

Convex combination maps

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Abstract

Piecewise linear maps over triangulations are used extensively in geometric modelling and computer graphics. This short note surveys recent progress on the important question of when such maps are one-to-one, central to which are *convex combination maps*.

1 Introduction

Piecewise linear maps over triangulations have several applications in geometric modelling and computer graphics. For example, Figure 1a shows a surface triangulation \mathcal{T} of a set of points (x_i, y_i, z_i) sampled from some unknown surface in \mathbb{R}^3 . A standard approach to fitting a smooth parametric surface $s(u, v)$ to these points is to first *parameterize* them, i.e., compute planar points (u_i, v_i) corresponding to the data points (x_i, y_i, z_i) . Then using some scattered data method, we find a parametric surface $s : \Omega \rightarrow \mathbb{R}^3$, defined over some suitable domain $\Omega \subset \mathbb{R}^2$ containing the points (u_i, v_i) , such that

$$s(u_i, v_i) \approx (x_i, y_i, z_i).$$

A choice of parameterization is shown in Figure 1b and a least squares surface approximation using bicubic B-splines is shown in Figure 1c.

Notice that the choice of parameter points (u_i, v_i) uniquely determines a piecewise linear map $\phi : D_{\mathcal{T}} \rightarrow \mathbb{R}^2$, where $D_{\mathcal{T}}$ is the union of the triangles in \mathcal{T} . In practice, a necessary requirement on ϕ to ensure adequate quality of the subsequent surface approximation $s(u, v)$ is that ϕ should be injective. In Figure 1b the mapping ϕ was taken to be a so-called *convex combination map*, which, as we will see later, is guaranteed to be one-to-one since the boundary of \mathcal{T} is mapped to a rectangle. Put another way, none of the triangles in Figure 1b are ‘folded over’. In fact further properties of the map are important, such as linear precision, and this was achieved in Figure 1b by using the so-called *shape-preserving* weights (the coefficients in the convex combinations). For a discussion of that, see [3].

Another application of piecewise linear maps is to image morphing. Image morphing can be carried out by continuously transforming one planar triangulation \mathcal{T}^0 (whose vertices represent feature points in the image) to another, \mathcal{T}^1 . Here we assume that there is a one-to-one correspondence between the vertices, edges, and triangles of \mathcal{T}^0 and \mathcal{T}^1 . We can view each intermediate triangulation $\mathcal{T}(t)$, $0 \leq t \leq 1$, where $\mathcal{T}(0) = \mathcal{T}^0$ and $\mathcal{T}(1) = \mathcal{T}^1$, as the image of a piecewise linear map $\phi(t) : D_{\mathcal{T}^0} \rightarrow D_{\mathcal{T}(t)}$. As with parameterizations, it is again important that $\phi(t)$ is one-to-one. Figure 2 shows the so-

called *convex combination morph* of [4] of two given planar triangulations: \mathcal{T}^0 appears on the left and \mathcal{T}^1 on the right. The two triangulations in the middle are $\mathcal{T}(1/3)$ and $\mathcal{T}(2/3)$. This morph ensures that $\phi(t)$ is one-to-one for all t in $[0, 1]$ and therefore $\mathcal{T}(t)$ has no ‘folded’ triangles at any time instant t .

Piecewise linear maps also arise in: texture mapping; numerical grid generation; and in setting up multiresolution frameworks, i.e., nested spaces of piecewise linear functions, for manifold surface triangulations in computer graphics.

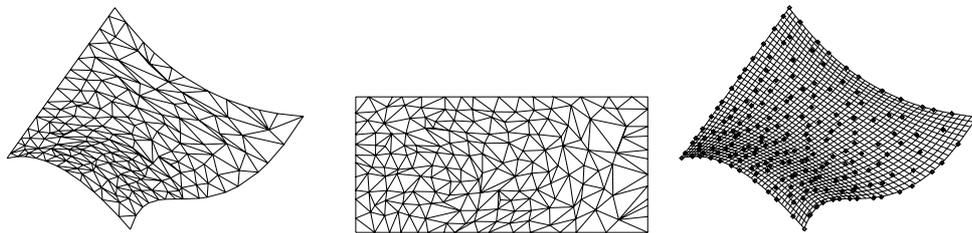


FIG. 1. Spatial triangulation (1a), Convex combination parameterization (1b), Bicubic spline approximation (1c)

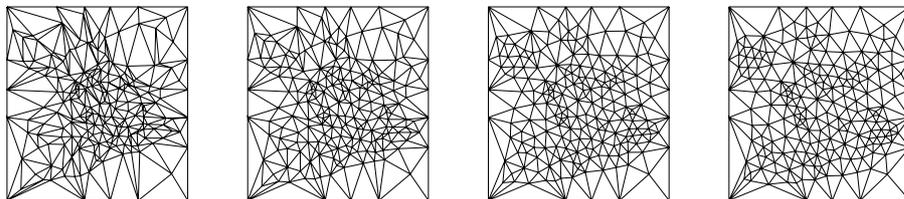


FIG. 2. Convex combination morph

2 Convex combination maps

For the sake of simplicity we will only discuss convex combination maps defined over planar triangulations even though all the results hold equally well when the domain of the map is a spatial triangulation such as that in Figure 1a. Thus let $\mathcal{T} = \{T_1, \dots, T_M\}$ be a simply-connected planar triangulation, with closed triangles $T_i \subset \mathbb{R}^2$, and let $D_{\mathcal{T}} = \bigcup_{T \in \mathcal{T}} T$, as in Figure 3. We will call a mapping $\phi : D_{\mathcal{T}} \rightarrow \mathbb{R}^2$ a **convex combination map** if it is piecewise linear over \mathcal{T} and, for every interior vertex v of \mathcal{T} , there exist weights $\lambda_{vw} > 0$, for $w \in N_v$, such that

$$\sum_{w \in N_v} \lambda_{vw} = 1,$$

and

$$\phi(v) = \sum_{w \in N_v} \lambda_{vw} \phi(w), \quad (1)$$

where N_v is the set of neighbours of v ; see Figure 3.

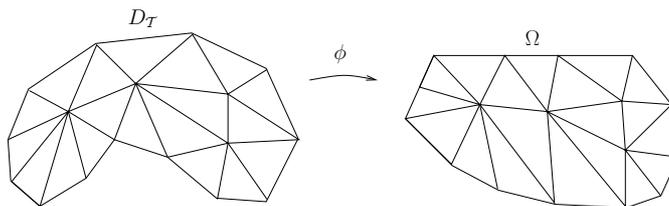


FIG. 3. Convex combination map

In applications, the mapped boundary vertices $\phi(v)$ are chosen first. Then the weights λ_{vw} are all specified according to some chosen strategy. Then finally the mapped interior vertices are found by treating the equations in (1) as a linear system.

Example 1. *If an interior vertex v of \mathcal{T} has five neighbours v_1, \dots, v_5 , then we might set*

$$\phi(v) = \frac{1}{4}\phi(v_1) + \frac{1}{8}\phi(v_2) + \frac{1}{8}\phi(v_3) + \frac{1}{4}\phi(v_4) + \frac{1}{4}\phi(v_5).$$

Until recently, the only theory behind convex combination maps was that of Tutte [8]. Working from a purely graph-theoretic point of view, Tutte proposed a so-called barycentric mapping for constructing straight line drawings of 3-connected graphs (which include triangulations). A barycentric mapping in our context is simply a convex combination map in which all the weights at each vertex are equal, i.e., $\lambda_{vw} = 1/d_v$, where d_v is the degree or valency of the vertex v . Thus for v in Example 1 we must have

$$\phi(v) = \frac{1}{5}\phi(v_1) + \frac{1}{5}\phi(v_2) + \frac{1}{5}\phi(v_3) + \frac{1}{5}\phi(v_4) + \frac{1}{5}\phi(v_5).$$

Tutte showed that a valid straight line drawing, i.e. one with no edge crossings, results from a barycentric mapping if the ‘boundary’ of the graph, a so-called ‘cycle’, is mapped to a convex polygon. However, as argued in [3], convex combination maps share all those properties of barycentric maps necessary for Tutte’s proof. Thus when interpreted in the right way and suitably generalized, Tutte’s theorem can be expressed in the following way.

Theorem 1. *Suppose $\phi : D_{\mathcal{T}} \rightarrow \mathbb{R}^2$ is a convex combination mapping which maps the n boundary vertices of \mathcal{T} cyclically into the n vertices of some n -sided convex polygon in the plane. Then ϕ is one-to-one.*

Despite this generalization, however, there are still two aspects of it which need to be improved from the point of view of applications and future research.

The first is that we would like to extend the theorem so that we can allow some, and indeed many, of the mapped boundary vertices to be collinear. Indeed in the application

to parameterization for surface fitting, it might be convenient to map all the boundary vertices of the given triangulation into the four sides of a rectangle, as in Figure 1b. This is because tensor-product splines surfaces are defined over rectangular domains. Collinearity will also often be desirable in morphing, as in Figure 2, and in most other applications. Thus a drawback of Theorem 1 is that it does not allow collinear vertices in the image boundary.

The second aspect is that we would like to simplify the proof in order to have some hope of establishing the injectivity of piecewise linear maps in even more general situations, such as when mapping to non-convex regions, or when some of the mapped vertices are constrained, for example. The fact that Tutte's proof relies on the non-existence of the Kuratowski subgraphs K_5 and $K_{3,3}$ in a planar graph illustrates its complexity.

It is these two improvements that are the focus of [5]. The main idea of [5] is the observation that Theorem 1 is very similar to a theorem on harmonic maps, referred to by Duren and Hengartner [2] as the Radó-Kneser-Choquet theorem, which was established in [7, 6, 1]. Recall that a mapping $\phi : D \rightarrow \mathbb{R}^2$, with $D \subset \mathbb{R}^2$ and $\phi = (u, v)$, is *harmonic* if both its components $u(x, y)$ and $v(x, y)$ satisfy the Laplace equation in D , i.e.,

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0;$$

see Figure 4.

Radó-Kneser-Choquet Theorem. *Suppose $\phi : D \rightarrow \mathbb{R}^2$ is a harmonic mapping which maps the boundary ∂D homeomorphically into the boundary $\partial\Omega$ of some convex region $\Omega \subset \mathbb{R}^2$. Then ϕ is one-to-one.*

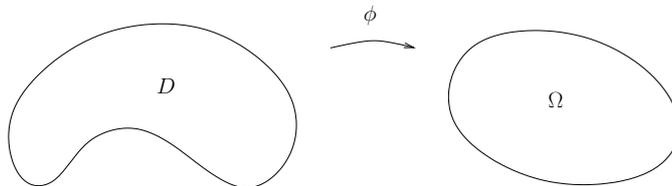


FIG. 4. Harmonic map

This suggested that a proof of Theorem 1 might be based on a proof of the Radó-Kneser-Choquet theorem, in particular the short proof of Kneser [6]. Kneser's proof begins by showing that ϕ is locally one-to-one in the sense that the Jacobian of ϕ ,

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

never vanishes. Kneser establishes this by supposing that the Jacobian is zero at some point $(x_0, y_0) \in D$. In that case there must be a straight line $ax + by + c = 0$ passing through the point $\phi(x_0, y_0)$ such that both partial derivatives of the function $h(x, y) = au(x, y) + bv(x, y) + c$ are zero at (x_0, y_0) . At the same time, the function $h : D \rightarrow \mathbb{R}$ is zero at (x_0, y_0) and has just two zeros along the boundary of D . Noting that $h(x, y)$

is a harmonic function, Kneser then uses the Nodal Lines theorem of Courant to argue that there are at least four zero contours of h emanating from (x_0, y_0) and due to the maximum principle for h , these four curves can never self-intersect nor intersect one another. Therefore all four curves must reach the boundary of D which is a contradiction.

These ideas were adapted in [5] to establish a much simpler proof of Theorem 1 than that of Tutte. No graph theory is needed. Instead, the discrete maximum principle for convex combination functions plays the role of the maximum principle for harmonic functions. Similar to Kneser's proof we show first that ϕ is locally one-to-one, except that we understand this to mean that the restriction of ϕ to any quadrilateral in \mathcal{T} is one-to-one, a quadrilateral being the union of two triangles sharing a common edge.

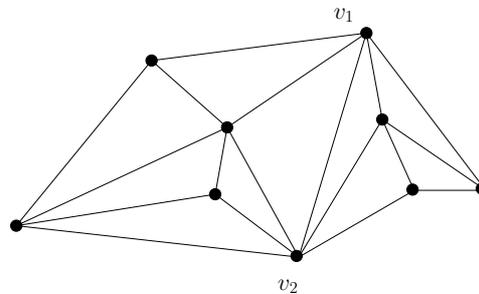


FIG. 5. Dividing edges

Moreover, Theorem 1 is generalized in [5] to allow collinear mapped boundary vertices. We call an edge $[v, w]$ of \mathcal{T} a *dividing edge* if both endpoints v and w are boundary vertices yet the edge $[v, w]$ itself is not contained in the boundary. For example in Figure 5, the only dividing edge in the triangulation is $[v_1, v_2]$. Dividing edges play a critical role because they partition the triangulation into subtriangulations \mathcal{T}_i , in each of which every convex combination function satisfies a discrete maximum principle in its strong form. The main result of [5] was the following.

Theorem 2. *Suppose \mathcal{T} is any triangulation and that $\phi : D_{\mathcal{T}} \rightarrow \mathbb{R}^2$ is a convex combination mapping which maps ∂D homeomorphically into the boundary $\partial\Omega$ of some convex region $\Omega \subset \mathbb{R}^2$. Then ϕ is one-to-one if and only if no dividing edge $[v, w]$ of \mathcal{T} is mapped by ϕ into $\partial\Omega$.*

3 Future research

Here is a list of topics for future research.

- A triangulation is a special (maximal) kind of planar graph. Can one extend Theorem 2 to other planar graphs, for example, rectangular grids? This is likely because Tutte's theory already holds for all 3-connected graphs.
- In what way can the theorem be extended from bivariate maps to trivariate ones?

- Can similar one-to-one maps be guaranteed when mapping closed surfaces of given topology to one another? For example, we would like to map a closed manifold triangulation, homeomorphic to a sphere, into a unit sphere injectively. Here each triangle in the triangulation would be mapped to a spherical triangle on the surface of the sphere.
- Can one find sufficient conditions for the injectivity of constrained maps, i.e., piecewise linear maps in which the image of certain interior points is specified in advance?
- Can one remove the requirement of having to map the boundary to a convex polygon and still ensure a one-to-one mapping under some weaker condition?
- Can the Radó-Kneser-Choquet theorem and Theorem 2 be combined as part of a single more general theorem?

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