Rational Cubic Implicitization

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Abstract. An explicit expression for the unique implicitization of any planar rational cubic Bézier curve is presented, the only exception being when the tangents at the ends of the curve are parallel. The representation is in the form of a cubic Bernstein-Bézier triangle and is invariant under affine transformations of the curve. The implicitization degenerates to zero if and only if the curve degenerates to a conic section.

§1. Introduction

Cubic and rational cubic Bézier curves are widely used in computer-aided design. Cubic polynomials are used frequently for approximation such as cubic Hermite interpolation and cubic spline approximation. With its additional two degrees of freedom, the rational cubic has also been used for approximation and both types of curve are highly suited for geometric modelling.

There are two ways of representing rational (and non-rational) Bézier curves: parametrically and implicitly. The former has traditionally been the most frequently used in computer-aided design while the other has received more attention in recent years. The parametric form is probably better suited for display and interactive manipulation while the implicit form is more useful when determining on which side of the curve a given point lies [1]. The intersection of two curves can be efficiently carried out if one is in parametric form while the other is in implicit form [4], since the parametric form of the first can be substituted into the equation of the second, thereby reducing the problem to that of finding the zeros of a polynomial. For example if one intersected two rational cubics in this manner, the degree of the polynomial would be nine.

The determination of the implicit representation of a planar rational cubic curve is consequently of considerable importance. Goldman, Sederberg, and Anderson [8] described the use of vector elimination and resultants for implicitizing planar rational polynomial curves of general degree. Sederberg, Anderson, and Goldman [10] looked specifically at rational cubic curves and, observing that they belong to the class of so-called monoid curves, found some simplification in the implicitization. In both papers, power bases are

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used for both the parametric and implicit representations. Meanwhile Sederberg [9] proposed a numerical method which is more geometrically intuitive, representing both curve and implicitization in terms of Bernstein bases.

In this paper we follow, initially, the method outlined in [9]. We begin with a planar rational cubic Bézier curve $\mathbf{r}$, assuming only that the end tangents $\mathbf{r}'(0)$ and $\mathbf{r}'(1)$ are non-parallel. We implicitize $\mathbf{r}$ in the form of a cubic Bernstein-Bézier triangular patch f, i.e. we find a non-zero $f : \mathbb{R}^2 \to \mathbb{R}$, defined in terms of barycentric coordinates, for which $f(\mathbf{r}(t)) = 0$ for all t. By a judicious choice of triangle, four of the coefficients of f are immediately zero.

Sederberg then chose five points on the curve in order to set up a system of five linear equations and to calculate numerically the coefficients of f. Instead we set up five equations directly by substituting the curve into f and demanding that the polynomial $f(\mathbf{r}(t))$ be zero. Mathematica was used to find explicit expressions for the coefficients and to simplify them. The (explicit) implicit form deduced in this way is very concise and is surely of use in geometric modelling and other applications. It is clearly more numerically stable to compute f from this explicit representation than to solve a linear system.

The original reason for computing the implicit representation analytically was the hope that it could be used for constructing various geometric Hermite interpolations to a rational cubic in the spirit of Dokken, Dæhlen, Lyche, and Mørken [2], and Floater [5,6,7]. The main idea used in [2,6,7] is that a polynomial or spline $\mathbf{q}$ is a good approximation to $\mathbf{r}$ if $f(\mathbf{q}(t))$ is small. Such a construction would form the basis of a conversion algorithm, converting rational cubic Bézier curves to polynomials spline curves. This is currently under investigation.

§2. The implicitization

Consider the general planar rational cubic Bézier curve

$$\mathbf{r}(t) = \frac{\sum_{i=0}^{3} B_i(t) w_i \mathbf{q}_i}{\sum_{i=0}^{3} B_i(t) w_i}, \qquad B_i(t) = \binom{3}{i} t^i (1-t)^{3-i}.$$

The control points $\mathbf{q}_i$ belong to $\mathbb{R}^2$ and we make the usual assumption that the weights w_i are all positive. Before deriving the implicitization, it will help to get $\mathbf{r}$ into a more convenient form.

First of all, it is usual in applications and makes no difference to the implicitization if one scales the weights and reparametrizes $\mathbf{r}$ so that it is in *standard form*, i.e. such that $w_0 = w_3 = 1$, [3]. Next, in order to choose an appropriate triangle, we make the further assumption that the vectors $\mathbf{q}_1 - \mathbf{q}_0$ and $\mathbf{q}_3 - \mathbf{q}_2$ are not parallel. Otherwise the method in this paper cannot be applied. With this assumption, let $\mathbf{p}_1$ be the intersection of the infinite straight lines through $\mathbf{q}_1 - \mathbf{q}_0$ and $\mathbf{q}_3 - \mathbf{q}_2$ and let $\mathbf{p}_0 = \mathbf{q}_0$ and $\mathbf{p}_2 = \mathbf{q}_3$. Then the points $\mathbf{p}_i$ will form the corners of the triangle that will be used to define barycentric coordinates. If we define the ratios λ_1 and λ_2 such that $\mathbf{q}_1 = (1-\lambda_1)\mathbf{p}_0 + \lambda_1\mathbf{p}_1$ and $\mathbf{q}_2 = (1-\lambda_2)\mathbf{p}_2 + \lambda_2\mathbf{p}_1$ one then finds that

$$\mathbf{r} = \frac{B_0\mathbf{p}_0 + B_1 w_1((1-\lambda_1)\mathbf{p}_0 + \lambda_1\mathbf{p}_1) + B_2 w_2((1-\lambda_2)\mathbf{p}_2 + \lambda_2\mathbf{p}_1) + B_3\mathbf{p}_2}{B_0 + B_1 w_1 + B_2 w_2 + B_3}. \tag{1}$$

The most important and most frequently occurring situation is shown in Figure 1 where both the λ_i lie in the interval (0,1), but this need not necessarily be the case. The implicitization method can equally well be applied to the three examples in Figure 2, where (a) $\lambda_1 > 1$, $\lambda_2 > 1$, (b) $\lambda_1 < 0$, $0 < \lambda_2 < 1$, (c) $\lambda_1 > 1$, $0 < \lambda_2 < 1$ respectively. However Figure 3 shows two curves where $\mathbf{q}_1 - \mathbf{q}_0$ and $\mathbf{q}_3 - \mathbf{q}_2$ are parallel and then the implicitization cannot be applied directly. The first curve can be implicitized by first subdividing it at any internal point and then implicitizing each subcurve. The same treatment can be applied to the second curve, though it is preferable to subdivide precisely at the inflection point so that each subcurve is convex.

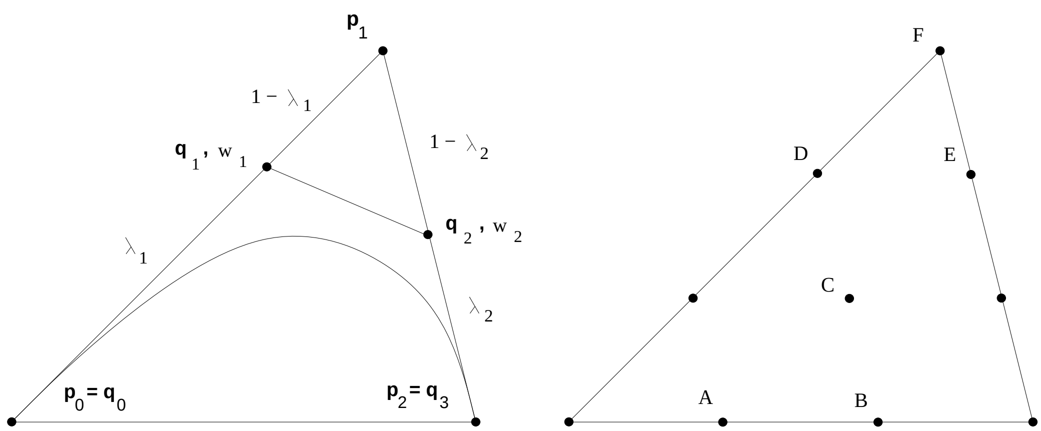

Fig. 1. $\mathbf{r}$ and its implicit representation f.

One last preprocessing step reduces the subsequent algebra somewhat. Define $\alpha_i = 3w_i(1-\lambda_i)$ and $\beta_i = 3w_i\lambda_i$ so that

$$\mathbf{r}(t) = \frac{(1-t)^3\mathbf{p}_0 + (1-t)^2 t(\alpha_1\mathbf{p}_0 + \beta_1\mathbf{p}_1) + (1-t)t^2(\alpha_2\mathbf{p}_2 + \beta_2\mathbf{p}_1) + t^3\mathbf{p}_2}{(1-t)^3 + (1-t)^2 t(\alpha_1+\beta_1) + (1-t)t^2(\alpha_2+\beta_2) + t^3}. \tag{2}$$

The weights w_i and ratios λ_i can be retrieved from α_i, β_i by noticing that $\lambda_i = \beta_i/(\alpha_i+\beta_i)$, $w_i = (\alpha_i+\beta_i)/3$.

The goal now is to find the implicit form of $\mathbf{r}$ as a Bernstein-Bézier triangular patch. Any point $(x, y) \in \mathbb{R}^2$ can be written uniquely in terms of the barycentric coordinates τ_0, τ_1, τ_2, where $\tau_0 + \tau_1 + \tau_2 = 1$, with respect to the triangle $\triangle\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$:

$$(x, y) = \tau_0\mathbf{p}_0 + \tau_1\mathbf{p}_1 + \tau_2\mathbf{p}_2.$$

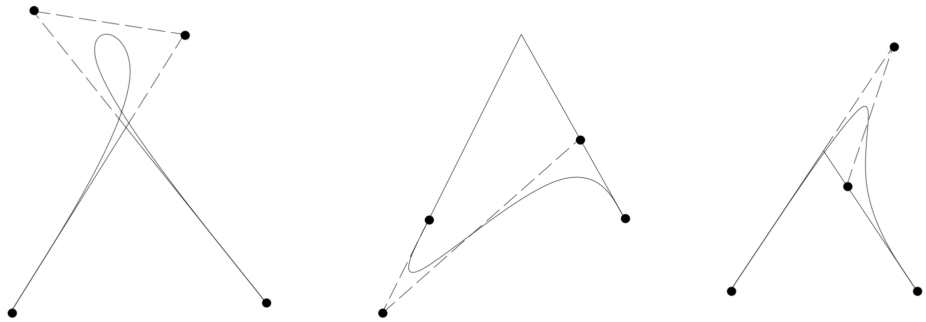

Fig. 2. Varying λ_1 and λ_2.

It is well-known [3] that the τ_i can be solved in terms of x and y;

$$\tau_0 = \frac{\text{area}(\mathbf{x}, \mathbf{p}_1, \mathbf{p}_2)}{\text{area}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)}, \qquad \tau_1 = \frac{\text{area}(\mathbf{p}_0, \mathbf{x}, \mathbf{p}_2)}{\text{area}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)}, \qquad \tau_2 = \frac{\text{area}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{x})}{\text{area}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)}.$$

We wish to compute a non-trivial bivariate function $f : \mathbb{R}^2 \to \mathbb{R}$, in the form

$$f(x, y) = \sum_{\substack{|\mathbf{i}|=3 \\ \mathbf{i}_j \geq 0}} B_{\mathbf{i},3}(\tau) a_{\mathbf{i}}, \qquad B_{\mathbf{i},3}(\tau) = \left(\frac{3!}{i_0! i_1! i_2!} \right) \tau_0^{i_0} \tau_1^{i_1} \tau_2^{i_2},$$

with $|\mathbf{i}| = i_0 + i_1 + i_2$, $\tau = (\tau_0, \tau_1, \tau_2)$, in such a way that $f(\mathbf{r}(t)) = 0$. The implicit representation of $\mathbf{r}$, unique up to a constant multiple, is then $f(x, y) = 0$. The solution is the following. Referring to Figure 1, let

$$\begin{aligned}
A &= -\beta_1^2(\beta_1 - \beta_2\alpha_2), \\
B &= -\beta_2^2(\beta_2 - \beta_1\alpha_1), \\
C &= -3\beta_1\beta_2 + 2\beta_1^2\alpha_1 + 2\beta_2^2\alpha_2 - \beta_1\beta_2\alpha_1\alpha_2, \\
D &= \alpha_2(\beta_1 - \beta_2\alpha_2), \\
E &= \alpha_1(\beta_2 - \beta_1\alpha_1), \\
F &= 1 - \alpha_1\alpha_2.
\end{aligned} \tag{3}$$

Theorem 1. *If*

$$f(x, y) = A\tau_0^2\tau_2 + B\tau_0\tau_2^2 + C\tau_0\tau_1\tau_2 + D\tau_0\tau_1^2 + E\tau_1^2\tau_2 + F\tau_1^3, \tag{4}$$

then $f(\mathbf{r}(t)) = 0$ *for all* t.

Proof: The barycentric coordinates of $\mathbf{r}$ in (2) are

$$\begin{aligned}
\tau_0 &= (1-t)^2((1-t) + t\alpha_1)/W, \\
\tau_1 &= (1-t)t((1-t)\beta_1 + t\beta_2)/W, \\
\tau_2 &= t^2((1-t)\alpha_2 + t)/W.
\end{aligned}$$

Fig. 3. When the implicitization method cannot be applied.

with

$$W = (1-t)^3 + (1-t)^2 t(\alpha_1+\beta_1) + (1-t)t^2(\alpha_2+\beta_2) + t^3.$$

Substituting these into (4) and collecting together coefficients of $t^i(1-t)^{9-i}$ we find

$$W^3 f(\mathbf{r}(t)) = \sum_{i=0}^{9} t^i(1-t)^{9-i} f_i$$

where $f_0 = f_1 = f_8 = f_9 = 0$, and

$$\begin{aligned}
f_2 &= \alpha_2 A + \beta_1^2 D, \\
f_3 &= (2\alpha_1\alpha_2+1)A + \beta_1\alpha_2 C + (2\beta_1\beta_2 + \alpha_1\beta_1^2)D + \beta_1^3 F, \\
f_4 &= (\alpha_1^2\alpha_2 + 2\alpha_1)A + \alpha_2^2 B + (\beta_1 + \alpha_2\beta_2 + \alpha_1\alpha_2\beta_1)C \\
&\quad + (\beta_2^2 + 2\alpha_1\beta_1\beta_2)D + \alpha_2\beta_1^2 E + 3\beta_1^2\beta_2 F, \\
f_5 &= \alpha_1^2 A + (\alpha_1\alpha_2^2 + 2\alpha_2)B + (\beta_2 + \alpha_1\beta_1 + \alpha_1\alpha_2\beta_2)C \\
&\quad + \alpha_1\beta_2^2 D + +(\beta_1^2 + 2\alpha_2\beta_1\beta_2)E + 3\beta_1\beta_2^2 F, \\
f_6 &= (2\alpha_1\alpha_2+1)B + \beta_2\alpha_1 C + (2\beta_1\beta_2 + \alpha_2\beta_2^2)E + \beta_2^3 F, \\
f_7 &= \alpha_1 B + \beta_2^2 E.
\end{aligned}$$

Substituting the expressions for $A,\ldots,F$ from (3) into the above it can be verified by hand that all the f_i are zero. The expressions for $A,\ldots,F$ were originally found by Mathematica by simultaneously solving the linear system of five equations $f_2 = f_3 = f_4 = f_6 = f_7 = 0$ in the five unknowns $A,\ldots,E$, leaving F fixed and equal to 1. The coefficients were afterwards multiplied by $1-\alpha_1\alpha_2$ so that they are all polynomials in α_1, α_2, β_1, and β_2. There are only five degrees of freedom since all coefficients in f can be scaled by a constant factor without altering its zeros. ■

In order to obtain the Bézier coefficients $a_{\mathbf{i}}$ for f it is only necessary to divide $A,\ldots,F$ by the appropriate factor $3!/(\tau_0!\tau_1!\tau_2!)$ and we find

$$a_{300} = a_{210} = a_{003} = a_{012} = 0,$$
$$a_{201} = A/3,\ a_{102} = B/3,\ a_{111} = C/6,\ a_{120} = D/3,\ a_{021} = E/3,\ a_{030} = F.$$

The advantage in having the curve meet the corners $\mathbf{p}_0$, $\mathbf{p}_2$ tangentially along the sides of the triangle is that four of the ten coefficients, f_0, f_1, f_8, and f_9, are immediately zero, reducing the complexity of the problem.

It is also worth pointing out that, since barycentric combinations are invariant under affine maps, λ_i, w_i, α_i, and β_i are invariant under affine transformations of $\mathbf{r}$. So f, as a function of the τ_i, is also invariant.

Example. *Let $\lambda_1 = \lambda_2 = 1/2$ and $w_1 = w_2 = 1$. Then the curve is a cubic polynomial and, since $\alpha_i = \beta_i = 3/2$, its implicitization is found to be*

$$a_{201} = 9/16,\ a_{102} = 9/16,\ a_{111} = 9/32,$$
$$a_{120} = -3/8,\ a_{021} = -3/8,\ a_{030} = -5/4.$$

§3. Degeneracy and conic sections

If f is to be of practical use, for example, to determine whether a given point in $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$ lies left, right or on the curve, it is desirable that f be positive on one side and negative on the other. The following simple analysis of f demonstrates that this is not always so even when $\mathbf{r}$ appears to be well-behaved. It turns out that $f \equiv 0$ precisely when $\mathbf{r}$ is a degree-raised rational quadratic Bézier curve, or conic section. It is therefore important to distinguish this case before implicitization and treat it separately. Proposition 3 below provides the necessary test. From a numerical point of view, one also needs to be careful when $\mathbf{r}$ is close to a conic, as f is then close to zero.

It is not surprising that f is zero when $\mathbf{r}$ is a conic since there is then no unique cubic implicitization. There are several solutions due to the surplus of degrees of freedom. However, if $\mathbf{r}$ is a conic, there is a unique (up to a scaling) quadratic implicitization which can be used instead. Indeed, if

$$\bar{\mathbf{r}}(t) = \frac{(1-t)^2\mathbf{p}_0 + 2t(1-t)v\mathbf{p}_1 + t^2\mathbf{p}_2}{(1-t)^2 + 2t(1-t)v + t^2},$$

then its implicitization [3,6] is

$$g(x,y) = \tau_1^2 - 4v^2\tau_0\tau_2. \tag{5}$$

The equivalence of f being zero to $\mathbf{r}$ being a conic section is provided by Propositions 2 and 3 below.

Proposition 2. *$f \equiv 0$ iff $\beta_1 = \alpha_2\beta_2$ and $\beta_2 = \alpha_1\beta_1$.*

Proof: Suppose that $f \equiv 0$, i.e. $A = B = \cdots = F = 0$. Since $F = 0$, neither α_1 nor α_2 can be zero, and because $D = \alpha_2(\beta_1 - \beta_2\alpha_2)$ and $E = \alpha_1(\beta_2 - \beta_1\alpha_1)$, it immediately follows that $\beta_1 = \alpha_2\beta_2$ and $\beta_2 = \alpha_1\beta_1$.

Suppose on the other hand that $\beta_1 - \alpha_2\beta_2 = 0$ and $\beta_2 - \alpha_1\beta_1 = 0$. If we let $\phi_1 = \beta_1 - \alpha_2\beta_2$ and $\phi_2 = \beta_2 - \alpha_1\beta_1$, one easily shows that

$$A = -\beta_1^2\phi_1, \quad B = -\beta_2^2\phi_2, \quad C = -\beta_2\phi_1 - \beta_1\phi_2 - \phi_1\phi_2,$$
$$D = \alpha_2\phi_1, \quad E = \alpha_1\phi_2, \quad F = (\phi_1 + \alpha_2\phi_2)/\beta_1.$$

Thus $f \equiv 0$. ■

In the following, it is important to remember that there is more than one way to express a conic as a rational cubic Bézier curve in standard form. The two operations of normalising (reparametrizing so that the curve is in standard form) and degree-raising are not commutative as pointed out by Farin [3]. With this in mind, it is to be expected that the subfamily of $\mathbf{r}$'s of the form (2) which are conics constitute a two-, rather than one-, parameter family. Put another way, there are two, rather than three, constraints on the four parameters α_1, α_2, β_1, and β_2.

Proposition 3. *$\mathbf{r}$ is a conic section iff $\beta_1 = \alpha_2\beta_2$ and $\beta_2 = \alpha_1\beta_1$.*

Proof: If $\mathbf{r}$ in (1) is a conic then it can be expressed most generally in the form

$$\mathbf{r}(u) = \frac{(1-u)^2 v_0 \mathbf{p}_0 + 2u(1-u)v_1\mathbf{p}_1 + u^2 v_2 \mathbf{p}_2}{(1-u)^2 v_0 + 2u(1-u)v_1 + u^2 v_2}. \tag{6}$$

If we then degree-raise $\mathbf{r}$ [3], we obtain

$$\mathbf{r} = \frac{B_0 v_0 \mathbf{p}_0 + B_1(v_0\mathbf{p}_0 + 2v_1\mathbf{p}_1)/3 + B_2(v_2\mathbf{p}_2 + 2v_1\mathbf{p}_1)/3 + B_3 v_2 \mathbf{p}_2}{B_0 v_0 + B_1(v_0 + 2v_1)/3 + B_2(v_2 + 2v_1)/3 + B_3 v_2}.$$

Now, scaling the weights and reparametrizing we obtain $\mathbf{r}$ in the form of (1) where

$$\lambda_1 = \frac{2v_1}{v_0 + 2v_1}, \quad \lambda_2 = \frac{2v_1}{v_2 + 2v_1}, \quad w_1 = \frac{v_0 + 2v_1}{3v_0^{2/3}v_2^{1/3}}, \quad w_2 = \frac{v_2 + 2v_1}{3v_0^{1/3}v_2^{2/3}}.$$

Consequently one finds

$$\alpha_1 = \frac{v_0^{1/3}}{v_2^{1/3}}, \quad \alpha_2 = \frac{v_2^{1/3}}{v_0^{1/3}}, \quad \beta_1 = \frac{2v_1}{v_0^{2/3}v_2^{1/3}}, \quad \beta_2 = \frac{2v_1}{v_0^{1/3}v_2^{2/3}}, \tag{7}$$

and $\beta_1 = \alpha_2\beta_2$ and $\beta_2 = \alpha_1\beta_1$. This proves the necessity.

For the sufficiency part, suppose $\beta_1 = \alpha_2\beta_2$ and $\beta_2 = \alpha_1\beta_1$. Then, letting $v_1 = 1$ it merely requires to solve for v_0 and v_2 in (7). Indeed solving in terms of the β_i one finds $v_0 = 2\beta_2/\beta_1^2$ and $v_2 = 2\beta_1/\beta_2^2$. Since all the steps in the necessity part of the proof are reversible, it is evident that $\mathbf{r}$ can be expressed as (6) with these values of v_i. ■

One cannot scale out the degeneracy in f in its most general form. However there is one noteworthy case in which a common factor can be removed, alleviating the degeneracy as $\mathbf{r}$ approaches a conic. In the symmetric case, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, the coefficients of f can be factorized and written as

$$\begin{gathered} A = B = -\beta^3(1-\alpha), \quad C = \beta^2(\alpha - 3)(1-\alpha), \\ D = E = \alpha\beta(1-\alpha), \quad F = (1+\alpha)(1-\alpha). \end{gathered}$$

Therefore all of them can be divided by $1-\alpha$ and one may define a new implicitization $\bar{f}$ with coefficients

$$\bar{A}=\bar{B}=-\beta^3, \quad \bar{C}=\beta^2(\alpha-3), \quad \bar{D}=\bar{E}=\alpha\beta, \quad \bar{F}=1+\alpha.$$

Due to Proposition 3, within this two-parameter subfamily of rational cubic curves, the conics are defined by $\alpha=1$ (while β remains arbitrary and determines the type of conic). Then

$$\bar{A}=\bar{B}=-\beta^3, \quad \bar{C}=-2\beta^2, \quad \bar{D}=\bar{E}=\beta, \quad \bar{F}=2,$$

and so $\bar{f}$ is non-trivial. Though $\bar{f}$ is not in general the same as g in (5) with degree raised to three.

§4. Uniqueness of branch

Suppose now that $\lambda_1, \lambda_2 \in (0,1)$. We wish to pose the question: is $\mathbf{r}$ the only branch of the algebraic curve $f=0$ in $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$, i.e. does f have any zeros in the triangle other than points on $\mathbf{r}$? This can be answered precisely in terms of the signs of $\phi_1=\beta_1-\alpha_2\beta_2$ and $\phi_2=\beta_2-\alpha_1\beta_1$.

Proposition 4. *Suppose $\lambda_1, \lambda_2 \in (0,1)$. Suppose further that ϕ_1 and ϕ_2 are non-zero. If ϕ_1 and ϕ_2 have the same sign then the points of $\mathbf{r}$ are the only zeros of f in $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$. If ϕ_1 and ϕ_2 have opposite sign then $f=0$ has another zero curve inside $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$.*

Proof: Since $\lambda_1, \lambda_2 \in (0,1)$, we see that all the α_i, β_i are positive.

Suppose ϕ_1 and ϕ_2 are both positive. Referring to the expressions for $A, \ldots, F$ in Proposition 2, one finds that $A, B < 0$, and $D, E, F > 0$ while the sign of C is undetermined. Now consider any straight line segment $\mathbf{l}$ starting at a point on side $\mathbf{p}_0\mathbf{p}_2$ and ending at $\mathbf{p}_1$. Along $\mathbf{l}$, f is a cubic polynomial whose four Bézier coefficients b_0, b_1, b_2, b_3, can be evaluated row by row from the $a_{\mathbf{i}}$. Due to the signs of $A, \ldots, F$, it follows that $b_0 < 0$, $b_2 > 0$, $b_3 > 0$. There is thus one sign-change in the sequence (b_i) so the variation-diminishing property for Bernstein polynomials implies that $f|\mathbf{l}$ has exactly one zero. By continuity of $\mathbf{r}$, this zero must be $\mathbf{r}(t)$ for some $t \in (0,1)$. By considering all possible segments $\mathbf{l}$ we find that the only zeros of f in $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$ are the points of $\mathbf{r}$. An identical argument applies to the case when ϕ_1 and ϕ_2 are both negative because the signs of A, B, D, E, F merely become reversed.

If ϕ_1 and ϕ_2 have opposite sign then so do A and B. This means that f has a zero somewhere along side $\mathbf{p}_0\mathbf{p}_2$. The signs of D and E are also opposite so that either D and F have opposite signs or E and F have. In either case f also has a zero on one of the sides $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_1\mathbf{p}_2$. Thus f certainly has at least one other zero curve inside $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$. ■

Thus if ϕ_1 and ϕ_2 have the same sign, f is positive on one side of $\mathbf{r}$ and negative on the other inside $\triangle \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2$. If either of the ϕ_i is zero, either A or B is zero and so f has a double point at either $\mathbf{p}_0$ or $\mathbf{p}_2$.

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