

Log-concavity of B-splines

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Abstract

Curry and Schoenberg showed that a B-spline is log-concave in its support by applying Brunn's theorem to a simplex. In this note we provide an alternative, 'analytic' proof of the log-concave property using only recursion formulas for B-splines and their first and second derivatives.

Keywords: B-splines, log-concavity.

1 Introduction

In a ground-breaking paper, Curry and Schoenberg [4] introduced B-splines over arbitrary knot vectors and derived many of their fundamental properties. One such property is that a B-spline is log-concave inside its support. They derived this firstly by showing that the values of a B-spline of degree d are the volumes of d -dimensional slices through an appropriate $(d+1)$ -simplex, and secondly by appealing to Brunn's theorem [1, p. 71] applied to a convex set in $\mathbb{R}^{d+1}$. This shows that the d -th root of the B-spline is concave, which implies that the log of the B-spline is also concave. Brunn's theorem can be derived from the Brunn-Minkowski inequality [10].

After completing their proof, Curry and Schoenberg wrote 'It is, of course, desirable to give an analytic proof which does not use the geometry of convex domains and Brunn's theorem'. No such proof seems to have appeared in the literature, and the purpose of this note is to provide one.

2 Log-concavity and m -concavity

Let (a, b) be an open interval and consider a positive function $f : (a, b) \rightarrow \mathbb{R}$. We say that f is *log-concave* if $\log(f)$ is concave in (a, b) . We will say that f is *m -concave* for some integer $m \geq 1$ if $f^{1/m}$ is concave in (a, b) .

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Lemma 1 Suppose $f : (a, b) \rightarrow \mathbb{R}$ is positive, C^1 , and piecewise C^2 . Then f is log-concave if

$$ff'' \leq (f')^2 \quad \text{in each piece,} \quad (1)$$

and m -concave, $m \geq 1$, if

$$ff'' \leq \frac{m-1}{m}(f')^2 \quad \text{in each piece.} \quad (2)$$

Proof. This follows from differentiation: in each piece,

$$(\log(f))' = \frac{f'}{f}, \quad (\log(f))'' = \frac{1}{f^2} (ff'' - (f')^2),$$

and

$$(f^{1/m})' = \frac{1}{m} f^{(1/m)-1} f', \quad (f^{1/m})'' = \frac{f^{(1/m)-2}}{m} (ff'' - \frac{m-1}{m} (f')^2).$$

□

From this we see that for a function f as in the lemma,

$$\text{concave} \equiv 1\text{-concave} \implies 2\text{-concave} \implies \dots \implies \text{log-concave}.$$

3 B-splines and recursion formulas

Let $n \geq 1$ and $d \geq 0$ and let $\mathbf{t} = (t_1, t_2, \dots, t_{n+d+1})$ be an associated knot vector, i.e. a real vector satisfying $t_i \leq t_{i+1}$ and $t_i < t_{i+d+1}$ for all i . Then the i -th B-spline of degree d , $i = 1, \dots, n$, is

$$B_{i,d}(x) := (t_{i+d+1} - t_i)[t_i, t_{i+1}, \dots, t_{i+d+1}](\cdot - x)_+^d, \quad x \in \mathbb{R}, \quad (3)$$

where $[t_i, \dots, t_{i+d+1}]$ denotes a divided difference and $u_+ := \max\{u, 0\}$. In what follows, we will make use of various recursion formulas. The first is the basic recursion formula found in [6] and [3]. For $d \geq 1$,

$$B_{i,d}(x) = \frac{x - t_i}{t_{i+d} - t_i} B_{i,d-1}(x) + \frac{t_{i+d+1} - x}{t_{i+d+1} - t_{i+1}} B_{i+1,d-1}(x). \quad (4)$$

Regarding first derivatives, it was also shown in [6] that

$$B'_{i,d}(x) = d \left(\frac{B_{i,d-1}(x)}{t_{i+d} - t_i} - \frac{B_{i+1,d-1}(x)}{t_{i+d+1} - t_{i+1}} \right). \quad (5)$$

Differentiating this leads to an expression for second derivatives. For $d \geq 2$,

$$B''_{i,d}(x) = d \left(\frac{B'_{i,d-1}(x)}{t_{i+d} - t_i} - \frac{B'_{i+1,d-1}(x)}{t_{i+d+1} - t_{i+1}} \right). \quad (6)$$

An alternative derivative formula was derived in [7] and [2]. For $d \geq 2$,

$$B'_{i,d}(x) = \frac{d}{d-1} \left(\frac{x - t_i}{t_{i+d} - t_i} B'_{i,d-1}(x) + \frac{t_{i+d+1} - x}{t_{i+d+1} - t_{i+1}} B'_{i+1,d-1}(x) \right). \quad (7)$$

4 Log-concavity of B-splines

B-splines of degrees 0 and 1 are concave (in their support), so we only need to prove log-concavity for degrees ≥ 2 .

Lemma 2 *Let $d \geq 2$. Then in each non-empty knot interval in (t_i, t_{i+d+1}) ,*

$$B_{i,d}B''_{i,d} - \frac{d-1}{d}(B'_{i,d})^2 = -dc_{i,d}(B_{i,d-1}B'_{i+1,d-1} - B_{i+1,d-1}B'_{i,d-1}), \quad (8)$$

where

$$c_{i,d} := \frac{t_{i+d+1} - t_i}{(t_{i+d} - t_i)(t_{i+d+1} - t_{i+1})}.$$

Proof. Let $f = B_{i,d}$, $g = B_{i,d-1}$, and $h = B_{i+1,d-1}$, and define

$$a(x) := \frac{x - t_i}{t_{i+d} - t_i} \quad \text{and} \quad b(x) := \frac{t_{i+d+1} - x}{t_{i+d+1} - t_{i+1}}.$$

Then using (4) and (6),

$$ff'' = d(ag + bh)(a'g' + b'h'),$$

and using both (5) and (7),

$$\frac{(d-1)}{d}(f')^2 = d(a'g + b'h)(ag' + bh'),$$

and therefore

$$ff'' - \frac{d-1}{d}(f')^2 = d(ab' - ba')(gh' - hg'),$$

and this expands to (8) because $ab' - ba' = -c_{i,d}$. □

Let us now consider the B-spline Wronskian of first order

$$W_{i,j,k}(x) := \begin{vmatrix} B_{i,k}(x) & B_{j,k}(x) \\ B'_{i,k}(x) & B'_{j,k}(x) \end{vmatrix}. \quad (9)$$

It is well known that for any $i < j$, and any degree $k \geq 1$,

$$W_{i,j,k}(x) \geq 0 \quad \text{and} \quad W_{i,j,k}(x) > 0 \text{ iff } B_{i,k}(x), B_{j,k}(x) > 0, \quad (10)$$

as this is a special case of the ‘oscillatory’ or ‘Hermite’ generalization of the total positivity of the B-spline basis; see [11, 12, 8]. In fact one can make a specific proof of (10) just using induction on k and the recurrences (4) and (7), starting from the easy case $k = 1$. This was the approach used in the proof of Proposition 5 of [9], where the non-negativity of $W_{i,j,k}$ was used to derive the hodograph property of rational spline curves in the B-spline basis.

Since $W_{i,i+1,d-1} \geq 0$, it follows from Lemma 2 that

$$B_{i,d}B''_{i,d} - \frac{d-1}{d}(B'_{i,d})^2 \leq 0 \quad (11)$$

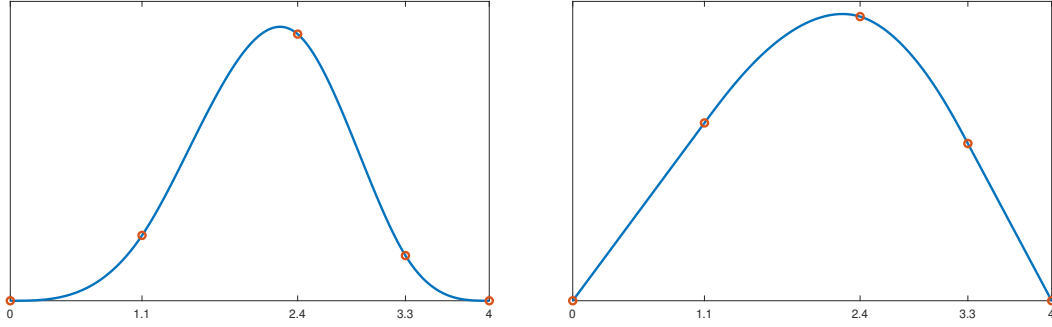


Figure 1: A cubic B-spline (left) and its cube root (right).

in any non-empty knot interval in (t_i, t_{i+d+1}) . Thus, by Lemma 1, if $B_{i,d}$ is C^1 in (t_i, t_{i+d+1}) then it is d -concave, and hence log-concave there. More precisely, the inequality (11) is strict in (t_{i+1}, t_{i+d}) and an equality in (t_i, t_{i+1}) and (t_{i+d}, t_{i+d+1}) . So $(B_{i,d}^{1/d})'' < 0$ in (t_{i+1}, t_{i+d}) and $(B_{i,d}^{1/d})'' = 0$ in (t_i, t_{i+1}) and (t_{i+d}, t_{i+d+1}) . The latter can also be seen from the fact that

$$B_{i,d}(x) = \frac{(x - t_i)^d}{\prod_{j=i+1}^{i+d} (t_j - t_i)}, \quad x \in (t_i, t_{i+1}), \quad (12)$$

and

$$B_{i,d}(x) = \frac{(t_{i+d+1} - x)^d}{\prod_{j=i+1}^{i+d} (t_{i+d+1} - t_j)}, \quad x \in (t_{i+d}, t_{i+d+1}), \quad (13)$$

which implies that the d -th root of $B_{i,d}$ is linear in (t_i, t_{i+1}) and (t_{i+d}, t_{i+d+1}) .

The only case in which $B_{i,d}$ is not C^1 in (t_i, t_{i+d+1}) is that

$$t_{i+1} = \cdots = t_{i+d} =: z.$$

In this case, due to (12) and (13),

$$B_{i,d}^{1/d} = \begin{cases} \frac{x - t_i}{z - t_i}, & t_i \leq x < z; \\ \frac{t_{i+d+1} - x}{t_{i+d+1} - z}, & z \leq x < t_{i+d+1}, \end{cases}$$

which is clearly concave.

Figure 1 shows a cubic B-spline and its cube root.

5 Final remark

A natural k -variate generalization of the B-spline arises as the volume function when projecting a $(d + k)$ -simplex orthogonally into $\mathbb{R}^k$, resulting in a k -variate piecewise polynomial of degree d , known simply as a simplex spline [7]. Brunn's theorem also applies in this more general setting and implies that simplex splines are log-concave; see [5]. An interesting question is whether there is also an 'analytic' proof of this.

References

- [1] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934.
- [2] K. R. Butterfield, *The computation of all the derivatives of a B-spline basis*, J. Inst. Maths Applies **17** (1976), 15–25.
- [3] M. G. Cox, *The numerical evaluation of B-splines*, J. Inst. Maths Applies **10** (1972), 134–149.
- [4] H. B. Curry and I. J. Schoenberg, *On Pólya frequency functions IV: the fundamental spline functions and their limits*, J. Analyse Math. **17** (1966), 71–107.
- [5] W. Dahmen and C. A. Micchelli, *On limits of multivariate B-splines*, J. Analyse Math. **39** (1981), 256–278.
- [6] C. de Boor, *On calculating with B-splines*, J. Approx. Theory **6** (1972), 50–62.
- [7] ———, *Splines as linear combinations of B-splines. A survey*, Approximation theory, II (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, eds.), Academic Press, 1976, pp. 1–47.
- [8] C. de Boor and R. DeVore, *A geometric proof of total positivity for spline interpolation*, Math. Comp. **45** (1985), 497–504.
- [9] M. S. Floater, *Evaluation and properties of the derivative of a NURBS curve*, Mathematical Methods in CAGD (T. Lyche and L. L. Schumaker, eds.), Academic Press, Boston, 1992, pp. 261–274.
- [10] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.
- [11] S. Karlin, *Total Positivity, Vol. I*, Stanford University Press, Stanford, California, 1968.
- [12] L. L. Schumaker, *Spline functions: basic theory*, Wiley, New York, 1981.