assumptions of Theorem 3.5 the asymptotics of u as t approaches T are analysed in Section 4. It is demonstrated that

$$\text{if } \gamma < (p-1)/2, \qquad \int_0^1 u^\gamma(x,t)dt \leq C_\gamma \qquad \text{for all } t \in (0,T) \tag{1.5}$$

and

$$\text{if } \gamma > (p-1)/2 \text{ and } u_0'' + u_0^p \geq 0, \qquad \int_0^1 u^\gamma(x,t)dt \to \infty \qquad \text{as } t \to T. \tag{1.6}$$

We also obtain information concerning the rate at which the maximum of u tends to the boundary. Letting $s(t)$ be any point such that $u(s(t), t) = \sup_{x\in\Omega} u(x,t)$, we find

$$C_1(T-t)^{1/3} \leq s(t) \leq C_2(T-t)^{1/(3+\delta)} \qquad \text{for any } \delta > 0. \tag{1.7}$$

In the case $p = 2$, these estimates are in agreement with Lacey (1984). Using formal methods he conjectured that

$$u(x,T) \sim -K_1 \frac{\ln x}{x^2} \text{ as } x \to 0$$

and (assuming $s(t)$ is unique)

$$s(t) \sim K_2\{-(T-t)\ln(T-t)\}^{1/3} \text{ as } t \to T.$$

Finally, in Section 5 the analytical results are extended to the more general equation

$$x^q u_t = u_{xx} + u^p.$$

Interestingly, the integral bounds (1.5), (1.6) are unchanged, i.e. independent of q. Meanwhile the bounds on $s(t)$ in (1.7) extend to

$$C_1(T-t)^{1/(q+2)} \leq s(t) \leq C_2(T-t)^{1/(q+2+\delta)} \qquad \text{for any } \delta > 0$$

and are independent of p. We also make some remarks on the likeness of

$$u_t = u_{xx} + \frac{u^p}{x^q}.$$

§2. Existence and uniqueness of solution

In this section it is demonstrated that a classical solution to (1.1)–(1.3) exists. Uniqueness follows easily from the maximum principle.

Our approach to existence is to define the new problem

$$xu_t = u_{xx} + u^p \qquad \text{in } (\epsilon, 1) \times (0, \infty) \tag{2.1}$$

$$u(\epsilon, t) = u(1, t) = 0 \qquad t > 0 \tag{2.2}$$

$$u(x, 0) = u_0(x) \geq 0 \qquad x \in (\epsilon, 1) \tag{2.3}$$

The initial data $u_0 : (\epsilon, 1) \to \mathbb{R}$ is simply the truncation of the initial data in (1.3). Even though the compatibility condition $u_0(\epsilon) = 0$ does not hold, it is well known that a $C^{2,1}$ solution u_ϵ of (2.1)–(2.3) exists on some time interval $(0, t_0)$, see Friedman (1964). The solution u_ϵ is continuous in $(\epsilon, 1] \times [0, t_0)$ and $[\epsilon, 1] \times (0, t_0)$. The idea in proving existence for (1.1)–(1.3) is that $\lim_{\epsilon \to 0} u_\epsilon(x, t) = u(x, t)$ is a good candidate for the solution. The remainder of Section 2 is devoted to proving that u is indeed the classical solution in $\Omega \times (0, t_0)$ and can be continued in t until either blow-up or infinity.

Lemma 2.1. *Let $1 > \epsilon_1 > \epsilon_2 > 0$ and suppose u_{ϵ_1} and u_{ϵ_2} are solutions of (2.1)–(2.3) on $(0, t_0)$. Then $u_{\epsilon_2} \geq u_{\epsilon_1}$ for all $x \in (\epsilon_1, 1)$, $t \in (0, t_0)$.*

Proof. Define $w = u_{\epsilon_2} - u_{\epsilon_1}$ in $(\epsilon_1, 1) \times (0, t_0)$. Then

$$w_t - \frac{1}{x} w_{xx} = \frac{1}{x}\{f(u_{\epsilon_2}) - f(u_{\epsilon_1})\} = \frac{f'(k)}{x} w$$

for some k between u_{ϵ_1} and u_{ϵ_2}. By the maximum principle, u_{ϵ_1} and u_{ϵ_2} are positive in their respective domains. In particular $u_{\epsilon_2} > 0$ when $x = \epsilon_1$. Therefore $w > 0$ at $x = \epsilon_1$. Further, $w = 0$ at $x = 1$ and $w = 0$ at $t = 0$. Hence by the maximum principle, $w > 0$ in $(\epsilon_1, 1) \times (0, t_0)$ as required. $\triangleleft$

Due to the monotonicity of the u_ϵ solutions, the pointwise limit of $\{u_\epsilon\}_{\epsilon > 0}$ as $\epsilon \to 0$ would be well defined were it not that t_0 depends on ϵ and $\{u_\epsilon\}_{\epsilon > 0}$ may tend to infinity at some point (x, t). In this context it is interesting to study

$$\begin{aligned} xu_t &= u^p, \\ u(x, 0) &= \sin \pi x. \end{aligned} \tag{2.4}$$

The solution can be found explicitly as

$$u(x, t) = \beta^\beta x^\beta / (T(x) - t)^\beta$$

where $T(x) = x/(p-1)u_0^{p-1}(x)$. For each $x \in (0, 1)$, there is an associated blow-up time $T(x)$. Now observe that since $u_0(x) \sim \pi x$ as $x \to 0$, $T(x) \to 0$ as $x \to 0$ when $p < 2$. Therefore there cannot be a solution of (2.4) in $C((0, 1) \times (0, t_0))$ for any $t_0 > 0$. We may conclude from this that for $1 < p < 2$ the diffusion term u_{xx} in (1.1)–(1.3) plays a vital role in smoothing out the solution. The next lemma shows that diffusion really does ensure existence.

Lemma 2.2. *There exists t_0 and an a priori bound $h \in C^{2,1}([0,1] \times [0,t_0])$ depending only on u_0 and p such that $\forall \epsilon > 0$ there exists a solution u_ϵ of (2.1)–(2.3) in $(\epsilon, 1) \times (0, t_0)$ with $u_\epsilon \leq h$.*

Proof. The idea in constructing h is to split the domain Ω into three parts $(0, \delta)$, $(\delta, 1-\delta)$, and $(1-\delta, 1)$. To show h is an upper solution for (1.1)–(1.3) we use the diffusion term h_{xx} in $(0, \delta)$, $(1-\delta, 1)$ and the term xh_t in $(\delta, 1-\delta)$ to overcome the nonlinearity h^p. Consider the following definitions:

(i) Let $\psi(x) = x(1-x)/2$.
(ii) Choose k_0 s.t. $k_0\psi(x) \geq u_0(x)$.
(iii) Choose $\delta \in (0, \frac{1}{2})$ s.t. $\delta \leq (2k_0)^{-(p-1)/p}$.
(iv) Let $k(t)$ solve $k(0) = k_0$ and $k' = k^p/4^{p-1}\delta$.
(v) Set $h(x,t) = k(t)\psi(x)$.
(vi) Define t_0 by $k(t_0) = \delta^{-p/(p-1)}$.

We claim that h is an upper solution for (1.1)–(1.3) in $\Omega \times (0, t_0)$. Substituting h into (1.1) gives

$$J = xh_t - h_{xx} - h^p = xk'\psi + k - k^p\psi^p.$$

For $x \in (0, \delta)$, $t \in (0, t_0)$,

$$\begin{aligned} J \geq& k - k^p\psi^p \\ \geq& k(1 - k^{p-1}\{\delta(1-\delta)/2\}^p) \\ \geq& k(1 - k^{p-1}\delta^p) \\ \geq& 0 \qquad \text{since } k^{p-1} \leq \delta^{-p} \text{ for } t \leq t_0. \end{aligned}$$

Exactly the same calculation holds for $x \in (1-\delta, 1)$. For $x \in (\delta, 1-\delta)$, $t \in (0, t_0)$,

$$\begin{aligned} J \geq& xk'\psi - k^p\psi^p \\ \geq& \psi(\delta k' - k^p/4^{p-1}) \\ =& 0 \qquad \text{by definition of } k. \end{aligned}$$

Finally let $\epsilon > 0$. Since h is an upper solution for (1.1)–(1.3) in $(0,1) \times (0, t_0)$ it is also an upper solution for (2.1)–(2.3) in $(\epsilon, 1) \times (0, t_0)$ (h is positive at $x = \epsilon$). Hence the solution u_ϵ of (2.1)–(2.3) exists in $(\epsilon, 1) \times (0, t_0)$ and $0 \leq u_\epsilon \leq h$, see Sattinger (1972). This ends the proof. $\triangleleft$

Due to Lemmas 2.1 and 2.2 a function u can be constructed as

$$u(x,t) = \begin{cases} \lim_{\epsilon \to 0} u_\epsilon(x,t) & \text{for } x \in (0,1],\ t \in [0, t_0); \\ 0 & \text{for } t \in [0, t_0). \end{cases} \tag{2.5}$$

We now show that u is a classical solution of (1.1)–(1.3).

Theorem 2.3. *Under the assumption that $u_0 \in C^{2+\alpha}(0,1)$, some $\alpha \in (0,1)$, u as given in (2.5) is a classical solution of (1.1)–(1.3) in $\Omega \times (0, t_0)$.*

Proof. It is required to prove that u is $C^{2,1}$ in $(0,1)\times(0,t_0)$ and continuous in $[0,1]\times[0,t_0)$.

Choose a point $(x_1, t_1) \in (0,1) \times (0, t_0)$. Then select a domain $Q = (a,b) \times (0, t_2)$ s.t. $0 < a < x_1 < b < 1$ and $0 < t_1 < t_2 < t_0$. Now since $u_\epsilon \to u$ pointwise in Q, $u_\epsilon \to u$ in any $L^q(Q)$. Further, as u_ϵ is uniformly bounded by h in Q, the L^q estimate of LSU (1968) yields

$$\int_Q |\Delta(u_\epsilon)|^q + |(u_\epsilon)_t|^q dxdt \le C$$

where C is independent of ϵ. Now choose q so that $q > 2/(1-\alpha)$. Then Sobolev's embedding gives

$$\|u_\epsilon\|_{C^\alpha(Q)} \le C'.$$

Finally, an application of the C^α estimate of LSU (1968) yields

$$\|u_\epsilon\|_{C^{2+\alpha,1+\alpha}(Q)} \le C''$$

where C'' depends on Q but not on ϵ. Now an appeal to the Ascoli-Arzela theorem shows $u \in C^{2+\alpha,1+\alpha}(Q)$ with $\|u\|_{C^{2+\alpha,1+\alpha}(Q)} \le C''$, see for example Friedman (1964). This shows that u is $C^{2,1}$ at (x_1, t_1) and continuous at $(x_1, 0)$. It is clear that u is continuous at the boundaries $x = 0$, $x = 1$ because of the estimate $0 \le u_\epsilon \le h$ and the sandwich lemma. ◃

Having proved existence, it is straightforward to prove uniqueness and positivity.

Proposition 2.4. *The solution u defined by (2.5) is unique and positive.*

Proof. We already noted the fact that $u_\epsilon \ge 0$ in $(\epsilon, 1) \times (0, t_0)$ and consequently, $u \ge 0$ in $(0,1) \times (0, t_0)$. An application of the (strong) maximum principle shows $u > 0$ in $(0,1) \times (0, t_0)$ unless $u_0 \equiv 0$.

To show u is unique, let u and v be two solutions of (1.1)–(1.3) in $(0,1) \times (0, t_0)$. Then, if $w = v - u$,

$$xw_t = w_{xx} + p\theta^{p-1}w$$

where $\theta = \lambda(x,t)u + (1 - \lambda(x,t))v$, for some $\lambda(x,t) \in (0,1)$. Further, $w = 0$ at $x = 0$, $x = 1$ and $w = 0$ at $t = 0$. Hence the maximum principle, modified to account for the degenerate coefficient of w_t, implies $w = 0$ in $(0,1) \times (0, t_0)$. See Friedman (1964). ◃

Thus far it has been proved that a unique classical solution exists in some time interval $(0, t_0)$. To complete the existence theory we prove a continuation theorem.

Theorem 2.5. *Let T be the supremum over t_0 for which there is a solution in $(0, t_0)$. Then there is a solution u in $(0, T)$. If $T < \infty$ then u is unbounded in $\Omega \times (0, T)$.*

Proof. Given any $(x_1, t_1) \in \Omega \times (0, T)$ we can choose a solution $v_{t_0}(x,t)$ in $\Omega \times (0, t_0)$ with $0 < t_1 < t_0 < T$. Then set $u(x_1, t_1) = v_{t_0}(x_1, t_1)$. This defines $u : \Omega \times (0, T) \to \mathbb{R}$. It is well defined by the uniqueness of v_{t_0} and u is a solution in $\Omega \times (0, T)$.

Now suppose $T < \infty$ and u is bounded in $\Omega \times (0, T)$. We need to show that u can be continued into a larger time interval $(0, T + t_0)$. This is done by extending the *a priori* bound h first. The only potential problem is that $u_x(0,t)$ may grow large as $t \to T$. We must show u is well behaved at $t = T$.

Suppose $u \leq M$ in $\Omega \times (0,T)$ and define $\psi(x) = Kx(1-x)$ where

$$K = \max\{M^p/2, \sup_{x\in\Omega}(u_0(x)/x(1-x))\}.$$

Then $\psi(x) \geq u_0(x)$ and $\psi = 0$ at $x = 0$, $x = 1$. Moreover

$$x\psi_t - \psi_{xx} - \{xu_t - u_{xx}\} = K/2 - u^p \geq K/2 - M^p \geq 0.$$

Therefore ψ is an upper solution of u in $\Omega \times (0,T)$. In particular u is bounded by $\psi(x)$ at $t = T$ (and so u is indeed well behaved at $(0,T)$).

Now as in Lemma 2.2 we can construct an upper solution $h(x,t)$ in $\Omega \times (T, T+t_0)$ of (1.1)–(1.2) with initial data $\psi(x)$ at $t = T$.

Finally set

$$H(x,t) = \begin{cases} \psi(x) & \text{for } 0 < t \leq T, \\ h(x,t) & \text{for } T \leq t \leq T + t_0. \end{cases}$$

Since H is an *a priori* upper bound for any u_ϵ in $\Omega \times (0, T+t_0)$, u_ϵ can be continued into $\Omega \times (0, T+t_0)$. By repeating theorem 2.3 but replacing h by H and t_0 by $T + t_0$ we find

$$u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t)$$

is a solution of (1.1)–(1.3) in $\Omega \times (0, T+t_0)$. This contradicts the definition of T. $\triangleleft$

To end this section an extra result is presented which will be required in Section 3.

Lemma 2.6. *The t derivative u_t of the solution u is continuous at $x = 0$ and $x = 1$.*

Proof. Since $(u_\epsilon)_t \to u_t$ as $\epsilon \to 0$ it is only necessary to find a uniform upper bound $h(x,t)$ for $(u_\epsilon)_t$ in $\Omega \times (0, t_0)$, any $t_0 < T$. The construction of h follows similar lines to Lemma 2.2.

Choose $t_0 \in (0,T)$. Then $u \leq M$ in $\Omega \times (0,t_0)$. Define h as follows:

(i) Let $\psi = \frac{1}{2}x(1-x)$.
(ii) Choose k_0 s.t. $k_0 x\psi(x) \geq u_0''(x) + u_0^p(x)$.
(iii) Choose $\delta \leq 1/pM^{p-1}$.
(iv) Let $k(t) = k_0 \exp(pM^{p-1}t/\delta)$.
(v) Set $h(x,t) = k(t)\psi(x)$.

Now we show h is an upper solution for $(u_\epsilon)_t$ in $\Omega \times (0,t_0)$. Differentiating (1.1) w.r.t. t and substituting h for u gives

$$J = xh_t - h_{xx} - pu^{p-1}h = xk'\psi + k - pu^{p-1}k\psi.$$

For $x \in (0, \delta)$, $t \in (0, t_0)$,

$$\begin{aligned} J &\geq k - pu^{p-1}k\psi \\ &\geq k(1 - pM^{p-1}\delta(1-\delta)/2) \\ &\geq k(1 - pM^{p-1}\delta) \\ &\geq 0. \end{aligned}$$

For $x \in (\delta, 1-\delta)$, $t \in (0, t_0)$,

$$\begin{aligned} J &\geq xk'\psi - pM^{p-1}k\psi \\ &\geq \psi(\delta k' - pM^{p-1}k) \\ &= 0. \end{aligned}$$

Thus h is an upper solution for u_t and also for $(u_\epsilon)_t$, any ϵ, in $\Omega \times (0, t_0)$. Letting $u_\epsilon \to u$ and using a similar procedure as in Theorem 2.3 it is shown that u_t is bounded and continuous at the boundary of Ω. $\triangleleft$

§3. The main theorem

The main object of this section is to ascertain the blow-up set. Before doing this (in Theorem 3.5) we show that u tends to infinity in finite time provided that u_0 is large enough. This is done by modifying the eigenfunction method due to Kaplan (1963).

Consider the linear boundary value problem

$$\begin{aligned} \phi'' &= -\lambda x\phi \qquad \text{in } \Omega, \\ \phi &= 0 \qquad \text{in } \partial\Omega. \end{aligned} \tag{3.1}$$

There is a unique solution

$$\phi(x) = kx^{1/2}J_{1/3}(\mu_1 x^{3/2}), \qquad \lambda = 9\mu_1^2/4 \tag{3.2}$$

for which $\phi(x) > 0$ and $\int_0^1 x\phi(x)dx = 1$. The constant k is $1/\int_0^1 x^{3/2}J_{1/3}(\mu_1 x^{3/2})dx$ and μ_1 is the first positive zero of $J_{1/3}$, the Bessel function of the first kind of order 1/3, see Abramowitz & Stegun (1965). That ϕ solves (3.1) is evident from the Bessel equation

$$x^2 J''_{1/3} + xJ'_{1/3} + (x^2 - 1/9)J_{1/3} = 0.$$

With ϕ and λ as defined in (3.2) a condition for blow-up is as follows.

Proposition 3.1. *Suppose u solves (1.1)–(1.3) and*

$$\int_0^1 x\phi(x)u_0(x)dx > \lambda^{1/(p-1)}.$$

Then u blows up in finite time.

Proof. Set

$$U(t) = \int_0^1 x\phi(x)u(x,t)dx.$$

Then multiplying (1.1) by ϕ and integrating over x leads to

$$\int_0^1 \phi x u_t = \int_0^1 \phi u_{xx} + \int_0^1 \phi u^p.$$

Integration by parts and Jensen's inequality imply

$$\begin{aligned}\left(\int_0^1 x\phi u\right)_t &= -\lambda \int_0^1 x\phi u + \int_0^1 \phi u^p \\ &\geq -\lambda \int_0^1 x\phi u + \int_0^1 x\phi u^p \\ &\geq -\lambda \int_0^1 x\phi u + \left(\int_0^1 x\phi u\right)^p.\end{aligned}$$

Therefore U is a solution of the initial value problem

$$U' \geq U(-\lambda + U^{p-1}), \qquad U(0) = \int_0^1 x\phi(x)u_0(x)dx.$$

By hypothesis, $U(0) > \lambda^{1/(p-1)}$ and therefore U tends to infinity in finite time. Hence u ceases to exist at some finite time T. By Theorem 2.5 there is a sequence (t_n) with $t_n \to T$ and $\sup_{x\in\Omega} u(x,t_n) \to \infty$ as $n \to \infty$. $\triangleleft$

The main purpose of this paper is to determine the blow-up set S. This is not immediately apparent from equation (1.1). It turns out that a change of variables can be made transforming (1.1) into a more recognisable form. The transformation is really a composition of two distinct transformations. The first, $u \mapsto u/x$ has the effect of turning the Dirichlet boundary condition into a Neumann boundary condition. The second, $x \mapsto (2/3)x^{3/2}$ is used to transform the operator $xu_t - u_{xx}$ into the well known $v_t - v_{yy}$ although terms in v_y are also introduced. This latter substitution is used in the analysis of Airy's equation $w'' = xw$.

Lemma 3.2. *Suppose u is the solution of (1.1)–(1.3) in $\Omega \times (0,T)$. Define $v(y,t)$ by*

$$u(x,t) = xv(y,t), \qquad y = (2/3)x^{3/2}. \tag{3.3}$$

Then v satisfies

$$v_t = v_{yy} + \frac{5}{3}\frac{v_y}{y} + ay^{(2/3)(p-2)}v^p \tag{3.4}$$

$$v_y(0,t) = v(2/3,t) = 0 \tag{3.5}$$

$$v(y,0) = u_0(x)/x \tag{3.6}$$

where $a = (3/2)^{(2/3)(p-2)}$. Also v is $C^{2,1}$ in $(0,2/3)\times(0,T)$, continuous in $[0,2/3]\times[0,T)$ and v_y is continuous at $y=0$.

Proof. The equation for v (3.4) follows from direct calculation. The only remaining property of v which is not obvious is $v_y(0,t) = 0$. By Lemma 2.6, we know u_t is continuous at $x = 0$ and so from (1.1), u_{xx} is continuous at $x = 0$ as well. Thus we can write

$$u(x,t) = u_x(0,t)x + \int_0^x \int_0^s u_{xx}(r,t)drds.$$

Then

$$v(y,t) = u(x,t)/x = u_x(0,t) + \frac{1}{x}\int_0^x \int_0^s u_{xx}(r,t)drds$$

and

$$\begin{aligned}\frac{\partial}{\partial x} v(y,t) = & -\frac{1}{x^2}\int_0^x \int_0^s u_{xx}(r,t)drds + \frac{1}{x}\int_0^x u_{xx}(s,t)ds \\ = & \frac{1}{x^2}\int_0^x s u_{xx}(s,t)ds.\end{aligned}$$

Therefore from (3.3),

$$v_y = x^{-1/2} v_x = x^{-5/2}\int_0^x s u_{xx}(s,t)ds.$$

Now from (1.1), $u_{xx}(0,t) = 0$. Moreover since $u^p \sim u_x^p(0,t)x^p$ as $x \to 0$ and $xu_t = o(x)$ as $x \to 0$ we have $u_{xx} = O(x^p)$ for $1 < p \leq 2$ and $u_{xx} = O(x^2)$ for $p > 2$. Therefore when $1 < p \leq 2$,

$$|v_y| \leq x^{-5/2}\int_0^x s(Ks^p)ds = \frac{Kx^{p-1/2}}{p+2} = Cy^{2(p-1/2)/3} \to 0 \qquad \text{as} \qquad y \to 0.$$

When $p > 2$,

$$|v_y| \leq x^{-5/2}\int_0^x s(Ks^2)ds = \frac{Kx^{3/2}}{4} = Cy \to 0 \qquad \text{as} \qquad y \to 0.$$

This ends the proof. ◁

Remark. Note that in the case $p = 2$, equation (3.4) simplifies to

$$v_t = v_{yy} + \frac{5}{3}\frac{v_y}{y} + v^2. \tag{3.7}$$

This is precisely

$$v_t = \Delta v + v^2 \tag{3.8}$$

where y is the radial coordinate except that the dimension is apparently 8/3 ! Even though the Laplacian operator is meaningless in 8/3 dimensions we would expect (3.7) and (3.8) to have similar properties.

The main reason for considering v is that there is a maximum principle for v_y if $p \leq 2$. This is because the coefficient $y^{(2/3)(p-2)}$ in equation (3.4) is *decreasing* in y for $p \leq 2$. For $p > 2$ the opposite is true.

Lemma 3.3. *Suppose* $1 < p \leq 2$ *and* $v_y(y,0) \leq 0$ *for* $y \in (0,2/3)$. *Then* $v_y < 0$ *for all* $y \in (0,2/3)$, $t \in (0,T)$.

Proof. Let $w = v_y$ and differentiate (3.4) with respect to y to obtain

$$w_t = w_{yy} + \frac{5}{3}\frac{w_y}{y} - \frac{5}{3}\frac{w}{y^2} + a\{\frac{2}{3}(p-2)y^{(2/3)(p-7/2)}v^p + py^{(2/3)(p-2)}v^{p-1}w\} \tag{3.9}$$

and since $1 < p \le 2$ we have

$$w_t - w_{yy} - \frac{5}{3}\frac{w_y}{y} \le bw$$

where b is bounded above in $(0, 2/3) \times (0, t_0)$, any $t_0 < T$. At the boundaries, $w = v_y = 0$ at $t = 0$ and since $v \ge 0$, $v_y < 0$ at $y = 2/3$. Finally, since $w \le 0$ at $t = 0$, the maximum principle shows $w < 0$ in $(0, 2/3) \times (0, t_0)$ and hence $w < 0$ in $(0, 2/3) \times (0, T)$ as required. ◁

Now suppose u blows up at $t = T$. Then $v = u/x \ge x$ blows up at T also. A consequence of Lemma 3.3 is that $y = 0$ is a blow-up point of v. The essence of proving Theorem 3.5 is showing that $y = 0$ is the *only* blow-up point of u. First we need a lemma. The idea is due to Mueller & Weissler (1985).

Lemma 3.4. *Suppose* $1 \le p < 2$ *and* $v_y(y, 0) \le 0$ *for* $y \in (0, 2/3)$. *Suppose* $y_0 \in (0, 2/3)$ *is a blow-up point of* v. *Then if* $0 < y_1 < y_0$, $v(y_1, t) \to \infty$ *as* $t \to T$.

Proof. Choose $y_1 \in (0, y_0)$ and let $y_2 = (y_1 + y_0)/2$. Since $v_y < 0$, it is clear that $\limsup_{t\to T} v(y_2, t) = \infty$ (note that it is not necessarily true that $\limsup_{t\to T} v(y_0, t) = \infty$). We show that $\lim_{t\to T} v(y_1, t) = \infty$. Observe that

$$v_t - v_{yy} = \frac{5}{3}\frac{v_y}{y} + ay^{(2/3)(p-2)}v^p \ge 2\frac{v_y}{y}$$

since $v_y < 0$. Define Ω' to be the ball in $\mathbb{R}^3$ of radius $2/3$, centre 0. Letting $y = |x|$ and choosing $\tau \in (0, T)$ we find $v(y, t) \ge w(x, t)$ where $w_t = \Delta w$ in Ω', $w \mid \partial\Omega' = 0$ and $w(|x|, \tau) = v(y, \tau)$. Denoting the Green's function for w by G, we have

$$w(x, t) = \int_{\Omega'} G(x, z, t - \tau) w(z, \tau) dz$$

and

$$v(y, t) \ge \int_{\Omega'} G(x, z, t - \tau) v(|z|, \tau) dz$$

for any x s.t. $y = |x|$. Now choose a sequence $(\tau_n) \subset (0, T)$ such that $\tau_n \to T$ and $v(y_2, \tau_n) \to \infty$ as $n \to \infty$. For any $t \in (\tau_n, T)$ we have

$$v(y_1, t) \ge \int_{\Omega'} G(x_1, z, t - \tau_n) v(|z|, \tau_n) dz$$

for any x_1 s.t. $y_1 = |x_1|$. Further, since $v_y < 0$ and $G \ge 0$,

$$v(y_1, t) \ge v(y_2, \tau_n) \int_{|x| \le y_2} G(x_1, z, t - \tau_n) dz.$$

But due to the asymptotics of G,

$$\int_{|x| \le y_2} G(x_1, z, t - \tau_n) \ge \frac{1}{2}$$

once $t - \tau_n$ is small enough. Hence letting $n \to \infty$ shows $v(y_1, t) \to \infty$ as $t \to T$. ◁

Now comes the main result. The method is based on that of Friedman & McLeod (1985) with refinements. Recall that S represents the set of blow-up points of u.

Theorem 3.5. *Suppose $1 < p \le 2$ and that the solution u of (1.1)–(1.3) blows up at $t = T$. If $\frac{\partial}{\partial x}(u_0(x)/x) \le 0$ for $x \in (0,1)$ then $S = \{0\}$.*

Proof. Since $v = u/x \ge u$, v blows up as $t \to T$. By Lemma 3.3, $y = 0$ is a blow-up point of v. Clearly if $y = 0$ is the *only* blow-up point of v then $x = 0$ is the only blow-up point of u.

Suppose $y_0 \in (0, 2/3)$ is a blow-up point of v. Choose y_1 s.t. $0 < y_1 < y_0$. Then, by Lemma 3.4, $v(y_1, t) \to \infty$ as $t \to T$. We show that this is not, in fact, the case.

Choose $\lambda \in (1, p)$ and define $J = v_y + \epsilon c(y) v^\lambda$ where $\epsilon > 0$ is to be determined. The function c is defined as

$$c(y) = \begin{cases} y & \text{if } 0 \le y \le y_0/3 \\ h(y) & \text{if } y_0/3 \le y \le 2y_0/3 \\ (y_0 - y)^3 & \text{if } 2y_0/3 \le y \le y_0 \end{cases}$$

where $h(y)$ is any positive function which completes the piecewise definition of c in such a way that $c \in C^2(0, y_0)$. It is important only that c is positive and has the appropriate behaviour near 0 and y_0.

Our task is to show $J \le 0$ in $(0, y_0) \times (t_0, T)$ for some $t_0 > 0$ via the maximum principle. From (3.4),

$$\begin{aligned} J_t - J_{yy} - \frac{5}{3}\frac{J_y}{y} =& -\frac{5}{3}\frac{v_y}{y^2} + \frac{2}{3}(p-2) a y^{(2/3)(p-7/2)} + a y^{(2/3)(p-2)} p v^{p-1} v_y \\ &+ \epsilon c \lambda v^{\lambda-1} v_t - \epsilon\{c'' v^\lambda + 2c'\lambda v^{\lambda-1} v_y + c\lambda(\lambda-1) v^{\lambda-2} v_y^2 + c\lambda v^{\lambda-1} v_{yy}\} \\ &- \frac{5}{3y}\{\epsilon c' v^\lambda + \epsilon c \lambda v^{\lambda-1} v_y\} \end{aligned}$$

now noting $p \le 2$,

$$\begin{aligned} \le& bJ - \epsilon c\{a y^{(2/3)(p-2)}(p-\lambda) v^{p+\lambda-1} \\ &+ \left(c'' + \frac{5}{3y}c' - \frac{5}{3y^2}c\right)\frac{v^\lambda}{c} - 2\epsilon c' \lambda v^{2\lambda-1}\}. \end{aligned}$$

Thus to show $J_t - J_{yy} - \frac{5}{3}\frac{J_y}{y} - bJ \le 0$ it is sufficient to show

$$a(p-\lambda) v^{p+\lambda-1}/y_1^{(2/3)(2-p)} \ge 2\epsilon\lambda c' v^{2\lambda-1} + \left(\frac{5}{3y^2} - \frac{5}{3y}\frac{c'}{c} - \frac{c''}{c}\right) v^\lambda. \tag{3.10}$$

Assuming $v \ge 1$, setting $\epsilon \le a(p-\lambda)/(4\lambda y_1^{(2/3)(2-p)} \max_{y\in(0,y_1)}\{c'(y)\})$ ensures

$$a(p-\lambda) v^{p+\lambda-1}/y_1^{(2/3)(2-p)} - 2\epsilon\lambda c' v^{2\lambda-1} \ge a(p-\lambda) v^{2\lambda-1}/2y_1^{(2/3)(2-p)}.$$

Further, by construction

$$\frac{5}{3y^2} - \frac{5}{3y}\frac{c''}{c} - \frac{c''}{c} \le M \qquad \text{in } (0, y_0)$$

for some bound $M > 0$. Therefore, since $v(y_1, t) \to \infty$ as $t \to T$ we can choose t_0 close enough to T so that (3.10) holds for $y = y_1$, $t \in (t_0, T)$. Since $v_y < 0$, (3.10) holds for any $y \in (0, y_1)$, $t \in (t_0, T)$.

Finally, to apply the maximum principle to J observe that $J \leq 0$ at $y = 0$ and $y = y_1$ since $c(0) = c(y_1) = 0$. To ensure $J \leq 0$ at $t = t_0$, $y \in (0, y_1)$ note that as $y \to 0$, $v_y = O(y^{(2/3)(p-1/2)})$, see Lemma 3.2. So for small enough y, $|v_y| \geq Ky$. Since $c(y)$ and $v(y,t)$ are bounded at $t = t_0$ it follows that $J \leq 0$ provided ϵ is chosen small enough. Thus by the maximum principle

$$v_y + \epsilon c(y) v^\lambda \leq 0 \qquad \text{in } (0, y_1) \times (t_0, T) \tag{3.11}$$

as required.

Now dividing (3.11) by v^λ and integrating w.r.t. y over $(0, y_1)$ gives

$$\frac{1}{v^{\lambda-1}(y_1,t)} - \frac{1}{v^{\lambda-1}(0,t)} \geq \epsilon(\lambda - 1) \int_0^{y_1} c(z)dz = \delta.$$

Hence $v(y_1, t) \leq \delta^{1/(\lambda-1)}$ which is a contradiction. $\triangleleft$

Blow-up at the boundary has been proved when $1 < p \leq 2$ and $(d/dx)(u_0(x)/x) \leq 0$. Particular examples of initial data u_0 for which $(d/dx)(u_0(x)/x) \leq 0$ are $\lambda \sin \pi x$ and $\lambda x(1-x)$, any $\lambda > 0$. More generally, $\frac{d^2 u_0}{dx^2} \leq 0 \implies \frac{d}{dx}(\frac{u_0}{x}) \leq 0$.

§4. Asymptotics near the singularity

In this section it is assumed that $1 < p \leq 2$, $(d/dx)(u_0(x)/x) \leq 0$ and u blows up at $t = T$.

Firstly, upper and lower integral bounds for u are calculated. These bounds largely determine the shape of u as $t \to T$.

Secondly, bounds on the the maximum of u are calculated. These determine the rate at which the maximum of u tends to 0. Let $s(t)$ be any $x(t) \in (0,1)$ such that $u(x(t), t) = \sup_{x \in (0,1)} u(x,t)$. We will show that

$$C_1(T-t)^{1/3} \leq s(t) \leq C_2(T-t)^{1/(3+\delta)} \qquad \text{for any } \delta > 0.$$

Integral bounds. The approach used to find the bounds is to find the corresponding bounds for v and then transfer them back to u via (3.3). To ease notation let $\alpha = (2/3)(2-p)$ so that $0 \leq \alpha < 2/3$ and let $a = (3/2)^{(2/3)(p-2)}$ as before.

Theorem 4.1. *Let u solve (1.1)–(1.3) with $1 < p \leq 2$ and $(d/dx)(u_0(x)/x) \leq 0$ where T is the blow-up time. Suppose that $u_0 \in C^2[0,1]$ and $u_0'' + u_0^p \geq 0$ in $(0,1)$. Then*

$$\int_0^1 u^\gamma(x,t)dx \to \infty \text{ as } t \to T \text{ provided } \gamma > (p-1)/2.$$

Proof. For this proof it is easier to treat v as a function of x rather than y. Let $u(x,t) = xv(x,t)$. Then

$$xv_t = v_{xx} + 2\frac{v_x}{x} + x^{p-1}v^p \tag{4.1}$$

with $v(x,0) = u_0(x)/x$ and $v(1,t) = 0$. At $x = 0$, the analysis in Lemma 3.2 shows

$$v_x = x^{-2} \int_0^x s u_{xx}(s,t) ds$$

and since $1 < p \le 2$,

$$|v_x| \le x^{-2} \int_0^x s(Ks^p) ds = \frac{Kx^p}{p+2} \to 0 \text{ as } x \to 0.$$

Also $v_x < 0$ in $(0,1) \times (0,T)$ since $v_y < 0$.

Fix $t \in (0,T)$. Multiply equation (4.1) by v_x and integrate w.r.t. x over $(0,x)$ to obtain

$$\int_0^x v_x x v_t = \frac{1}{2} v_x^2 + 2\int_0^x \frac{v_x^2}{x} + \int_0^x x^{p-1} v^p v_x.$$

Now by the maximum principle $u_t > 0$ and so $v_t = u_t/x > 0$. Further, since $v_x < 0$ we find

$$\begin{aligned}\frac{1}{2}v_x^2 \le & -\frac{1}{p+1}\int_0^x x^{p-1}(v^{p+1})_x \\ = & -\frac{1}{p+1}\left[x^{p-1}v^{p+1}\right]_0^x + \frac{p-1}{p+1}\int_0^x x^{p-2}v^{p+1} \le \frac{p-1}{p+1}\bar{v}^{p+1}\int_0^x x^{p-2} \le \frac{\bar{v}^{p+1}x^{p-1}}{p+1}\end{aligned}$$

where $\bar{v}(t)$ denotes $v(0,t)$. Thus

$$-v_x \le C x^{(p-1)/2} \bar{v}^{(p+1)/2}.$$

Integrating over $(0,x)$ gives

$$\bar{v} - v \le \frac{2Cx^{(p+1)/2}\bar{v}^{(p+1)/2}}{p+1}. \tag{4.2}$$

Define $\bar{x}(t)$ to be the unique point in $(0,1)$ at which $v(\bar{x}(t),t) = \bar{v}(t)/2$. Then letting $x = \bar{x}$ in (4.2),

$$\bar{x} \ge K \bar{v}^{-(p-1)/(p+1)} \tag{4.3}$$

for some $K > 0$. Finally,

$$\begin{aligned}\int_0^1 u^\gamma(x,t)dx =& \int_0^1 x^\gamma v^\gamma(x,t)dx \\ \ge& \frac{\bar{v}^\gamma}{2^\gamma}\int_0^{\bar{x}} x^\gamma dx \\ =& \frac{\bar{v}^\gamma \bar{x}^{\gamma+1}}{2^\gamma(\gamma+1)} \\ \ge& \frac{K^{\gamma+1}}{2^\gamma(\gamma+1)}\bar{v}^{\gamma-(\gamma+1)(p-1)/(p+1)}.\end{aligned}$$

Therefore $\int_0^1 u^\gamma(x,t)dx \to \infty$ as $t \to T$ provided

$$\gamma - \frac{(\gamma+1)(p-1)}{p+1} > 0$$

which is equivalent to $\gamma > (p-1)/2$. $\triangleleft$

To obtain the corresponding upper bound an improvement is made on the inequality (3.11) used to show 1-point blow-up.

Lemma 4.2. *Let u solve (1.1)–(1.3) on $(0,1)\times(0,T)$ with $(d/dx)(u_0(x)/x) \le 0$ and $1 < p \le 2$. For any $\lambda \in (1,p)$ there exists $\epsilon > 0$ such that*

$$v_y \le -\epsilon y^{1-\alpha} v^\lambda \tag{4.4}$$

in $(0,1/3)\times(T/2,T)$.

Proof. Recall the equation (3.4) for v. Fix $\lambda \in (1,p)$ and set $J = v_y + \epsilon y^{1-\alpha}v^\lambda$ where ϵ is yet to be determined. Then

$$\begin{aligned} J_t - J_{yy} - \frac{5}{3}\frac{J_y}{y} =& -\frac{5}{3}\frac{v_y}{y^2} + \frac{apv^{p-1}v_y}{y^\alpha} - \frac{\alpha a v^p}{y^{\alpha+1}} + \epsilon y^{1-\alpha}\lambda v^{\lambda-1}v_t \\ & - \epsilon\{-\alpha(1-\alpha)y^{-\alpha-1}v^\lambda + 2(1-\alpha)y^{-\alpha}\lambda v^{\lambda-1}v_y \\ & + y^{1-\alpha}[\lambda(1-\lambda)v^{\lambda-2}v_y^2 + \lambda v^{\lambda-1}v_{yy}]\} \\ & - \frac{5}{3}\frac{\epsilon}{y}\{(1-\alpha)y^{-\alpha}v^\lambda + y^{1-\alpha}\lambda v^{\lambda-1}v_y\} \end{aligned}$$

substituting $v_y = J - \epsilon y^{1-\alpha}v^\lambda$ and using $v_y^2 \ge 0$,

$$\begin{aligned} \le& bJ + \frac{5}{3}\epsilon y^{-\alpha-1}v^\lambda - \epsilon a p y^{1-2\alpha}v^{p+\lambda-1} - \alpha a y^{-\alpha-1}v^p \\ & + \epsilon a\lambda y^{1-2\alpha}v^{p+\lambda-1} + \epsilon\alpha(1-\alpha)y^{-\alpha-1}v^\lambda \\ & + 2\epsilon^2(1-\alpha)\lambda y^{1-2\alpha}v^{2\lambda-1} - \frac{5}{3}(1-\alpha)\epsilon y^{-\alpha-1}v^\lambda \end{aligned}$$

where b is bounded above in $(0,1/3)\times(T/2,t_0)$, $t_0 < T$,

$$\begin{aligned} =& bJ - \epsilon y^{1-2\alpha}\{a(p-\lambda)v^{p+\lambda-1} - 2\epsilon(1-\alpha)\lambda v^{2\lambda-1}\} \\ & - y^{-\alpha-1}\left\{\alpha a v^p - \epsilon\left(\frac{5}{3}\alpha + \alpha(1-\alpha)\right)v^\lambda\right\} \\ =& bJ - \epsilon y^{1-2\alpha}v^{2\lambda-1}\{a(p-\lambda)v^{p-\lambda} - 2\epsilon(1-\alpha)\lambda\} \\ & - \epsilon\alpha y^{-\alpha-1}v^\lambda\left\{av^{p-\lambda} - \epsilon\left(\frac{8}{3} - \alpha\right)\right\} \\ \le& 0 \end{aligned}$$

as long as ϵ is small enough. Note $\inf_{(0,1/3)\times(T/2,T)} v > 0$.

To apply the maximum principle to J observe $J = 0$ at $y = 0$. Further, it is known that v is bounded at $y = 1/3$ for all $t \in (T/2,T)$ by Theorem 3.5. Also v_y is bounded above by a negative constant for all $t \in (T/2, T)$. To understand this let $\bar{v}$ be the solution (for all $t > 0$) of

$$\bar{v}_t = \bar{v}_{yy} + \frac{5}{3}\frac{\bar{v}_y}{y} \tag{4.5}$$

with conditions (3.5) and (3.6). A comparison of equations (4.5) and (3.4) shows that $\bar{v}$ is a lower solution for v. Then differentiating (4.5) w.r.t. y and comparing with (3.9) shows that $\bar{v}_y$ is an upper solution for v_y. Thus $v_y \le -\delta$ in $\{1/3\} \times (T/2, T)$. So we can ensure $J \le 0$ at $y = 1/3$ by choosing ϵ even smaller if necessary. Finally, to ensure $J \le 0$ in $(0, 1/3) \times \{T/2\}$ recall $v_y = O(y^{(2/3)(p-1/2)}) = O(y^{1-\alpha})$ as $y \to 0$, see Theorem 3.5. So letting ϵ be small enough implies $J \le 0$ in $(0, 1/3) \times \{T/2\}$.

Hence the maximum principle applies to show $J \le 0$ in $(0, 1/3) \times (T/2, T)$ as required. ◁

By using (4.4) an upper bound for u can be computed.

Theorem 4.3. *Let u solve (1.1)–(1.3) on $(0,1) \times (0,T)$ where $(d/dx)(u_0(x)/x) \le 0$ and $1 < p \le 2$. Then, provided $0 < \gamma < (p-1)/2$, there exists a bound C_γ such that*

$$\int_0^1 u^\gamma(x,t)dx \le C_\gamma \qquad \text{for all } t \in (0,T).$$

Proof. Let $\lambda \in (1, p)$. Then by Lemma 4.2 there is $\epsilon > 0$ with

$$v_y \le -\epsilon y^{1-\alpha} v^\lambda$$

in $(0, 1/3) \times (T/2, T)$. Now dividing by v^λ and integrating from 0 to y yields

$$v \le K_\lambda y^{-(2-\alpha)/(\lambda-1)}.$$

To transfer back to u, let $v = u/x$, $y = (2/3)x^{3/2}$ and remembering $\alpha = (2/3)(2-p)$ we find

$$u \le C_\lambda x^{-(2+p-\lambda)/(\lambda-1)}.$$

Therefore

$$\int_0^1 u^\gamma(x,t)dx \le C_\lambda^\gamma x^{-\gamma(2+p-\lambda)/(\lambda-1)}dx \le C_\gamma$$

provided

$$\gamma < \frac{\lambda - 1}{2 + p - \lambda}. \tag{4.6}$$

But (4.6) holds for any $\gamma < (p-1)/2$ if λ is chosen close enough to p. ◁

Bounds on $s(t)$. It appears difficult to show that at any given t, the maximum of u over x in $(0,1)$ is attained at a unique point (there is no maximum principle for u_x). Despite this, we can obtain the following bounds. Let $s_1(t) \in (0,1)$ be the infimum over $s \in (0,1)$ for which $u(s,t) = \sup_{x\in\Omega} u(x,t)$. Similarly, let $s_2(t)$ be the supremum over all such s. It will be shown that

$$C_1(T-t)^{1/3} \le s_1(t) \le s_2(t) \le C_2(T-t)^{1/(3+\delta)} \qquad \text{for any } \delta > 0.$$

Lemma 4.4. *Suppose $u_0'' + u_0^p \geq 0$. There exists $\epsilon > 0$ such that $x^{p-1}u_t \geq \epsilon u^p$ for any x in $(0,1)$, $t \in (t_0, T)$.*

Proof. There are difficulties in applying the maximum principle to both $u(x,t)$ and $v(y,t)$. Instead we work with $v(x,t) = u(x,t)/x$.

Now set $J = v_t - \epsilon v^p$. Then from equation (4.1),

$$xJ_t - J_{xx} - 2\frac{J_x}{x} - x^{p-1}pv^{p-1}J = \epsilon p(p-1)v^{p-2}v_x^2 \geq 0.$$

As $v_x(0,t) = 0$, so $J_x(0,t) = 0$. Also $J(1,t) = 0$ and $J(x,t_0) \geq 0$ if ϵ is chosen small enough. Hence, by the maximum principle, $J \geq 0$ and consequently $x^{p-1}u_t \geq \epsilon u^p$ as required. ◁

Theorem 4.5. *Suppose $1 < p \leq 2$, $(d/dx)(u_0(x)/x) \leq 0$ and u blows up at $t = T$. Under the additional assumption that $u_0'' + u_0^p \geq 0$, there exists $C_1 > 0$ such that*

$$s_1(t) \geq C_1(T-t)^{1/3} \qquad \text{for all } t \in (0,T).$$

Proof. Fix $t \in (0,T)$. Let $\bar{x}(t)$ be the least member of $(0,1)$ for which $u_x(\bar{x}(t),t) = 0$. Clearly $\bar{x}(t) \leq s_1(t)$. Now multiply (1.1) by u_x and integrate over $(x,\bar{x})$ to obtain

$$0 \leq \int_x^{\bar{x}} xu_tu_x = \int_x^{\bar{x}} (u_x^2)_x/2 + \int_x^{\bar{x}} u^p = -\frac{u_x^2}{2} + \frac{u^{p+1}(\bar{x},t)}{p+1} - \frac{u(x,t)}{p+1}.$$

Therefore

$$u_x \leq Ku^{(p+1)/2}(\bar{x},t) \qquad \text{for } x \in (0,\bar{x}).$$

Integrating over $(0,\bar{x})$ shows

$$\bar{x} \geq Cu^{-(p-1)/2}(\bar{x},t). \tag{4.7}$$

Now from Lemma 4.4, $x^{p-1}u_t \geq \epsilon u^p$. Integrating w.r.t. t over (t,T) implies

$$u \leq \frac{Dx^{2/(p-1)}}{(T-t)^{1/(p-1)}}. \tag{4.8}$$

Now combining (4.7) and (4.8) yields

$$\bar{x}(t) \geq C_1(T-t)^{1/3}$$

which completes the proof. ◁

To obtain an upper bound for $s_2(t)$ another lemma is needed.

Lemma 4.6. *Let $\bar{u}(t) = u(s_2(t),t)$. Then $\bar{u}$ is Lipschitz continuous a.e. and satisfies*

$$s_2\bar{u}' \leq \bar{u}^p.$$

Proof. It is straightforward to adapt the proof by Friedman & McLeod (1985) regarding $u_t = u_{xx} + u^p$ to the equation (1.1). ◁

Theorem 4.7. *Let u be the solution of (1.1)-(1.3) where $1 < p \leq 2$, $(d/dx)(u_0(x)/x) \leq 0$ and suppose u blows up at $t = T$. Then given $\delta > 0$ there exists $C_2 > 0$ such that*

$$s_2(t) \leq C_2(T-t)^{1/(3+\delta)}.$$

Proof. Let $\bar{u}(t) = u(s_2(t), t)$. Then from Lemma 4.6 and Theorem 4.5,

$$\frac{\bar{u}'}{\bar{u}^p} \leq \frac{1}{s_2} \leq \frac{1}{s_1} \leq \frac{1}{C_1(T-t)^{1/3}}.$$

Integrating over (t, T) implies

$$\bar{u}^{-(p-1)} \leq K_1(T-t)^{2/3}. \tag{4.9}$$

Now from Lemma 4.2, for any $\lambda \in (1, p)$,

$$v_y \leq \epsilon y^{1-\alpha} v^{\lambda}$$

for some $\epsilon > 0$. Substituting $u = xv$, $y = (2/3)x^{3/2}$, $\alpha = (2/3)(2-p)$ gives

$$xu_x \leq u - \epsilon \left(\frac{2}{3}\right)^{(2/3)(p-1/2)} x^{p-\lambda+2} u^{\lambda}.$$

Now at $x = s_2$, $u_x = 0$ and therefore

$$\bar{u} \leq K_2 s_2^{-(p-\lambda+2)/(\lambda-1)}. \tag{4.10}$$

Eliminating $\bar{u}$ from (4.9) and (4.10) yields

$$s_2(t) \leq C_\lambda(T-t)^{(2/3)/\{(p-1)(p-\lambda+2)\}}.$$

Finally we can arrange that

$$\frac{2(\lambda-1)}{3(p-1)(p-\lambda+2)} = \frac{1}{3+\delta}$$

by an appropriate choice of λ. $\lhd$

§5. Generalisations

To complete this paper the previous analysis is extended (with very little effort) to equation (1.4). Thus the new problem is

$$x^q u_t = u_{xx} + u^p \qquad \text{in } \Omega \times (0,\infty) \tag{5.1}$$

$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{5.2}$$

$$u(x,0) = u_0(x) \geq 0 \qquad x \in \Omega \tag{5.3}$$

where $p > 1$ and $q > 0$.

It is not necessary to go through much detail since the proofs are almost identical. Existence and uniqueness of solution can be demonstrated in exactly the same way as before.

To obtain asymptotic results concerning u the key transformation is now

$$u(x,t) = xv(y,t), \qquad y = \frac{2}{q+2}x^{(q+2)/2}.$$

Then

$$\begin{aligned} v_t =& v_{yy} + \frac{q+4}{q+2}\frac{v_y}{y} + ay^{2(p-q-1)/(q+2)}v^p \\ v(y,0) =& u_0(x)/x \end{aligned}$$

with $a = \{(q+2)/2\}^{2(p-q-1)/(q+2)}$. Note the "dimension" is now $2(q+3)/(q+2)$ which is strictly between 2 and 3 for all $q > 0$, c.f. the case $q = 1$ when the dimension is $8/3$.

The behaviour of v near $y = 0$ maybe different depending on p and q. In all cases $v_y = 0$, $v_y \to -D$ as $y \to 0$, or $v_y \to -\infty$ as $y \to 0$. Following Lemma 3.2,

$$v_y = x^{-q/2}v_x = x^{-q/2-2}\int_0^x su_{xx}(s,t)ds.$$

Now the behaviour of u_{xx} is found from (5.1) where $x^q u_t = O(x^{q+1})$ and $u^p = O(x^p)$ as $x \to 0$. Therefore when $p \leq q+1$,

$$|v_y| \leq x^{-q/2-2}\int_0^x s(Ks^p)ds = \frac{Kx^{p-q/2}}{p+2} = Cy^{2(p-q/2)/(q+2)}.$$

Thus $v_y \to 0$ if $1 < p \leq 2$, $v_y \to -\infty$ if $p < q/2$, and $v_y \to -D$ if $p = q/2$ (v_y is negative near $y = 0$ since u_{xx} is also).

When $p > q+1$, $|u_{xx}| \leq Kx^{q+1}$ so that

$$|v_y| \leq x^{-q/2-2}\int_0^x s(Ks^{q+1})ds = \frac{Kx^{q/2+1}}{q+3} = Cy \to 0 \qquad \text{as} \qquad y \to 0.$$

Theorem 5.1. *Suppose $1 < p \leq q+1$ and that the solution of (5.1)–(5.3) blows up at $t = T$. If $\frac{\partial}{\partial x}(u_0(x)/x) \leq 0$ for $x \in (0,1)$ then $S = \{0\}$.*

Proof. We show v blows up only at $y = 0$. By the maximum principle, $v_y < 0$, see Lemma 3.3. Now suppose $y_0 \in (0, 2/(q+2))$ is a blow-up point of v. By an argument similar to Lemma 3.4, we can choose any $y_1 \in (0, y_0)$ such that $v(y_1,t) \to \infty$ as $t \to T$.

Choose $\lambda \in (1, p)$ and define $J = v_y + \epsilon c(y) v^\lambda$. As in Theorem 3.5, the maximum principle can be applied to J to deduce

$$v_y + \epsilon c(y) v^\lambda \leq 0 \qquad \text{in } (0, y_1) \times (t_0, T).$$

This leads to a contradiction, see Theorem 3.5. $\triangleleft$

Numerical evidence given by Stuart & Floater (1989) indicates that when $p > q + 1$, blow-up occurs away from the boundary.

Integral bounds. The asymptotic bounds in Section 4 generalise in a straightforward manner. Letting $\alpha = 2(q + 1 - p)/(q + 2)$, Theorem 4.1 now becomes Theorem 5.2.

Theorem 5.2. *Let u solve (5.1)–(5.3) with $1 < p \leq q+1$ and $(d/dx)(u_0(x)/x) \leq 0$ where T is the blow-up time. Suppose that $u_0 \in C^2[0,1]$ and $u_0'' + u_0^p \geq 0$ in $(0,1)$. Then*

$$\int_0^1 u^\gamma(x,t)dx \to \infty \text{ as } t \to T \text{ provided } \gamma > (p-1)/2.$$

By obtaining the bound $v_y \leq -\epsilon y^{1-\alpha} v^\lambda$ we can extend Theorem 4.3.

Theorem 5.3. *Let u solve (5.1)–(5.3) on $(0,1) \times (0,T)$ where $(d/dx)(u_0(x)/x) \leq 0$ and $1 < p \leq q + 1$. Then provided $0 < \gamma < (p-1)/2$, there exists a bound C_γ such that*

$$\int_0^1 u^\gamma(x,t)dx \leq C_\gamma \text{ for all } t \in (0,T).$$

Bounds on $s(t)$. Let $s_1(t)$ (respectively $s_2(t)$) be the infimum (respectively supremum) of $s \in (0,1)$ for which $u(s,t) = \sup_{x\in(0,1)} u(x,t)$. Lemma 4.4 holds as before when x is replaced by x^q. The same transformation $v(x,t) = u(x,t)/x$ is required. Note that $v_x(0,t) = 0$ for any $p > 1$, $q > 0$ despite $v_y(0,t)$ being infinite if $p > q/2$. Using the inequality $x^{p-1}u_t \geq \epsilon u^p$ leads to Theorem 5.4.

Theorem 5.4. *Suppose $1 < p \leq q + 1$, $(d/dx)(u_0(x)/x) \leq 0$, and T is the blow-up time for u. Under the added assumption that $u_0'' + u_0^p \geq 0$, there exists $C_1 > 0$ such that*

$$s_1(t) \geq C_1(T-t)^{1/(q+2)} \qquad \text{for all } t \in (0,T).$$

A simple extension of Lemma 4.6 is to show $s_2^q \bar{u}' \leq \bar{u}^p$. Then Theorem 4.7 can be generalised.

Theorem 5.5. *Let u be the solution of (5.1)–(5.3) where $1 < p \leq q+1$, $(d/dx)(u_0(x)/x) \leq 0$ and suppose u blows up at $t = T$. Then given $\delta > 0$ there exists $C_2 > 0$ such that*

$$s_2(t) \leq C_2(T-t)^{1/(q+2+\delta)}.$$

As a final remark it is worth noting that the equation

$$u_t = u_{xx} + \frac{u^p}{x^q}$$

has similar properties to (5.1) though there might not be a classical solution for large q and small p (especially $p \leq q-2$).

Letting $u(x,t) = xv(x,t)$ yields

$$v_t = v_{xx} + 2\frac{v_x}{x} + x^{p-q-1}v^p.$$

As for (5.1) the sign of the quantity $p-q-1$ is critical in determining the blow-up set. By showing that $v_x < 0$ it follows that blow-up is at the boundary when $p \leq q+1$ and $\frac{d}{dx}(u_0(x)/x) \leq 0$. Conversely we expect blow-up to be in $(0,1)$ when $p > q+1$.

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§6. References

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