Analysis of Hermite subdivision using piecewise polynomials

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Abstract

In this paper we begin to explore a new method of analyzing the regularity of Hermite subdivision schemes that are defined from local polynomial interpolants. The idea of the method is to view the limit of the scheme as the limit of splines formed by these local interpolants rather than as the limit of polygons. We demonstrate the success of the method by obtaining the precise Hölder regularity of the simple, but non-trivial scheme in which the data are uniformly spaced and the refinement rule is defined by quintic interpolation of four values and two derivatives.

Math Subject Classification: 65D05, 65D17
Keywords: Interpolatory subdivision, Hermite subdivision, piecewise polynomials.

1 Introduction

Subdivision is now established as one of the standard methods of generating curves and surfaces in computer-aided geometric design and computer graphics, and is closely related to multiresolution analysis for signal and image processing [7, 1].

Many subdivision schemes have the advantage of being simple and efficient to implement. However, apart from special cases such as spline subdivision, it is often difficult to establish, mathematically, the smoothness, or regularity of the limit curve or surface. This is usually the case for *interpolatory* subdivision schemes whose limit functions interpolate the initial data. For curves, a well known family of such schemes is based on local Lagrange polynomial interpolation of odd degree, studied by Dubuc and Deslauriers [3, 2, 5]. Rioul [14] derived the precise Hölder regularity of the scheme for each degree up to 19 from the spectral radius of a certain matrix.

If the initial data consists both of values and derivatives up to some order, it is natural to work with so-called *Hermite* subdivision in which one maintains both values and derivatives

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at each level of subdivision. Various Hermite schemes have been studied and analyzed in the literature [13, 4, 11, 10, 6, 15, 12]. However, even for schemes that are defined simply by the evaluation of locally fitted Hermite interpolants, there does not appear to be any known general procedure for determining the regularity.

Recently, in [8], the regularity of the cubic Dubuc-Deslauriers scheme was analyzed by viewing the limit function as the limit of piecewise (cubic) polynomials, or splines, rather than as the limit of polygons, and, at least in cases when the data are irregularly spaced, this turns out to simplify and improve on previous analysis. The purpose of this paper is to report that at least for one specific Hermite scheme, regularity can be determined using a similar approach. We study the scheme whose refinement rule is defined by quintic interpolation of four values and two derivatives. We give a complete regularity analysis for uniformly spaced data, and show that the precise Hölder regularity is $C^{2+\alpha}$, where

$$\alpha = -\log_2(3/4) \approx 0.4150.. \tag{1}$$

2 The scheme

Starting from function values $f_k = f(x_k) \in \mathbb{R}$, $k = -2, -1, \ldots, n+2$, and derivatives $d_k = f'(x_k) \in \mathbb{R}$, $k = -1, 0, \ldots, n+1$, for some $n \geq 1$, where

$$x_{-2} < x_{-1} < \cdots < x_{n+2},$$

we initialize the scheme by setting $x_{0,k} = x_k$, $g_{0,k} = f_k$, and $m_{0,k} = d_k$, and, for each level of subdivision $j \in \mathbb{N}_0$ we generate data by the rules

$$\begin{aligned} g_{j+1,2k} &= g_{j,k}, \\ m_{j+1,2k} &= m_{j,k}, \\ g_{j+1,2k+1} &= p_{j,k-1}(x_{j+1,2k+1}), \\ m_{j+1,2k+1} &= p'_{j,k-1}(x_{j+1,2k+1}), \end{aligned} \tag{2}$$

where $x_{j+1,2k} = x_{j,k}$ and $x_{j+1,2k+1}$ is any point between $x_{j,k}$ and $x_{j,k+1}$ and $p_{j,k-1}$ is the unique quintic polynomial such that

$$\begin{aligned} p_{j,k-1}(x_{j,i}) &= g_{j,i}, \qquad i = k-1, \ldots, k+2, \\ p'_{j,k-1}(x_{j,i}) &= m_{j,i}, \qquad i = k, k+1. \end{aligned}$$

We refer to the grid of points $X = \{x_{j,k}\}$ as *regular* if $x_{j,k} = 2^{-j}k$ and as *semi-regular* if the points $x_{0,k} = x_k$ are arbitrary but $x_{j+1,2k+1} = (x_{j,k} + x_{j,k+1})/2$. An example of the limit function in the regular case is shown in Figure 1, together with its first and second derivatives. A corresponding example for non-uniformly spaced data is shown in Figure 2. We have implemented the scheme using the Newton form of $p_{j,k-1}$ rather than the Hermite form, and then the scheme is easy to apply both for regular and semi-regular grids.

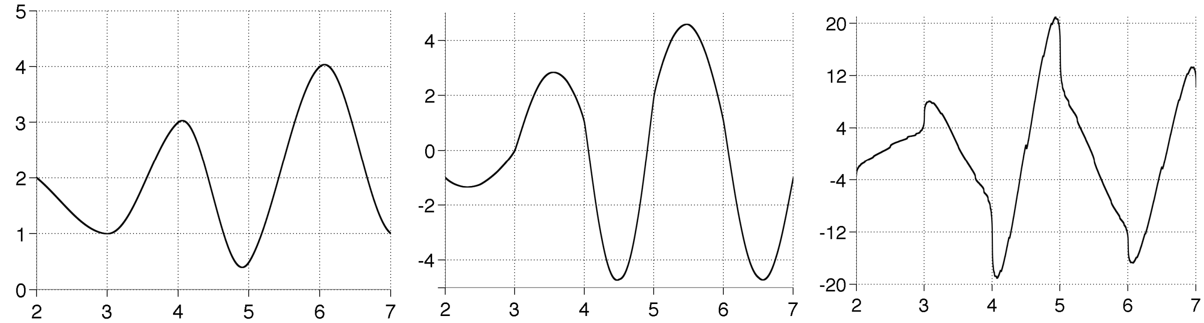

Figure 1: Limit function g with g' and g''.

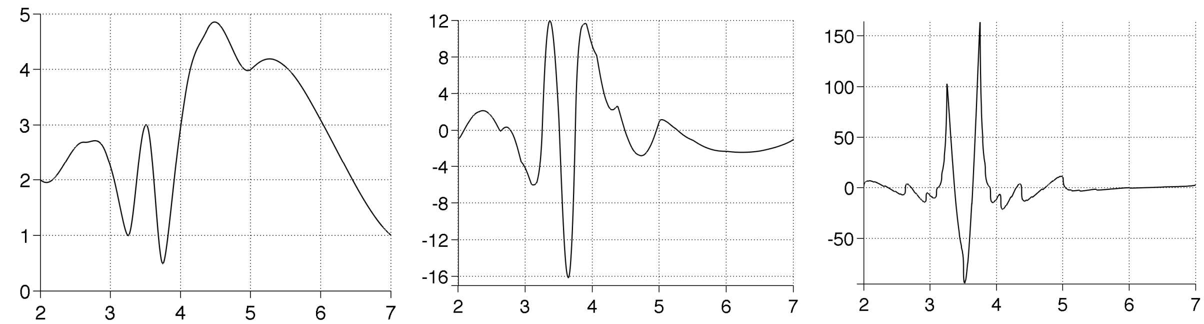

Figure 2: Limit function g for a nonuniform grid, with g' and g''.

Some lengthy calculatons show that in the regular case, with $x_{j,k} = 2^{-j}k$, the third and fourth equations of (2) reduce to

$$\begin{aligned} g_{j+1,2k+1} &= a_1(g_{j,k} + g_{j,k+1}) + b_1(g_{j,k-1} + g_{j,k+2}) + 2^{-j}c_1(m_{j,k+1} - m_{j,k}), \\ m_{j+1,2k+1} &= 2^j a_2(g_{j,k+1} - g_{j,k}) + 2^j b_2(g_{j,k+2} - g_{j,k-1}) + c_2(m_{j,k} + m_{j,k+1}), \end{aligned} \tag{3}$$

where

$$(a_1, b_1, c_1) = \frac{1}{128}(63, 1, -18), \qquad (a_2, b_2, c_2) = \frac{1}{192}(297, 1, -54). \tag{4}$$

This scheme is a special case of the 4-point family of schemes 'HS41' analyzed by Dubuc and Merrien [4]. Theorem 3 of [4] gives a sufficient condition for the convergence of this family, i.e., that the scheme has a C^1 limit function. We have done some numerical computations and found that the conditions hold for (3) when using the matrix norm $\|\cdot\|_\theta$ of Sec. 2.2. of [4] for various values of θ, for example, $\theta = 3$. However, the exact regularity of the HS41 schemes was not determined in [4]. A general framework for analyzing the regularity of Hermite subdivision schemes of arbitrary order has been proposed by Merrien and Sauer [12]. Hölder regularity is, however, not discussed in [12], and no attempt is made to determine the exact regularity of (3).

3 Piecewise polynomials

The regularity analysis is much more involved for non-uniformly spaced data and we only treat regular grids and assume from now on that $x_{j,k} = 2^{-j}k$. We follow the same basic approach used in [8]. For each j we form a spline from the quintic polynomials at level j used to define the scheme. Letting $I_{j,k} = [x_{j,k}, x_{j,k+1}]$, we define

$$s_j(x) = p_{j,k-1}(x), \qquad x \in I_{j,k},$$

and so s_j is a C^1 piecewise quintic polynomial, with

$$s_j(x_{j,k}) = g_{j,k}, \qquad s_j'(x_{j,k}) = m_{j,k}. \tag{5}$$

Thus the limit function of the scheme, if it exists, is the limit of these splines,

$$g(x) := \lim_{j\to\infty} s_j(x), \qquad x \in [0, n],$$

and its existence and smoothness can be analyzed from the convergence of the s_j and their derivatives. As we will see, this analysis depends on certain divided differences defined by the scheme; see Sec. 6.1 of [9]. We will work with two kinds of divided difference of each order $r \geq 0$. Due to (5) we can, for simplicity, define these as divided differences of the function s_j. For fixed j, with $x_k := x_{j,k}$, they are

$$\mu_{j,k}^{[r]} = [\underbrace{x_k^{(2)}, x_{k+1}^{(2)}, \ldots}_{r+1}]s_j,$$
$$\nu_{j,k}^{[r]} = [\underbrace{x_k, x_{k+1}^{(2)}, x_{k+2}^{(2)}, \ldots}_{r+1}]s_j,$$

with $x_i^{(2)}$ denoting that x_i appears twice in the sequence. To be more precise, consider the two cases that r is even or odd. When $r = 2p$,

$$\mu_{j,k}^{[2p]} = [x_k^{(2)}, \ldots, x_{k+p-1}^{(2)}, x_{k+p}]s_j,$$
$$\nu_{j,k}^{[2p]} = [x_k, x_{k+1}^{(2)}, \ldots, x_{k+p}^{(2)}]s_j,$$

and when $r = 2p + 1$,

$$\mu_{j,k}^{[2p+1]} = [x_k^{(2)}, \ldots, x_{k+p}^{(2)}]s_j,$$
$$\nu_{j,k}^{[2p+1]} = [x_k, x_{k+1}^{(2)}, \ldots, x_{k+p}^{(2)}, x_{k+p+1}]s_j.$$

These divided differences satisfy the usual recursions, i.e.,

$$\mu_{j,k}^{[0]} = g_{j,k}, \quad \nu_{j,k}^{[0]} = g_{j,k}, \quad \mu_{j,k}^{[1]} = m_{j,k},$$

and for $p \geq 1$,

$$\mu_{j,k}^{[2p]} = 2^j(\nu_{j,k}^{[2p-1]} - \mu_{j,k}^{[2p-1]})/p,$$
$$\nu_{j,k}^{[2p]} = 2^j(\mu_{j,k+1}^{[2p-1]} - \nu_{j,k}^{[2p-1]})/p,$$

and for $p \geq 1$ and $p \geq 0$ respectively,

$$\mu_{j,k}^{[2p+1]} = 2^j(\nu_{j,k}^{[2p]} - \mu_{j,k}^{[2p]})/p,$$
$$\nu_{j,k}^{[2p+1]} = 2^j(\mu_{j,k+1}^{[2p]} - \nu_{j,k}^{[2p]})/(p+1).$$

For convenience we will also use symbols for differences of divided differences,

$$\tilde{\mu}_{j,k}^{[r]} = \nu_{j,k}^{[r-1]} - \mu_{j,k}^{[r-1]},$$
$$\tilde{\nu}_{j,k}^{[r]} = \mu_{j,k+1}^{[r-1]} - \nu_{j,k}^{[r-1]},$$

and the following notation for certain nodal polynomials,

$$\chi_{j,k}(x) = (x - x_{j,k})(x - x_{j,k+1})^2(x - x_{j,k+2}),$$
$$\phi_{j,k}(x) = (x - x_{j,k})\chi_{j,k}(x),$$
$$\psi_{j,k}(x) = \chi_{j,k}(x)(x - x_{j,k+2}).$$

The main ingredient in the analysis is the following formula for the difference between successive piecewise polynomials.

Lemma 1 *For $j \geq 0$,*

$$s_{j+1}(x) - s_j(x) = \begin{cases} -\tilde{\nu}_{j+1,2k-1}^{[6]}\phi_{j+1,2k}(x), & \text{for } x \in I_{j+1,2k}, \\ \tilde{\mu}_{j+1,2k}^{[6]}\psi_{j+1,2k}(x), & \text{for } x \in I_{j+1,2k+1}. \end{cases}$$

Proof. Suppose $x \in I_{j+1,2k}$ (the case when $x \in I_{j+1,2k+1}$ is similar). Then

$$s_{j+1}(x) - s_j(x) = p_{j+1,2k-1}(x) - p_{j,k-1}(x).$$

Since these two polynomials have the same values at $x_{j+1,2k}$, $x_{j+1,2k+1}$ and $x_{j+1,2k+2}$, and the same first derivatives at $x_{j+1,2k}$ and $x_{j+1,2k+1}$, their difference is divisible by the quintic $\phi_{j+1,2k}$, and so

$$p_{j+1,2k-1}(x) - p_{j,k-1}(x) = \alpha\phi_{j+1,2k}(x)$$

where α is the difference between their leading coefficients, namely

$$\alpha = \nu_{j+1,2k-1}^{[5]} - \mu_{j,k-1}^{[5]} = \nu_{j+1,2k-1}^{[5]} - \mu_{j+1,2k}^{[5]} = -\tilde{\nu}_{j+1,2k-1}^{[6]}.$$

□

By differentiating the formula in the lemma r times, $0 \leq r \leq 5$, it follows that there is a constant $C > 0$ such that

$$|s_{j+1}^{(r)}(x) - s_j^{(r)}(x)| \leq C 2^{j(r-5)} \max(|\tilde{\nu}_{j+1,2k-1}^{[6]}|, |\tilde{\mu}_{j+1,2k}^{[6]}|), \qquad \text{for } x_{j,k} < x < x_{j,k+1}. \tag{6}$$

As we will see, by estimating the growth rate of the differences $\tilde{\mu}_{j,k}^{[6]}$ and $\tilde{\nu}_{j,k}^{[6]}$, this bound will be used, in the cases $r = 0, 1$, to show that g is C^1. We will also use the bound, in the case $r = 2$, to show that g is C^2, but need in addition a bound on the jumps in the second derivatives of the s_j.

Lemma 2 *For $j \geq 0$,*

$$s''_{j,+}(x_{j,k}) - s''_{j,-}(x_{j,k}) = 2^{-3j+1}(\tilde{\mu}_{j,k-1}^{[6]} - \tilde{\nu}_{j,k-2}^{[6]}). \tag{7}$$

Proof. Observe that

$$s''_{j,+}(x_{j,k}) - s''_{j,-}(x_{j,k}) = d''(x_{j,k}),$$

where d is the quintic,

$$d(x) = p_{j,k-1}(x) - p_{j,k-2}(x).$$

If q is the cubic such that

$$q(x_{j,i}) = g_{j,i}, \quad i = k-1, k, k+1, \qquad q'(x_{j,k}) = m_{j,k},$$

then we can express $p_{j,k-1}$ in Newton form as

$$p_{j,k-1}(x) = q(x) + (\nu_{j,k-1}^{[4]} + \nu_{j,k-1}^{[5]}(x - x_{j,k+1}))\chi_{j,k-1}(x),$$

and $p_{j,k-2}$ as

$$p_{j,k-2}(x) = q(x) + (\mu_{j,k-1}^{[4]} + \nu_{j,k-2}^{[5]}(x - x_{j,k-1}))\chi_{j,k-1}(x).$$

Since

$$\chi_{j,k-1}(x_{j,k}) = \chi'_{j,k-1}(x_{j,k}) = 0, \qquad \chi''_{j,k-1}(x_{j,k}) = -2^{-2j+1},$$

it follows that

$$\begin{aligned} d''(x_{j,k}) &= -2^{-2j+1}(\nu_{j,k-1}^{[4]} - \mu_{j,k-1}^{[4]} - 2^{-j}\nu_{j,k-1}^{[5]} - 2^{-j}\nu_{j,k-2}^{[5]}) \\ &= -2^{-3j+1}(2\mu_{j,k-1}^{[5]} - \nu_{j,k-1}^{[5]} - \nu_{j,k-2}^{[5]}), \end{aligned}$$

which reduces to (7). □

4 The derived scheme of highest order

The second step of the analysis is to estimate the growth rate of the differences $\tilde{\mu}_{j,k}^{[6]}$ and $\tilde{\nu}_{j,k}^{[6]}$, which we do by deriving the subdivision scheme they satisfy. First we derive the scheme for the lower order differences $\mu_{j,k}^{[5]}$ and $\nu_{j,k}^{[5]}$. To do this we use the following 'chain' lemma to replace one divided difference of order five by a linear combination of other divided differences of the same order.

Lemma 3 *For any function f defined on the distinct points $\mathbf{y} = (y_0, y_1, \ldots, y_n)$,*

$$[y_0, y_n]f = \sum_{i=0}^{n-1} \frac{y_{i+1} - y_i}{y_n - y_0} [y_i, y_{i+1}]f. \tag{8}$$

Proof. Since

$$f(y_n) - f(y_0) = \sum_{i=0}^{n-1} (f(y_{i+1}) - f(y_i)),$$

we have

$$(y_n - y_0)[y_0, y_n]f = \sum_{i=0}^{n-1} (y_{i+1} - y_i)[y_i, y_{i+1}]f,$$

which gives the result. □

We will use three cases of the chain lemma, noting that shifts and scalings of the points in $\mathbf{y}$ do not change the ratios in (8). The choices $\mathbf{y} = (1, 4, 3)$, $\mathbf{y} = (1, 0, 3)$, and $\mathbf{y} = (1, 0, 4, 3)$ give

$$[1, 3]f = \frac{1}{2}(3[1, 4]f - [3, 4]f), \tag{9}$$

$$[1, 3]f = \frac{1}{2}(-[0, 1]f + 3[0, 3]f), \tag{10}$$

$$[1, 3]f = \frac{1}{2}(-[0, 1]f + 4[0, 4]f - [3, 4]f). \tag{11}$$

Lemma 4 *The fifth order derived scheme is*

$$\begin{bmatrix} \mu_{j+1,2k}^{[5]} \\ \nu_{j+1,2k}^{[5]} \\ \mu_{j+1,2k+1}^{[5]} \\ \nu_{j+1,2k+1}^{[5]} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 \\ 7 & -4 & 1 \\ -8 & 20 & -8 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} \nu_{j,k-1}^{[5]} \\ \mu_{j,k}^{[5]} \\ \nu_{j,k}^{[5]} \end{bmatrix}. \tag{12}$$

Proof. From the definition of the scheme (2) it follows that

$$\mu_{j+1,2k}^{[5]} = \nu_{j,k-1}^{[5]},$$

which is the first of the four equations in (12). To derive the second equation, identity (9) implies that

$$\nu^{[5]}_{j+1,2k} = \frac{1}{2}(3\alpha^{[5]}_{j+1,2k} - \rho^{[5]}_{j+1,2k}), \tag{13}$$

where, with $x_k := x_{j,k}$,

$$\rho^{[5]}_{j,k} = [x_k, x_{k+1}, x^{(2)}_{k+2}, x_{k+3}, x_{k+4}]s_j,$$

and

$$\alpha^{[5]}_{j,k} = [x_k, x^{(2)}_{k+1}, x^{(2)}_{k+2}, x_{k+4}]s_j.$$

From the definition of the scheme (2),

$$\alpha^{[5]}_{j+1,2k} = \nu^{[5]}_{j,k-1}, \tag{14}$$

and using the identity (11),

$$\rho^{[5]}_{j+1,2k} = \frac{1}{2}(-\nu^{[5]}_{j,k-1} + 4\mu^{[5]}_{j,k} - \nu^{[5]}_{j,k}), \tag{15}$$

and substituting (14) and (15) into (13) gives the second equation in (12).

The fourth equation in (12) is a kind of reversal of the second. With

$$\beta^{[5]}_{j,k} = [x_k, x^{(2)}_{k+2}, x^{(2)}_{k+3}, x_{k+4}]s_j,$$

the identity (10) gives

$$\nu^{[5]}_{j+1,2k+1} = \frac{1}{2}(-\rho^{[5]}_{j+1,2k} + 3\beta^{[5]}_{j+1,2k}),$$

and since

$$\beta^{[5]}_{j+1,2k} = \nu^{[5]}_{j,k},$$

from the definition of the scheme, and using (15) again we obtain the fourth equation.

To derive the third equation in (12), we can use the identity (11) to show that

$$\mu^{[5]}_{j+1,2k+1} = \frac{1}{2}(-\nu^{[5]}_{j+1,2k} + 4\rho^{[5]}_{j+1,2k} - \nu^{[5]}_{j+1,2k+1}),$$

and then substitute in the expressions we have already found for the three terms on the right. □

From this, we obtain the scheme for the next differences,

$$\begin{bmatrix} \tilde{\mu}^{[6]}_{j+1,2k} \\ \tilde{\nu}^{[6]}_{j+1,2k} \\ \tilde{\mu}^{[6]}_{j+1,2k+1} \\ \tilde{\nu}^{[6]}_{j+1,2k+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -3 & 1 \\ 15 & -9 \\ -9 & 15 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \tilde{\nu}^{[6]}_{j,k-1} \\ \tilde{\mu}^{[6]}_{j,k} \end{bmatrix}, \tag{16}$$

and hence a bound on their growth rate: since the largest ℓ_1 norm of the four rows of this matrix is 6,

$$\max_k \max\{|\tilde{\mu}^{[6]}_{j,k}|, |\tilde{\nu}^{[6]}_{j,k}|\} \leq C6^j. \tag{17}$$

5 Regularity

We now come to the main results of the paper.

Theorem 1 *The scheme has a C^2 limit function g.*

Proof. Combining the bounds (6) and (17) implies

$$\sup_{x_{j,k}<x<x_{j,k+1}} |s_{j+1}^{(r)}(x) - s_j^{(r)}(x)| \leq C\lambda_r^j, \qquad 0 \leq r \leq 5, \tag{18}$$

where

$$\lambda_r = 3 \cdot 2^{r-4}.$$

Since both s_j and s_j' are continuous it follows that

$$\|s_{j+1}^{(r)} - s_j^{(r)}\| \leq C\lambda_r^j, \qquad r = 0, 1,$$

where $\|\cdot\|$ denotes the maximum norm over the interval $[0, n]$. Since $\lambda_0 < 1$ the sequence $(s_j)_j$ is therefore Cauchy in this norm with a continuous limit function g. Further, since $\lambda_1 < 1$ the sequence $(s_j')_j$ is also Cauchy in this norm with a continuous limit function $g^{[1]}$. By a standard integration technique (see for example [8]), one can show that it follows that g is differentiable with $g' = g^{[1]}$.

Finally, even though the functions $s_{j,+}''$ and $s_{j,-}''$ are not in general continuous, the sequences $(s_{j,+}'')_j$ and $(s_{j,-}'')_j$ are both pointwise Cauchy because $\lambda_2 < 1$. Moreover, the jumps are controlled by Lemma 2, which shows that

$$s_{j,+}''(x) - s_{j,-}''(x) \to 0$$

as $j \to \infty$ for every dyadic point. So the analysis used in [8], adapted to this Hermite case, shows that both sequences $(s_{j,+}'')_j$ and $(s_{j,-}'')_j$ convergence to the same continuous limit function, $g^{[2]}$, and, similar to the first derivative case, one can show that g must then have a second derivative and that $g'' = g^{[2]}$. □

To derive the Hölder regularity of g'' we first estimate the growth rate of the third derivative of s_j.

Lemma 5 *There is a constant C such that for all $j \geq 0$ and $x \in (x_{j,k}, x_{j,k+1})$,*

$$|s_j'''(x)| \leq C\lambda_3^j. \tag{19}$$

Proof. For any $x \in (x_{j,k}, x_{j,k+1})$, (18) with $r = 3$ implies

$$\left|s_j'''(x) - s_{j-1}'''(x)\right| \leq C\lambda_3^{j-1},$$

and applying this repeatedly gives

$$\left|s_j'''(x) - s_0'''(x)\right| \leq C\sum_{i=0}^{j-1} \lambda_3^i \leq 2C\lambda_3^j,$$

which implies (19). □

Theorem 2 *The function g'' is Hölder continuous with exponent $\alpha = -\log_2(3/4)$.*

Proof. Let $x, y \in [0, n]$ be such that $0 < y - x < 1$. Then there exists $j \geq 1$ such that the number r of dyadic points $x_{j,k}$ that are in the open interval (x, y) is either 2 or 3. Thus, fixing this j, there is some k such that

$$x_{j,k} \leq x < x_{j,k+1} < \cdots < x_{j,k+r} < y \leq x_{j,k+r+1},$$

with $r \in \{2, 3\}$. Let $y_0 = x$, $y_i = x_{j,k+i}$ for $i = 1, \ldots, r$, and $y_{r+1} = y$. Then by the triangle inequality,

$$|g''(y) - g''(x)| \leq |g''(y) - s''_{j,-}(y)| + |s''_{j,-}(y) - s''_{j,+}(x)| + |s''_{j,+}(x) - g''(x)|, \tag{20}$$

and

$$\begin{aligned}|s''_{j,-}(y) - s''_{j,+}(x)| &\leq \sum_{i=0}^{r} |s''_{j,-}(y_{i+1}) - s''_{j,+}(y_i)| + \sum_{i=1}^{r} |s''_{j,+}(y_i) - s''_{j,-}(y_i)| \\ &= \sum_{i=0}^{r} |(y_{i+1} - y_i) s'''_j(\xi_i)| + \sum_{i=1}^{r} |s''_{j,+}(y_i) - s''_{j,-}(y_i)|\end{aligned}$$

for some $\xi_i \in (y_i, y_{i+1})$. By Lemma 5,

$$|(y_{i+1} - y_i) s'''_j(\xi_i)| \leq 2^{-j} C \lambda_3^j = C \lambda_2^j,$$

and from Lemma 2 and (17),

$$|s''_{j,+}(y_i) - s''_{j,-}(y_i)| \leq C_2 \lambda_2^j,$$

and so, since $r \leq 3$,

$$|s''_{j,-}(y) - s''_{j,+}(x)| \leq C_3 \lambda_2^j.$$

To deal with the other two terms in (20), observe that from (18),

$$\max(|g''(y) - s''_{j,-}(y)|, |s''_{j,+}(x) - g''(x)|) \leq C_4 \lambda_2^j.$$

Thus,

$$|g''(y) - g''(x)| \leq C_5 \lambda_2^j.$$

Since $r \geq 2$, we also have $y - x \geq 2^{-j}$, and so for any $\alpha \in (0, 1)$,

$$\frac{|g''(y) - g''(x)|}{(y - x)^\alpha} \leq C_5 (2^\alpha \lambda_2)^j,$$

which is bounded by a constant independent of j if $2^\alpha \lambda_2 \leq 1$, i.e., if

$$\alpha \leq -\log_2(\lambda_2) = -\log_2(3/4).$$

□

This Hölder estimate is optimal.

Theorem 3 *The Hölder exponent* $\alpha_0 = -\log_2(3/4)$ *of* g'' *is the largest possible.*

Proof. Suppose, on the contrary, that g'' is Hölder continuous with exponent $\alpha = \alpha_0 + \delta$ for some $\delta > 0$. Then there is a constant C such that

$$|g''(y) - g''(x)| \leq C|y - x|^{\alpha}$$

for $x, y \in [0, n]$ with $|y - x| < 1$. Then for any j and k,

$$|\tilde{\mu}^{[3]}_{j,k}| = |\nu^{[2]}_{j,k} - \mu^{[2]}_{j,k}| = |g''(y) - g''(x)|/2$$

for some $x, y \in (x_{j,k}, x_{j,k+1})$ and so

$$|\tilde{\mu}^{[3]}_{j,k}| \leq C2^{-j\alpha}.$$

Similarly,

$$|\tilde{\nu}^{[3]}_{j,k}| = |\mu^{[2]}_{j,k+1} - \nu^{[2]}_{j,k}| = |g''(y) - g''(x)|/2$$

for some $x \in (x_{j,k}, x_{j,k+1})$ and $y \in (x_{j,k+1}, x_{j,k+2})$ and so also

$$|\tilde{\nu}^{[3]}_{j,k}| \leq C2^{-j\alpha}.$$

Then we deduce, recursively, that

$$\max(|\tilde{\mu}^{[6]}_{j,k}|, |\tilde{\nu}^{[6]}_{j,k}|) \leq C_2 2^{j(3-\alpha)} = C_2 6^j 2^{-j\delta}. \tag{21}$$

On the other hand, the second and third equations in the scheme (16) in the case $k = -1$ are

$$\begin{bmatrix} \tilde{\nu}^{[6]}_{j+1,-2} \\ \tilde{\mu}^{[6]}_{j+1,-1} \end{bmatrix} = A \begin{bmatrix} \tilde{\nu}^{[6]}_{j,-2} \\ \tilde{\mu}^{[6]}_{j,-1} \end{bmatrix},$$

where

$$A = \frac{1}{4} \begin{bmatrix} 15 & -9 \\ -9 & 15 \end{bmatrix},$$

whose eigenvalues are $\mu_1 = 6$ and $\mu_2 = 3/2$, with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T, \qquad v_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T.$$

Therefore, with initial data chosen so that

$$\tilde{\nu}^{[6]}_{0,-2} = 1, \qquad \tilde{\mu}^{[6]}_{0,-1} = -1,$$

we have

$$\tilde{\nu}^{[6]}_{j,-2} = 6^j, \qquad \tilde{\mu}^{[6]}_{j,-1} = -6^j,$$

which contradicts (21). □

6 Final remarks

We have shown that for regular grids the scheme has regularity $C^{2+\alpha}$ with α in (1). Our numerical tests and a preliminary mathematical analysis strongly indicate that for semi-regular grids, the scheme is again at least C^2.

There are several ways to generalize the scheme. One possibility is to choose any $r \geq 1$ and define a Hermite scheme in which derivatives of all orders $\leq r$ are maintained and base the refinement on fitting a polynomial of degree $\leq 2r+3$ to match values and derivatives up to order r at $x_{j,k}$ and $x_{j,k+1}$ but only match values at $x_{j,k-1}$ and $x_{j,k+2}$. From our numerical tests the regularity of this scheme appears to be at least C^{r+1} for small r. There is a natural sequence of splines in this scheme and a relatively simple expression for the differences between them. The challenge for the regularity analysis is to bound these differences and to derive the highest order divided difference scheme. We plan to work on this in the future.

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