Analysis of Curve Reconstruction by Meshless Parameterization

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Abstract: This paper proposes and analyzes a method called *meshless parameterization* for reconstructing curves from unordered point samples. The method solves a linear system of equations based on convex combinations so as to map the sampled points into corresponding parameter values, whose natural ordering provides the ordering of the points. Using the theory of M-matrices, we derive natural conditions on the point sample which guarantee the correct ordering. A sufficient condition is that the underlying curve be tangent-continuous and free of self-intersections and that the sample is dense enough.

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1. Introduction

This paper studies *curve reconstruction*, the construction of a curve which approximates an unknown curve c in $\mathbb{R}^d$ given only a sample of unordered points from c. The simplest form of curve reconstruction is to fit a polygonal curve to the sampled points and then the only relevant question is how to *order* the given points. If the points are sampled densely enough from c we expect that a good algorithm will be able to order the points correctly, i.e., in the order in which they are visited by c.

We propose here a new method of curve reconstruction, called *meshless parameterization*, which is based on mapping the points into the real axis, by solving a sparse linear system, and ordering them there. Though this method requires that the first and last points, with respect to the curve, are known, it is simple and fast and applies equally well to curves in any space dimension. It also has the advantage that a polygonal curve will result no matter how badly the points are sampled.

Meshless parameterization was recently introduced in [**6**] as a method for *surface* reconstruction in the form of an interpolatory triangulation. In this case the points were mapped into a plane where they can be triangulated easily. The algorithm was shown empirically to be successful in a variety of numerical examples, both in [**6**] and [**7**].

In the current paper we describe how meshless parameterization can be applied to reconstruct curves as well as surfaces, and moreover, unlike in [**6**], we give a theoretical justification for the method. We derive sufficient discrete conditions on the sampled points which ensure that the points are ordered correctly.

2. Meshless Parameterization

Most existing methods for curve and surface reconstruction are based on concepts from computational geometry, such as Voronoi diagrams, Delaunay triangulations, and minimal

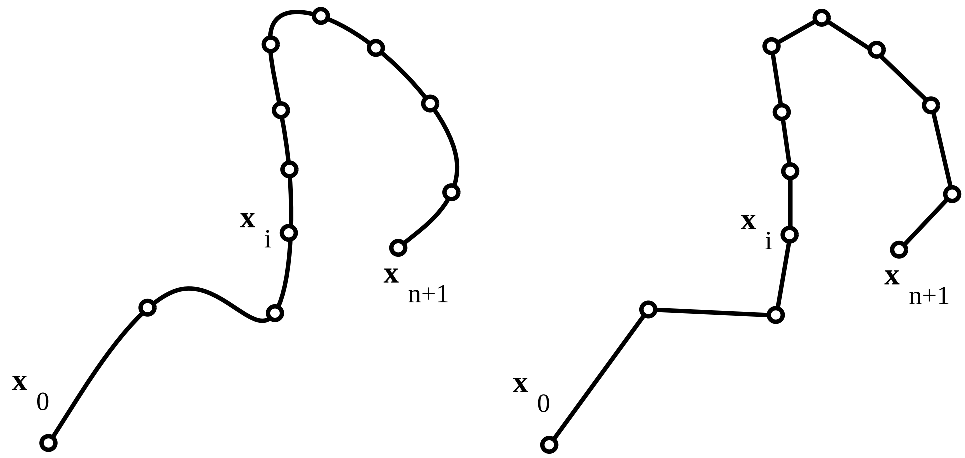

Figure 1. Curve reconstruction

spanning trees; see for example [**1,2,3,4,5,8**]. However, the method of meshless parameterization is more 'numerical' in nature and is based on estimating the parameterization of the underlying curve or surface.

Let us see how this works for curve reconstruction. We first suppose that the unknown curve is represented parametrically as a mapping $c : [a, b] \to \mathbb{R}^d$, in which case each sampled point x_i of a sample

$$x_1, x_2, \ldots, x_n,$$

is an evaluation of c at some parameter value τ_i in the parameter domain $[a, b]$, thus $x_i = c(\tau_i)$. Clearly, if we knew the parameter values $\tau_1, \tau_2, \ldots, \tau_n$, we could simply order them in increasing sequence and this would provide the correct ordering of the points x_i with respect to the curve c. The idea of meshless parameterization is simply to *estimate* the ordering of the unknown parameter values τ_i. It would clearly be a hopeless task to estimate the values τ_i themselves because no curve has a unique parametric representation. However a well-defined parametric representation $c : [a, b] \to \mathbb{R}^d$ of a curve is a *bijective* mapping. So provided the start and end points of the curve are fixed, the ordering of the points of the curve is always the same.

The method begins by letting x_0 and x_{n+1} be the end points of the curve, $x_0 = c(a)$ and $x_{n+1} = c(b)$, which could, alternatively, be two of the points in the sample. We then compute a sequence of parameter values $t_0, t_1, \ldots, t_{n+1}$ in $\mathbb{R}$ from the corresponding points $x_0, x_1, \ldots, x_{n+1}$ in $\mathbb{R}^d$. We simply let t_0 and t_{n+1} be any real values such that $t_0 < t_{n+1}$, for example $t_0 = 0$ and $t_{n+1} = 1$. We estimate the remaining parameter values by demanding, for each $i = 1, \ldots, n$, that the parameter value t_i corresponding to the data point x_i should be a convex combination of the parameter values t_j corresponding to data points x_j close to x_i. In this way we obtain a system of n linear equations

$$t_i = \frac{\sum_{j \in N_i} \lambda_{ij} t_j}{\sum_{j \in N_i} \lambda_{ij}}, \qquad i = 1, \ldots, n, \tag{2.1}$$

in the n unknowns $t_1, \ldots, t_n$, where N_i is a (small) subset of the indices $0, 1, \ldots, i-1, i+1, \ldots, n, n+1$, and the *weights* λ_{ij} are strictly positive real numbers. We will study in

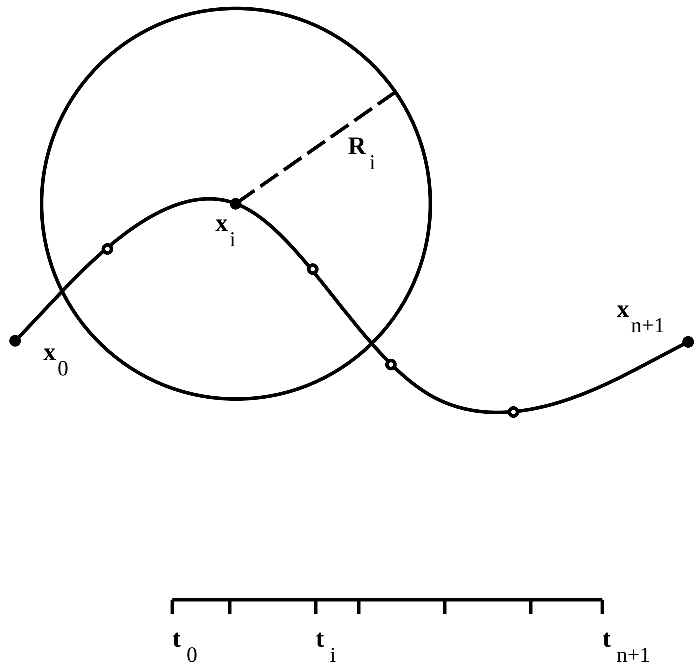

Figure 2. Meshless parameterization

particular weights which depend only on the distance between the data points, as these were found to be suitable in the examples in [**6**] and [**7**]. Thus we will consider the case

$$\lambda_{ij} = \phi_i \left(\frac{||x_j - x_i||}{R_i} \right), \tag{2.2}$$

where the value R_i is positive, and the function $\phi : [0, \infty) \to \mathbb{R}$ is positive in $[0, 1)$ and zero in $[1, \infty)$. Then

$$N_i = \{ j : 0 < ||x_j - x_i|| < R_i \},$$

and the neighbours x_j, $j \in N_i$, are those sample points contained in the ball around x_i of radius R_i, as illustrated in Figure 2. Thus if the R_i are small enough, the equation system (2.1) will be sparse and fast to solve.

We note that $\phi(||x - x_i||/R_i)$ is a radial function of x and such functions have often been used in methods for scattered data approximation [**9**], where such methods are often referred to as meshless.

It remains to choose ϕ and R_i. In [**6**], good results for surfaces were found in many cases by simply taking R_i to be constant and letting

$$\phi(r) = \begin{cases} 1/r & \text{if } 0 < r < 1; \\ 0 & \text{if } r \geq 1, \end{cases} \tag{2.3}$$

which is not even continuous over $(0, \infty)$. One could certainly design ϕ so that it is continuous and smoother, such as in Figure 3, for then the solutions t_i to the linear system (2.1) depend continuously or smoothly on the data points x_i. However, the sufficient

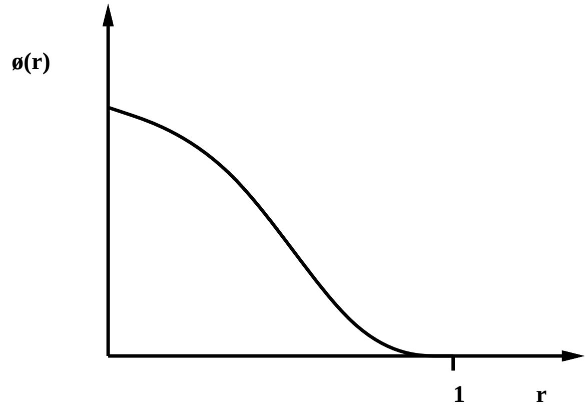

Figure 3. A possible choice of ϕ

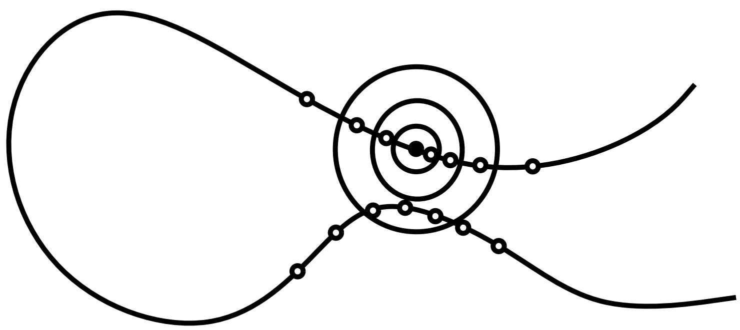

Figure 4. Choosing the radius R_i

conditions we derive later, which ensure that the t_i are correctly ordered, depend not on the smoothness of ϕ, but only on the assumption that ϕ is monotonically decreasing, as is the case in (2.3).

Letting R_i be some constant R is not, however, well suited to point samples with varying density, and even if the density is quite uniform, it is not clear how to choose R. For example, in Figure 4, three balls have been drawn around one data point. As will become clear in later sections, the best choice of the three is the middle sized radius. With the smallest radius we risk having a linear system which is not solvable, while the largest radius may yield an incorrect ordering of the parameter points.

A solution to these problems is to vary the R_i so that the set N_i is of constant size. We can choose some constant integer $k > 0$, and then for each i, we let R_i be the distance from x_i to the k-th nearest point to x_i (which may not necessarily be unique). In [**7**], it was found for surfaces that $k = 10$ and $k = 20$ were often reasonable choices in practice.

3. Monotonicity of the Solutions

In order to justify meshless parameterization as a good method for curve reconstruction we will derive conditions on the sampling which guarantee that the ordering of the points will be reproduced. Put another way, we will derive conditions which imply that if the sampled points $x_1, \ldots, x_n$ from c are already correctly ordered, then the solutions t_i to (2.1) are monotonically increasing,

$$t_0 < t_1 < \ldots < t_n < t_{n+1}.$$

Initially we derive conditions on the weights λ_{ij} themselves. Note that the equations in (2.1) can also be written as

$$\sum_{j \in N_i} \lambda_{ij}(t_j - t_i) = 0, \qquad i = 1, \ldots, n,$$

or, if we define $\lambda_{ij} = 0$ for $j \notin N_i$, as

$$\sum_{j=0}^{n+1} \lambda_{ij}(t_j - t_i) = 0, \qquad i = 1, \ldots, n. \tag{3.1}$$

We can express the latter system in the matrix form

$$At = b, \tag{3.2}$$

where

$$A = \begin{pmatrix} \sum_{j \neq 1} \lambda_{1j} & -\lambda_{12} & \cdots & -\lambda_{1n} \\ -\lambda_{21} & \sum_{j \neq 2} \lambda_{2j} & & \vdots \\ \vdots & & \ddots & -\lambda_{n-1,n} \\ -\lambda_{n1} & \cdots & -\lambda_{n,n-1} & \sum_{j \neq n} \lambda_{nj} \end{pmatrix}, \tag{3.3}$$

and

$$t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}, \qquad b = \begin{pmatrix} \lambda_{1,0} t_0 + \lambda_{1,n+1} t_{n+1} \\ \lambda_{2,0} t_0 + \lambda_{2,n+1} t_{n+1} \\ \vdots \\ \lambda_{n,0} t_0 + \lambda_{n,n+1} t_{n+1} \end{pmatrix}. \tag{3.4}$$

We note that the matrix A is *diagonally dominant* in the sense that

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \tag{3.5}$$

for all $1 \leq i \leq n$. We also observe that all the off-diagonal entries a_{ij}, $i \neq j$, are non-positive. This kind of matrix occurs frequently in the numerical solution of differential equations, and we will draw on a well-known result from that field which will be useful for establishing some basic facts about the system (3.1) as well as studying monotonicity of the solutions.

Recall that a square matrix A of order $n \geq 2$ is *irreducible* if for any two integers $i, j \in \{1, \ldots, n\}$, there exists a sequence of non-zero elements of A of the form

$$a_{i,i_1}, a_{i_1,i_2}, a_{i_2,i_3}, \ldots, a_{i_{m-1},j}.$$

For example, if all the elements $a_{i,i+1}$ and $a_{i,i-1}$ of A are non-zero, then A is irreducible. We say that an $n \times n$ matrix A is *irreducibly diagonally dominant* if A is irreducible and diagonally dominant, with strict inequality in (3.5) for at least one row i. The following result is given in [**10**].

Theorem 3.1. *If $A = (a_{ij})$ is a real, irreducibly diagonally dominant $n \times n$ matrix with $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ii} > 0$ for all $1 \leq i \leq n$, then A is nonsingular and $A^{-1} > 0$.*

We note here that by $A^{-1} > 0$, we mean that every element of A^{-1} is (strictly) positive. The matrix A in Theorem 3.1 is an example of an M-matrix. A real $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \leq 0$ for all $i \neq j$ is an *M-matrix* if A is nonsingular and $A^{-1} \geq 0$. The matrix A in Theorem 3.1 is also clearly *monotone*, namely a real square matrix with the property that $Ax \geq 0$ implies $x \geq 0$.

We use Theorem 3.1 to establish first a simple condition on the weights λ_{ij} to ensure that the solution exists and that all the parameter values lie in the interval $[t_0, t_{n+1}]$.

Theorem 3.2. *Suppose that $t_0 < t_{n+1}$ and that $\lambda_{i,i+1} > 0$ and $\lambda_{i,i-1} > 0$ for $i = 1, \ldots, n$. Then the linear system (3.1) has a unique solution and*

$$t_0 < t_i < t_{n+1}, \qquad i = 1, 2, \ldots, n.$$

Proof: The matrix A in (3.3) is diagonally dominant and since the entries $a_{i,i+1}$ and $a_{i,i-1}$ are non-zero, A is irreducible. Since, moreover, $\lambda_{1,0} > 0$ and $\lambda_{n,n+1} > 0$, the first and last rows of A are strictly diagonally dominant. The condition that $\lambda_{i,i+1} > 0$ also ensures that A has a positive diagonal. Therefore A satisfies the conditions of Theorem 3.1 and so A is nonsingular and (3.1) has a unique solution $t_1, \ldots, t_n$.

Now let $u_i = t_i - t_0$ and observe that the variables $u_0, \ldots, u_{n+1}$ satisfy the linear system

$$Au = c,$$

which is identical to (3.2) except that t_i is replaced by u_i. Since $u_0 = 0$ and $u_{n+1} > 0$, the column vector c is non-negative, and its last element is positive since

$$\lambda_{n,0}u_0 + \lambda_{n,n+1}u_{n+1} \geq \lambda_{n,n+1}u_{n+1} > 0.$$

Therefore, since from Theorem 3.1, $A^{-1} > 0$, we deduce that $u = A^{-1}c > 0$, so that $u_i > 0$ for all $i = 1, \ldots, n$, implying that $t_i > t_0$ for all $1 \leq i \leq n$. In a similar way, we deduce that $v_i = t_{n+1} - t_i > 0$ for all $1 \leq i \leq n$. ■

Next we turn to the important question of when the solutions t_i are monotonically increasing. First of all we characterize this in the first non-trivial case $n = 2$, when there are two unknowns t_1 and t_2.

Theorem 3.3. *Suppose that $n = 2$ and the conditions of Theorem 3.2 hold. Then $t_0 < t_1 < t_2 < t_3$ if and only if*

$$\begin{vmatrix} \lambda_{10} & \lambda_{13} \\ \lambda_{20} & \lambda_{23} \end{vmatrix} > 0. \tag{3.6}$$

Proof: We introduce a new set of variables $u_i = t_{i+1} - t_i$ for $i = 0, 1, 2$ and we notice from Theorem 3.2 that $u_0, u_2 > 0$ and it remains to compute u_1 in terms of t_0 and t_3. From (3.1), we find

$$-\lambda_{10}u_0 + \lambda_{12}u_1 + \lambda_{13}(u_1 + u_2) = 0,$$

$$-\lambda_{20}(u_0 + u_1) - \lambda_{21}u_1 + \lambda_{23}u_2 = 0,$$

and letting $u = t_3 - t_0$ and substituting $u_2 = u - u_0 - u_1$, we have

$$-(\lambda_{10} + \lambda_{13})u_0 + \lambda_{12}u_1 + \lambda_{13}u = 0,$$

$$-(\lambda_{20} + \lambda_{23})u_0 - (\lambda_{20} + \lambda_{21} + \lambda_{23})u_1 + \lambda_{23}u_2 = 0.$$

Finally, eliminating u_0 from these two equations we find

$$u_1 = \frac{(\lambda_{10}\lambda_{23} - \lambda_{20}\lambda_{13})u}{(\lambda_{10} + \lambda_{12} + \lambda_{13})(\lambda_{20} + \lambda_{21} + \lambda_{23}) - \lambda_{12}\lambda_{21}},$$

which means that $u_1 > 0$ is equivalent to (3.6). ■

Next we look at the case of general n and derive a sufficient condition for monotonicity of the t_i. We begin by letting

$$u_i = t_{i+1} - t_i, \qquad i = 0, 1, \ldots, n,$$

and notice from Theorem 3.2 that $u_0 > 0$, and $u_n > 0$. We will show, under a natural condition on the weights, that $u_1, \ldots, u_{n-1}$ are positive by showing that they satisfy a set of $n - 1$ equations with similar properties to those of (3.1). From (3.1),

$$-\sum_{j=0}^{i-1} \lambda_{ij} \sum_{k=j}^{i-1} u_k + \sum_{j=i+1}^{n+1} \lambda_{ij} \sum_{k=i}^{j-1} u_k = 0,$$

and exchanging summation signs we find

$$-\sum_{j=0}^{i-1} \sum_{k=0}^{j} \lambda_{ik}u_j + \sum_{j=i}^{n} \sum_{k=j+1}^{n+1} \lambda_{ik}u_j = 0, \tag{3.7}$$

This gives us n equations in the variables $u_1, \ldots, u_{n-1}$, which is one equation too many. This suggests forming a set of $n - 1$ equations by subtracting the $(i + 1)$-th equation in (3.7) from the i-th, giving

$$-\sum_{j=0}^{i-1} b_{ij}u_j + d_iu_i - \sum_{j=i+1}^{n} c_{ij}u_j = 0,$$

where

$$b_{ij} = \sum_{k=0}^{j} (\lambda_{ik} - \lambda_{i+1,k}), \quad c_{ij} = \sum_{k=j+1}^{n+1} (\lambda_{i+1,k} - \lambda_{ik}), \quad d_i = \sum_{k=i+1}^{n+1} \lambda_{ik} + \sum_{k=0}^{i} \lambda_{i+1,k}.$$

Viewing $u_1, \dots, u_{n-1}$ as the unknowns, we can write this as

$$Bu = d, \tag{3.8}$$

where

$$B = \begin{pmatrix} d_1 & -c_{12} & \cdots & -c_{1,n-1} \\ -b_{21} & d_2 & & \vdots \\ \vdots & & \ddots & \\ & & & -c_{n-2,n-1} \\ -b_{n-1,1} & \cdots & -b_{n-1,n-2} & d_{n-1} \end{pmatrix},$$

and

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}, \qquad d = \begin{pmatrix} b_{1,0}u_0 + c_{1,n}u_n \\ b_{2,0}u_0 + c_{2,n}u_n \\ \vdots \\ b_{n-1,0}u_0 + c_{n-1,n}u_n \end{pmatrix}.$$

Theorem 3.4. *Suppose the conditions of Theorem 3.2 hold. Then $t_0, \dots, t_{n+1}$ are monotonically increasing if, for all $1 \le i \le n-1$,*

(i) $b_{ij} \ge 0$ *for* $0 \le j < i$,

(ii) $c_{ij} \ge 0$ *for* $i < j \le n$, *and*

(iii) $b_{i,i-1} > 0$ *and* $c_{i,i+1} > 0$.

Proof: The column vector d in (3.8) is clearly non-negative, and since $b_{10} > 0$ and $c_{n-1,n} > 0$, the first and last elements of d are positive. So in order to show that $u = B^{-1}d > 0$, it is sufficient to show that $B^{-1} > 0$.

Since $\lambda_{i,i-1}$ and $\lambda_{i,i+1}$ are both positive for $1 \le i \le n$, also d_i is positive for $1 \le i \le n-1$, and by assumption, all off-diagonal elements of B in (3.8) are non-positive. Moreover, since $b_{i,i-1} > 0$ for $2 \le i \le n-1$ and $c_{i,i+1} > 0$ for $1 \le i \le n-2$, we see that B is irreducible. Next we notice that B is diagonally dominant in its *columns*, because, for $1 \le j \le n-1$, the sum of the elements in the j-th column is

$$\begin{aligned} s_j &= -\sum_{i=1}^{j-1} c_{ij} + d_j - \sum_{i=j+1}^{n-1} b_{ij} \\ &= -\sum_{k=j+1}^{n+1} (\lambda_{j,k} - \lambda_{1k}) + d_j - \sum_{k=0}^{j} (\lambda_{j+1,k} - \lambda_{nk}) \\ &= \sum_{k=j+1}^{n+1} \lambda_{1k} + \sum_{k=0}^{j} \lambda_{nk} \ge 0. \end{aligned}$$

Moreover, this inequality is strict for $j = 1$ and $j = n-1$, because

$$s_1 \geq \lambda_{12} > 0, \qquad \text{and} \qquad s_{n-1} \geq \lambda_{n,n-1} > 0.$$

Therefore B^T is irreducibly diagonally dominant and Theorem 3.1 shows that $(B^T)^{-1} > 0$ and so $B^{-1} > 0$. ■

From the definition of b_{ij} and c_{ij}, a simpler but stronger sufficient condition for monotonicity is clearly the following.

Corollary 3.5. *Suppose the conditions of Theorem 3.2 hold. Then $t_0, \ldots, t_{n+1}$ are monotonically increasing if, for all $1 \leq i \leq n-1$,*
(i) $\lambda_{ij} \geq \lambda_{i+1,j}$ for $0 \leq j < i$,
(ii) $\lambda_{ij} \leq \lambda_{i+1,j}$ for $i+1 < j \leq n$, and
(iii) $\lambda_{i,i-1} > \lambda_{i+1,i-1}$ and $\lambda_{i,i+2} < \lambda_{i+1,i+2}$.

4. Radial Weights

Finally, we specialize the results of the previous section to the case where we are given the sample of points $x_0, x_1, \ldots, x_{n+1}$ in $\mathbb{R}^d$ and the weights λ_{ij} are radial, defined in (2.2). We denote by $B(x, r)$ the open ball in $\mathbb{R}^d$ with centre x and radius r.

Corollary 4.1. *Suppose λ_{ij} is given by (2.2) and that ϕ is monotonically decreasing in its support* $[0,1]$. *If*

$$||x_{i\pm1} - x_i|| < R_i, \qquad \textit{for } i = 1, 2, \ldots, n, \tag{4.1}$$

and if, for $i = 1, \ldots, n-1$,

$$\frac{||x_j - x_i||}{R_i} < \frac{||x_j - x_{i+1}||}{R_{i+1}}, \qquad \textit{for } j < i \textit{ s.t. } x_j \in B(x_{i+1}, R_{i+1}), \tag{4.2}$$

and

$$\frac{||x_j - x_i||}{R_i} > \frac{||x_j - x_{i+1}||}{R_{i+1}}, \qquad \textit{for } j > i+1 \textit{ s.t. } x_j \in B(x_i, R_i), \tag{4.3}$$

then the linear system (2.1) has a unique solution and $t_0, \ldots, t_{n+1}$ are monotonically increasing, implying that the ordering $x_0, x_1, \ldots, x_{n+1}$ will be reproduced.

Proof: From (2.2), the condition (4.1) ensures the condition of Theorem 3.2. Clearly, if $x_j \notin B(x_{i+1}, R_{i+1})$, condition (i) of Corollary 3.5 is satisfied. Otherwise, since ϕ is monotonically decreasing, that condition is implied by the inequality in (4.2). By making the inequality in (4.2) strict, the first inequality of (iii) of Corollary 3.5 is also satisfied. The remaining conditions in Corollary 3.5 follows from condition (4.3) in a similar way. ■

Condition (4.1) simply demands that the two neighbours x_{i-1} and x_{i+1} of x_i should be close enough, or conversely that R_i should be chosen large enough. Let us give a geometric interpretation of the two other conditions in the case $R_i = R$. For each $i = 1, \ldots, n-1$, the equation

$$||x - x_i|| = ||x - x_{i+1}||,$$

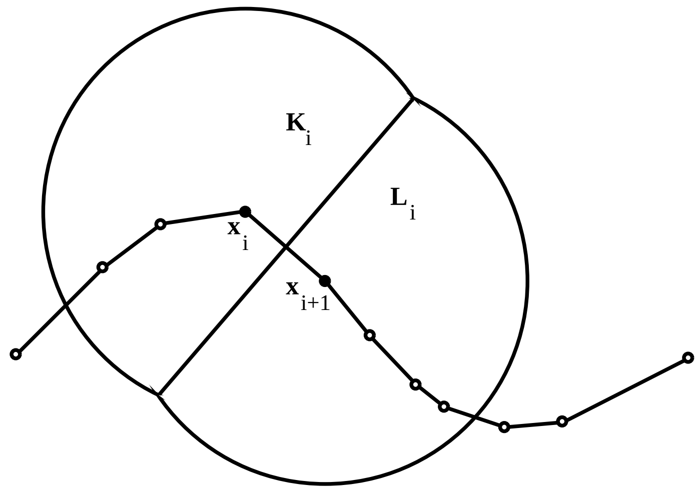

Figure 5. Sufficient condition (4.4)

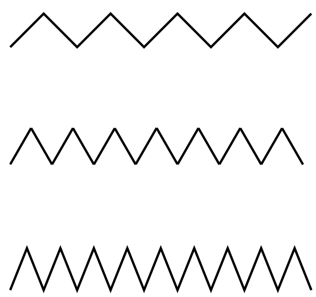

Figure 6. Monotonicity for zig-zagging polygons

for x in $\mathbb{R}^d$, which is the limiting case of the two inequalities in (4.2) and (4.3) with $R_i = R_{i+1}$, defines a hyperplane P_i which is perpendicular to the line segment $[x_i, x_{i+1}]$ and passes through its midpoint. This hyperplane partitions the union of the two balls $B(x_i, R)$ and $B(x_{i+1}, R)$ into two halves. Let K_i be the (closed) half containing x_i and L_i the closed half containing x_{i+1}. Then conditions (4.2) and (4.3) are clearly equivalent to the condition

$$x_j \notin L_i \quad \text{for } j < i \qquad \text{and} \qquad x_j \notin K_i \quad \text{for } j > i+1. \tag{4.4}$$

This condition is satisfied in Figure 5.

An instructive example of these conditions is to let the points $x_0, \ldots, x_{n+1}$ be the ordered vertices of the planar "zig-zagging" polygons of Figure 6. In order to satisfy condition (4.1), we must simply take the radius R of the ball to be any value larger than the length of the segments of the polygons. Due to symmetry considerations we would then expect $t_0, \ldots, t_{n+1}$ to be monotonically increasing. However, condition (4.4) is evidently satisfied if and only if the angle between successive line segments is greater than $\pi/3$. Thus the condition holds for the first polygon in Figure 6, but not the third. The second polygon has the largest angle, namely $\pi/3$, for which the condition fails.

For general, varying R_i, a similar geometric interpretation of conditions (4.2) and (4.3) can be made. In the general case, the hyperplane P_i becomes a hypersphere, the loci of points x in $\mathbb{R}^d$ satisfying the equation

$$\frac{||x - x_i||}{R_i} = \frac{||x - x_{i+1}||}{R_{i+1}}.$$

It is not difficult to see that if R_i is taken to be the distance of the k-th nearest sample point to x_i with k fixed, then the boundaries of the two balls $B(x_i, R_i)$ and $B(x_{i+1}, R_{i+1})$ must intersect. Thus the hypersphere P_i divides the union of $B(x_i, R_i)$ and $B(x_{i+1}, R_{i+1})$ into two regions K_i and L_i as before, and conditions (4.2) and (4.3) reduce again to (4.4).

5. Final remarks

We have derived sufficient conditions, namely (4.1) and (4.4), on a sequence of points $x_0, x_1, \ldots, x_{n+1}$ which guarantee that meshless parameterization will correctly reproduce their ordering. A topic of future research is to convert these conditions into conditions on how densely the points are sampled from a given curve c. However it is not difficult to see that conditions (4.1) and (4.4) will hold provided the curve c is tangent continuous and free of self-intersections, and the points are sampled densely enough. For in that case, the points will locally be close to lying in a straight line.

More precise conditions might be formulated in terms of the distance of points on c to the medial axis of c, a formulation which has been used by Amenta, Bern, and Eppstein for giving success criteria for the crust and β-skeleton curve reconstruction methods [**1**].

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