Local and Global Convexity Preservation

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Abstract. This paper concerns systems of functions which generate convex curves from convex control polygons. We discuss necessary and sufficient conditions for a system to preserve local convexity, global convexity and π-convexity.

§1. Introduction

Convexity is an important property of planar curves in geometric modelling but it has several interpretations. In this paper we emphasize locally convex curves and explore systems of functions on an interval $[a, b]$ which generate locally convex curves from locally convex control polygons. We also treat and compare globally convex and π-convex curves.

By locally convex we mean, roughly speaking, a planar parametric curve which has no inflections and keeps turning anticlockwise (or clockwise) with respect to its parameter. A locally convex curve is free to turn through any angle. In contrast, a globally convex curve is limited to turn through an angle of not more than 2π as it forms part of a closed convex curve. Global convexity preservation has recently been studied in [6] and by Carnicer and García-Esnaola [3]. We will call curves which are globally convex and which turn through an angle of at most π, π-convex. Such curves occur frequently, for example as the graphs of convex functions on an interval, or more generally as such graphs after a suitable rotation of axes. The preservation of convex functions and π-convex curves has been studied by Carnicer, García-Esnaola, and Peña [4,5].

In Section 2 we define what we mean by systems which preserve local convexity of order k and show that totally positive blending systems with a local support property belong to this category. A B-spline basis of degree $\leq k$ provides an example of such systems. In Section 3 we define globally convex curves and polygons and discuss conditions for global convexity preservation. Finally in Section 4 we propose a simple definition of π-convex curves and we characterize π-convexity preservation.

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§2. Local Convexity

Denoting the determinant of two vectors $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$ by $[p, q] := p_1 q_2 - p_2 q_1$, the usual way to specify local convexity of a twice differentiable planar curve $c : [a, b] \to \mathbb{R}^2$ is by the condition

$$[c'(t), c''(t)] \geq 0, \qquad a \leq t \leq b. \tag{1}$$

However we prefer to make a definition which does not require differentiability. We will say that an arbitrary curve $c : [a, b] \to \mathbb{R}^2$ is locally convex if there exists $\epsilon > 0$ such that

$$[c(s) - c(r), c(t) - c(s)] \geq 0, \qquad a \leq r < s < t \leq b, \quad t - r \leq \epsilon. \tag{2}$$

A frequently occurring kind of curve $p : [a, b] \to \mathbb{R}^2$ in geometric modelling takes the form

$$p(t) = \sum_{i=0}^{n} P_i u_i(t) \tag{3}$$

where $P_0, \ldots, P_n$ are points in $\mathbb{R}^2$ and $u_0, \ldots, u_n$ are functions defined on the interval $[a, b]$. The polygonal arc $P_0, \ldots, P_n$ is the control polygon of p and the sequence $u_0, \ldots, u_n$ is often referred to as a system, denoted by $(u_0, \ldots, u_n)$. We say that $(u_0, \ldots, u_n)$ is a blending system if for $t \in [a, b]$, $u_i(t) \geq 0$, for $i = 0, \ldots, n$ and $\sum_{i=0}^{n} u_i(t) = 1$. Two common examples of blending systems are the Bernstein and B-spline bases. These two bases are also totally positive [1,7]. The system $(u_0, \ldots, u_n)$ is said to be totally positive if all minors of its collocation matrices

$$M \begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix} = \big(u_j(t_i)\big)_{0 \leq i \leq n,\ 0 \leq j \leq n}$$

are nonnegative for $a \leq t_0 < \cdots < t_n \leq b$. As observed by Goodman [7], totally positive blending systems enjoy several shape-preserving properties; the curves they generate (Bézier and B-spline curves in the two examples above) tend to mimic the shapes of their control polygons.

If we identify the control polygon $P_0, \ldots, P_n$ with the piecewise linear curve $P : [0, 1] \to \mathbb{R}^2$ defined as

$$P(t) = (i + 1 - nt)P_i + (nt - i)P_{i+1}, \qquad t \in [\frac{i}{n}, \frac{i+1}{n}], \tag{4}$$

for $i = 0, 1, \ldots, n-1$, then it is not difficult to show from (2) that P is locally convex if and only if

$$[P_i - P_{i-1}, P_{i+1} - P_i] \geq 0, \qquad 1 \leq i \leq n - 1. \tag{5}$$

This leads us to a natural generalization. Let us say that the control polygon $P_0, \ldots, P_n$ is locally convex of order k, $k \geq 1$, if

$$[P_i - P_{i-1}, P_j - P_{j-1}] \geq 0, \qquad 1 \leq i < j \leq n, \qquad j - i \leq k. \tag{6}$$

When $k = 1$, condition (6) reduces to condition (5). We also note that local convexity of order k implies local convexity of order $k-1$.

Correspondingly let us say that a system $(u_0, \ldots, u_n)$ preserves local convexity of order k if the curve $p(t)$ in (3) is locally convex when its control polygon is locally convex of order k. We note that local convexity preservation of order k implies local convexity preservation of order $k+1$.

Next we characterize local convexity preservation in terms of the signs of minors of collocation matrices. Let

$$v_i(t) := \sum_{j=i}^{n} u_j(t), \qquad i = 0, 1, \ldots, n. \tag{7}$$

We note that $v_0 \equiv 1$ in the event that $(u_0, \ldots, u_n)$ is a blending system.

Proposition 2.1. *The blending system $(u_0, \ldots, u_n)$ (i) preserves local convexity of order k, if and only if (ii) $\exists \epsilon > 0$ such that for $a \le r < s < t \le b$, $t - r \le \epsilon$, and $1 \le i < j \le n$,*

$$\det M \begin{pmatrix} 1, v_i, v_j \\ r, s, t \end{pmatrix} \quad \begin{cases} \ge 0 & j - i \le k, \\ = 0 & j - i > k. \end{cases}$$

Let us remark that from standard properties of determinants,

$$\det M \begin{pmatrix} 1, v_i, v_j \\ r, s, t \end{pmatrix} = \sum_{\alpha=0}^{i-1} \sum_{\beta=i}^{j-1} \sum_{\gamma=j}^{n} \det M \begin{pmatrix} u_\alpha, u_\beta, u_\gamma \\ r, s, t \end{pmatrix}. \tag{8}$$

So the determinant in Proposition 2.1 is nonnegative if $(u_0, \ldots, u_n)$ is totally positive.

Proof: If $p(t)$ is the curve in (3) then using the fact that $\sum_i u_i = 1$,

$$p(t) = P_0 + \sum_{i=1}^{n} v_i(t)(P_i - P_{i-1}) \tag{9}$$

and it follows after a couple of lines of algebra (c.f. Theorem 5.2 of [2]) that

$$[p(s) - p(r), p(t) - p(s)] = \sum_{1 \le i < j \le n} \det M \begin{pmatrix} 1, v_i, v_j \\ r, s, t \end{pmatrix} [P_i - P_{i-1}, P_j - P_{j-1}]. \tag{10}$$

Suppose that $(u_0, \ldots, u_n)$ satisfies property (ii) and choose ϵ as in (ii). Then if the control polygon of p is locally convex of order k we find from (10) that $[p(s) - p(r), p(t) - p(s)] \ge 0$ for $a \le r < s < t \le b$ and $t - r \le \epsilon$ and so p is locally convex.

Conversely suppose $(u_0, \ldots, u_n)$ preserves local convexity of order k. For $1 \le i < j \le n$, choose $P_0 = \ldots = P_{i-1} = (0,0)$, $P_i = \ldots = P_{j-1} = (1,0)$, and $P_j = \ldots = P_n = (1,1)$. Since the control polygon of p is locally convex of order

k, p must be locally convex and so $\exists \epsilon_{ij} > 0$ such that for $a \le r < s < t \le b$ and $t - r \le \epsilon_{ij}$,

$$0 \le [p(s) - p(r), p(t) - p(s)] = \det M \begin{pmatrix} 1, v_i, v_j \\ r, s, t \end{pmatrix}.$$

Moreover, if $j - i > k$, the control polygon defined by $P_0 = \ldots = P_{i-1} = (0,0)$, $P_i = \ldots = P_{j-1} = (1,0)$, and $P_j = \ldots = P_n = (1,-1)$ is also locally convex of order k. Thus again p is locally convex and so $\exists \hat{\epsilon}_{ij} > 0$ such that for $a \le r < s < t \le b$ and $t - r \le \hat{\epsilon}_{ij}$,

$$0 \le [p(s) - p(r), p(t) - p(s)] = -\det M \begin{pmatrix} 1, v_i, v_j \\ r, s, t \end{pmatrix}.$$

Hence, letting $\epsilon = \min\{\epsilon_{ij}, \hat{\epsilon}_{ij}\}_{i,j}$ proves (ii). □

Using the characterization of Proposition 2.1, we can show that a totally positive blending system preserves local convexity provided that the supports of the functions do not overlap too much.

Corollary 2.2. *If $(u_0, \ldots, u_n)$ is a totally positive blending system and if $\mathrm{supp}(u_\alpha) \cap \mathrm{supp}(u_\beta) = \emptyset$ for $0 \le \alpha, \beta \le n$ and $|\beta - \alpha| > k+1$, then $(u_0, \ldots, u_n)$ preserves local convexity of order k.*

Proof: Let

$$\epsilon = \min\{d(\mathrm{supp}(u_\alpha), \mathrm{supp}(u_\beta)) : |\beta - \alpha| > k+1\},$$

where $d(A,B) := \min\{|s-t| : s \in A, t \in B\}$ for compact subsets A and B of $\mathbb{R}$. Then for $a \le r < s < t \le b$ and $t - r \le \epsilon$,

$$\det M \begin{pmatrix} u_\alpha, u_\beta, u_\gamma \\ r, s, t \end{pmatrix} = 0, \qquad 0 \le \alpha < \beta < \gamma \le n, \qquad \gamma - \alpha > k+1. \tag{11}$$

Indeed, since $t - r \le d(\mathrm{supp}(u_\alpha), \mathrm{supp}(u_\gamma))$, by hypothesis, either $[r,t]$ and $\mathrm{Int}(\mathrm{supp}(u_\alpha))$ are disjoint or $[r,t]$ and $\mathrm{Int}(\mathrm{supp}(u_\gamma))$ are. So either the first or third column of the matrix in equation (11) is zero.

Due to identity (8) it follows that $(u_0, \ldots, u_n)$ satisfies property (ii) of Proposition 2.1. □

Proposition 2.1 and Corollary 2.2 enable us to determine precisely which B-splines bases preserve local convexity of a given order. We note that Goodman showed in Theorem 1 of [8] that, under certain conditions, the number of inflections in a B-spline curve is bounded by the the number of inflections in the control polygon. Local convexity preservation of B-splines is clearly related to that theorem because locally convex curves are curves with no inflections. Letting

$$\tau_0 \le \tau_1 \le \ldots \le \tau_d < \tau_{d+1} < \ldots < \tau_{n+1} \le \ldots \le \tau_{n+d+1}$$

be a sequence of knots (any interior ones being simple), $n \ge d$, the B-splines

$$N_{i,d}(t) = (\tau_{i+d+1} - \tau_i)[\tau_i, \ldots, \tau_{i+d+1}](\cdot - t)_+^d, \qquad i = 0, 1, \ldots, n, \tag{12}$$

of degree d constitute a totally positive blending system $(N_{0,d}, \ldots, N_{n,d})$ on the interval $[\tau_d, \tau_{n+1}]$. Moreover, recalling the fact that the support of $N_{i,d}$ is the interval $[\tau_i, \tau_{i+d+1}]$, we are able to deduce the following statement.

Corollary 2.3. *The system* $(N_{0,d}, \ldots, N_{n,d})$ *preserves local convexity of order* k *if and only if either (i)* $k \geq d$ *or (ii)* $k = d-1$ *and* $n = d$.

Proof: Let $u_i = N_{i,d}$, $i = 0, \ldots, n$. We know that $\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset$ when $|i-j| > d+1$. So by Corollary 2.2, $(u_0, \ldots, u_n)$ preserves local convexity of order d. Furthermore, if $n = d$ then $|i-j| \leq d$ for all $i, j \in \{0, \ldots, n\}$ and so similarly, by Corollary 2.2, $(u_0, \ldots, u_n)$ preserves local convexity of order $d-1$.

Conversely, we first show that $(u_0, \ldots, u_n)$ does not preserve local convexity of order $d-2$. For, if $\tau_d < r < s < t < \tau_{d+1}$, the Schoenberg-Whitney conditions imply that the determinant

$$\det M \begin{pmatrix} u_0, u_\beta, u_d \\ r, s, t \end{pmatrix}$$

is (strictly) positive for all integers β, $0 < \beta < d$ and therefore, from equation (8),

$$\det M \begin{pmatrix} 1, v_1, v_d \\ r, s, t \end{pmatrix} > 0.$$

Therefore, due to Proposition 2.1, the system $(u_0, \ldots, u_n)$ does not preserve local convexity of order $d-2$ and therefore neither of orders $\leq d-2$.

If $n > d$ and if $\tau_d < r < \tau_{d+1} < t < \tau_{d+2}$ then the Schoenberg-Whitney conditions imply that the determinant

$$\det M \begin{pmatrix} u_0, u_\beta, u_{d+1} \\ r, \tau_{d+1}, t \end{pmatrix}$$

is positive for $0 < \beta < d+1$ and so

$$\det M \begin{pmatrix} 1, v_1, v_{d+1} \\ r, \tau_{d+1}, t \end{pmatrix} > 0.$$

Therefore, we deduce from Proposition 2.1 that $(u_0, \ldots, u_n)$ does not preserve local convexity of order $d-1$. □

Thus a quadratic Bézier curve, which is a special case of a B-spline curve with $n = d = 2$, is locally convex when its control polygon is. However a quadratic B-spline curve is in general not. This is due to the singular situation at the the internal knot τ_i when $P_{i-2} = P_{i-1}$.

§3. Global Convexity

In this section we discuss global convexity, a stronger property than local convexity. We say that a curve $c : [a, b] \to \mathbb{R}^2$ is globally convex if

$$[c(s) - c(r), c(t) - c(s)] \geq 0, \qquad a \leq r < s < t \leq b.$$

Clearly, from (2), a globally convex curve is also locally convex. It was shown in [6] that a control polygon $P_0, \ldots, P_n$ in $\mathbb{R}^2$, regarded as the piecewise linear curve P in (4), is globally convex if and only if

$$[P_j - P_i, P_k - P_j] \geq 0, \qquad 0 \leq i < j < k \leq n.$$

There are other definitions of global convexity which are equivalent except in pathological cases. For a discussion of several definitions see [3]. In contrast to local convexity preservation, it turns out that *all* totally positive blending systems preserve global convexity. Indeed, by substituting the expression for the determinant in equation (8) into equation (10) and interchanging the summations over i, j with those over α, β, γ we find

$$[p(s) - p(r), p(t) - p(s)] = \sum_{0 \leq \alpha < \beta < \gamma \leq n} \det M \begin{pmatrix} u_\alpha, u_\beta, u_\gamma \\ r, s, t \end{pmatrix} [P_\beta - P_\alpha, P_\gamma - P_\beta]$$

for $r, s, t \in [a, b]$. This identity can alternatively be deduced from a simple application of the Cauchy-Binet theorem as in [6].

A precise characterization for global convexity preservation is as yet unknown when $n \geq 4$. However it was shown in [6] that a blending system (u_0, u_1, u_2) preserves global convexity if and only if

$$\det M \begin{pmatrix} u_0, u_1, u_2 \\ r, s, t \end{pmatrix} \geq 0, \qquad a \leq r < s < t \leq b,$$

and a blending system (u_0, u_1, u_2, u_3) preserves global convexity if and only if for $a \leq r < s < t \leq b$,

$$\det M \begin{pmatrix} u_0, u_1, u_2 + u_3 \\ r, s, t \end{pmatrix} \geq 0, \quad \det M \begin{pmatrix} u_0, u_1 + u_2, u_3 \\ r, s, t \end{pmatrix} \geq 0,$$
$$\det M \begin{pmatrix} u_0 + u_1, u_2, u_3 \\ r, s, t \end{pmatrix} \geq 0, \quad \det M \begin{pmatrix} u_3 + u_0, u_1, u_2 \\ r, s, t \end{pmatrix} \geq 0.$$

§4. π-convexity

In this last section let us analyze an even stronger property of a planar curve. Let us say that a differentiable curve $c : [a, b] \to \mathbb{R}^2$ is π-convex if

$$[c'(s), c'(t)] \geq 0, \qquad a \leq s < t \leq b.$$

Using the fact that

$$[c(s) - c(r), c(t) - c(s)] = \int_r^s \int_s^t [c'(x), c'(y)]\, dx\, dy,$$

we see that a differentiable π-convex curve is also globally convex and indeed

$$\pi\text{-convex} \quad \Rightarrow \quad \text{globally convex} \quad \Rightarrow \quad \text{locally convex}.$$

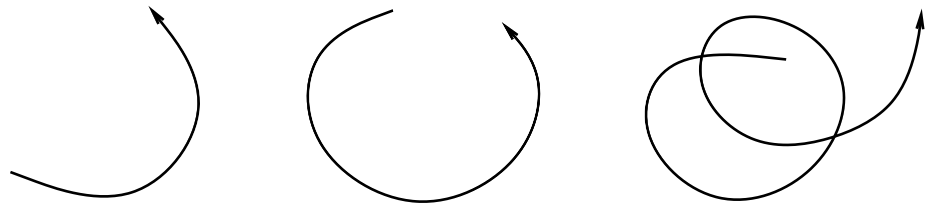

Fig. 1. π-convex, globally convex, and locally convex curves.

Figure 1 shows from left to right, π-convex, globally convex, and locally convex curves. A natural corresponding definition of π-convexity for a control polygon $P_0, \ldots, P_n$ is

$$[P_i - P_{i-1}, P_j - P_{j-1}] \geq 0, \qquad 0 \leq i < j \leq n.$$

We remark that from (6), this condition is equivalent to local convexity of order n. On the other hand, local convexity of order k is equivalent to every subpolygon $P_i, \ldots, P_{i+k}$ being π-convex, for $i = 0, \ldots, n-k$. Making now the obvious definition of π-convexity preservation we find the following.

Proposition 4.1. *A differentiable blending system $(u_0, \ldots, u_n)$ on $[a, b]$ preserves π-convexity if and only if*

$$\det M \begin{pmatrix} v_i', v_j' \\ s, t \end{pmatrix} \geq 0, \qquad a \leq s < t \leq b, \quad 1 \leq i < j \leq n, \tag{13}$$

where the functions v_i are defined in (7).

Proof: Differentiating equation (9) we find

$$\begin{aligned} [p'(s), p'(t)] &= \sum_{i,j=1}^{n} v_i'(s) v_j'(t) [P_i - P_{i-1}, P_j - P_{j-1}] \\ &= \sum_{1 \leq i < j \leq n} \det M \begin{pmatrix} v_i', v_j' \\ s, t \end{pmatrix} [P_i - P_{i-1}, P_j - P_{j-1}]. \end{aligned}$$

Thus condition (13) is sufficient for π-convexity preservation. Conversely, if $(u_0, \ldots, u_n)$ preserves π-convexity and $1 \leq i < j \leq n$, let $P_0, \ldots, P_n$ be the control polygon defined by $P_0 = \ldots = P_{i-1} = (0, 0)$, $P_i = \ldots = P_{j-1} = (1, 0)$, and $P_j = \ldots = P_n = (1, 1)$. Since $P_0, \ldots, P_n$ is π-convex we deduce that

$$0 \leq [p'(s), p'(t)] = \det M \begin{pmatrix} v_i', v_j' \\ s, t \end{pmatrix}$$

and so condition (13) is fulfilled. $\square$

Proposition 4.1 is closely related to Corollary 3.4 of [5] since π-convexity is similar to the concept of 'geometric convexity' discussed in [5].

At first, like condition (ii) of Proposition 2.1, condition (13) may seem obscure as it is defined in terms of the auxiliary functions v_i rather than u_i. Nevertheless in the case of B-splines of degree at least two (and therefore C^1), condition (13) holds. For if $u_i = N_{i,d}$, $i = 0, \ldots, n$, $d \geq 2$, and $(N_{0,d}, \ldots, N_{n,d})$ is the B-spline basis in (12), the well-known identity for derivatives of B-splines (see de Boor [1], page 138) implies

$$v_i'(t) = \sum_{j=i}^{n} N_{i,d}'(t) = \frac{d}{\tau_{i+d} - \tau_i} N_{i,d-1}(t)$$

and so

$$\begin{vmatrix} v_i'(s) & v_j'(s) \\ v_j'(t) & v_j'(t) \end{vmatrix} = \frac{d^2}{(\tau_{i+d} - \tau_i)(\tau_{j+d} - \tau_j)} \begin{vmatrix} N_{i,d-1}(s) & N_{j,d-1}(s) \\ N_{j,d-1}(t) & N_{j,d-1}(t) \end{vmatrix}.$$

The latter determinant is nonnegative when $1 \leq i < j \leq n$ and $a \leq s < t \leq b$ because the B-splines $(N_{1,d-1}, \ldots, N_{n,d-1})$ are totally positive.

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