Transfinite mean value interpolation in general dimension

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Abstract

Mean value interpolation is a simple, fast, linearly precise method of smoothly interpolating a function given on the boundary of a domain. For planar domains, several properties of the interpolant were established in a recent paper by Dyken and the second author, including: sufficient conditions on the boundary to guarantee interpolation for continuous data; a formula for the normal derivative at the boundary; and the construction of a Hermite interpolant when normal derivative data is also available. In this paper we generalize these results to domains in arbitrary dimension.

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1 Introduction

Mean value coordinates and mean value interpolation have been developed in a series of papers [4, 7, 12, 11] and have been used for mesh parameterization, mesh deformation, and image warping in computer graphics and geometric modelling. Further work on mean value coordinates, Wachspress coordinates, and other related topics can be found in [1, 5, 6, 8, 13, 14, 15, 16, 17].

In a recent paper [3], several properties of mean value interpolation over arbitrary curves were derived, and a Hermite interpolant was constructed, based on a formula for the normal derivative. In this paper we extend the results of [3] from $\mathbb{R}^2$ to arbitrary space dimension $\mathbb{R}^n$, including the important surface case $\mathbb{R}^3$. Similar to the planar case, the weight function used to construct the Hermite interpolant behaves like a smoothed version of the signed distance function and could be used as the weight function for the web-spline method of [9, 10].

2 Lagrange interpolation

2.1 Interpolation on convex domains

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex domain. For any point $\mathbf{x}$ in Ω and any unit vector $\mathbf{v} \in \mathbb{R}^n$, let $\mathbf{p}(\mathbf{x}, \mathbf{v})$ denote the unique point of intersection between the boundary $\partial\Omega$ and the semi-infinite line $\{\mathbf{x} + \lambda\mathbf{v} : \lambda \geq 0\}$, and let $\rho(\mathbf{x}, \mathbf{v})$ be the Euclidean distance $\|\mathbf{p}(\mathbf{x}, \mathbf{v}) - \mathbf{x}\|$. The mean value interpolant $g : \Omega \to \mathbb{R}$, as proposed in [7, 12], is

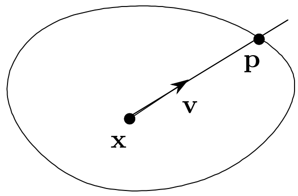

Figure 1: Notation.

$$g(\mathbf{x}) = \int_S \frac{f(\mathbf{p}(\mathbf{x}, \mathbf{v}))}{\rho(\mathbf{x}, \mathbf{v})}\, d\mathbf{v} \Big/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \int_S \frac{1}{\rho(\mathbf{x}, \mathbf{v})}\, d\mathbf{v}. \tag{1}$$

Here, $S \subset \mathbb{R}^n$ is the unit sphere, $S = \{\mathbf{v} : \|\mathbf{v}\| = 1\}$, and $d\mathbf{v}$ is the element of area at $\mathbf{v} \in S$.

2.2 A boundary integral formula

If a parametric representation of $\partial\Omega$ is available, the two integrals in (1) can be converted to integrals over the parameters of the surface, as observed by Ju, Schaefer, and Warren [12]. To this end, we make the following definitions. If $\mathbf{w}^1, \ldots, \mathbf{w}^n$ are vectors in $\mathbb{R}^n$ and $\mathbf{w}^i = (w_1^i, \ldots, w_n^i)^T$, we define the scalar and vector determinants

$$\det(\mathbf{w}^1, \ldots, \mathbf{w}^n) := \begin{vmatrix} w_1^1 & \cdots & w_1^n \\ w_2^1 & \cdots & w_2^n \\ \vdots & & \vdots \\ w_n^1 & \cdots & w_n^n \end{vmatrix}, \quad \mathbf{det}(\mathbf{w}^1, \ldots, \mathbf{w}^{n-1}) := \begin{vmatrix} \mathbf{e}_1 & w_1^1 & \cdots & w_1^{n-1} \\ \mathbf{e}_2 & w_2^1 & \cdots & w_2^{n-1} \\ \vdots & \vdots & & \vdots \\ \mathbf{e}_n & w_n^1 & \cdots & w_n^{n-1} \end{vmatrix},$$

where

$$\mathbf{e}_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})^T.$$

For example, in $\mathbb{R}^2$, $\det(\mathbf{a}, \mathbf{b}) = \mathbf{a} \times \mathbf{b}$, the 2-dimensional cross product $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$, and $\mathbf{det}(\mathbf{a}) = \mathrm{rot}(\mathbf{a}) = (a_2, -a_1)$, the rotation of $\mathbf{a}$ through an angle of $-\pi/2$. In $\mathbb{R}^3$, $\mathbf{det}(\mathbf{a}, \mathbf{b}) = \mathbf{a} \times \mathbf{b}$, the three-dimensional cross product of $\mathbf{a}$ and $\mathbf{b}$, and $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ where $\cdot$ denotes the scalar product.

Suppose that $\mathbf{s} = (s_1, \ldots, s_{n-1}) : D \to \partial\Omega$ is a parameterization of the surface $\partial\Omega$ with parameter domain $D \subset \mathbb{R}^{n-1}$. We will assume that $\mathbf{s}$ is a regular parameterization, by which we mean that $\mathbf{s}$ is continuously differentiable and that at every point $\mathbf{t} = (t_1, \ldots, t_{n-1}) \in D$, the first order partial derivatives $D_i\mathbf{s} := \partial\mathbf{s}/\partial t_i$, $i = 1, 2, \ldots, n-1$, are linearly independent. Thus the vector

$$\mathbf{s}^{\perp}(\mathbf{t}) := \mathbf{det}(D_1\mathbf{s}(\mathbf{t}), \ldots, D_{n-1}\mathbf{s}(\mathbf{t}))$$

is non-zero, and is normal to the tangent plane at $\mathbf{s}(\mathbf{t})$. We make the convention that $\mathbf{s}^\perp$ points outwards. Thus, since Ω is convex, if $\mathbf{x}$ is any point in Ω,

$$(\mathbf{s}(\mathbf{t}) - \mathbf{x}) \cdot \mathbf{s}^\perp(\mathbf{t}) > 0.$$

Theorem 1 *The function g can be written as*

$$g(\mathbf{x}) = \int_D w(\mathbf{x},\mathbf{t}) f(\mathbf{s}(\mathbf{t}))\, d\mathbf{t} \bigg/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \int_D w(\mathbf{x},\mathbf{t})\, d\mathbf{t}, \tag{2}$$

where

$$w(\mathbf{x},\mathbf{t}) = \frac{(\mathbf{s}(\mathbf{t}) - \mathbf{x}) \cdot \mathbf{s}^\perp(\mathbf{t})}{\|\mathbf{s}(\mathbf{t}) - \mathbf{x}\|^{n+1}}. \tag{3}$$

This formula is useful because it provides a way of numerically computing the value of g at a point $\mathbf{x}$ by sampling the surface $\mathbf{s}$ and its first derivatives and applying numerical integration.

Proof. It suffices to prove the formula for ϕ in (2). With $\mathbf{x} \in \Omega$ fixed, we can express $\mathbf{v}$ in (1) as $\mathbf{v} = \mathbf{r}(\mathbf{t})/r(\mathbf{t})$, where $\mathbf{r}(\mathbf{t}) := \mathbf{s}(\mathbf{t}) - \mathbf{x}$ and $r(\mathbf{t}) = \|\mathbf{r}(\mathbf{t})\|$, and changing the variable of integration from $\mathbf{v}$ to $\mathbf{t}$ gives

$$\phi(\mathbf{x}) = \int_D \frac{1}{r(\mathbf{t})} \left\| \left(\frac{\mathbf{r}}{r}\right)^\perp (\mathbf{t}) \right\| d\mathbf{t}.$$

To simplify this integral, we will show that there is some $\mu \in \mathbb{R}$ such that $(\mathbf{r}/r)^\perp = \mu \mathbf{r}$. To see this observe that for $i = 1, 2, \ldots, n-1$,

$$D_i r = D_i \left(\sum_{j=1}^n r_j^2 \right)^{1/2} = \frac{\mathbf{r} \cdot D_i \mathbf{r}}{r},$$

and therefore

$$D_i \left(\frac{\mathbf{r}}{r}\right) = \frac{D_i \mathbf{r}}{r} - \frac{(\mathbf{r} \cdot D_i \mathbf{r})\mathbf{r}}{r^3} = \frac{D_i \mathbf{s}}{r} - \frac{(\mathbf{r} \cdot D_i \mathbf{s})\mathbf{r}}{r^3}. \tag{4}$$

Using this expression we see that since the n vectors $\mathbf{r}, D_1\mathbf{s}, \ldots, D_{n-1}\mathbf{s}$ are linearly independent, so too are the $n-1$ vectors $D_i(\mathbf{r}(\mathbf{t})/r(\mathbf{t}))$, $i = 1, \ldots, n-1$. Also using (4), we see that

$$\mathbf{r} \cdot D_i \left(\frac{\mathbf{r}}{r}\right) = 0, \qquad i = 1, \ldots, n-1,$$

and so, since also

$$(\mathbf{r}/r)^\perp \cdot D_i \left(\frac{\mathbf{r}}{r}\right) = 0, \qquad i = 1, \ldots, n-1,$$

the linear independence of $D_i(\mathbf{r}/r)$, $i = 1, \ldots, n-1$ implies that $(\mathbf{r}/r)^\perp$ and $\mathbf{r}$ are scalar multiples of each other. Thus $\mathbf{d} = \mu \mathbf{r}$, as claimed, and therefore

$$\phi(\mathbf{x}) = \int_D |\mu(\mathbf{t})|\, d\mathbf{t},$$

where

$$\mu = \frac{\mathbf{r}\cdot\mathbf{d}}{\mathbf{r}\cdot\mathbf{r}} = \frac{\det(\mathbf{r}, D_1(\mathbf{r}/r), \ldots, D_{n-1}(\mathbf{r}/r))}{r^2} = \frac{\det(\mathbf{r}, D_1\mathbf{s}, \ldots, D_{n-1}\mathbf{s})}{r^{n+1}},$$

which is the same as $(\mathbf{r}\cdot\mathbf{s}^{\perp})/r^{n+1}$ which is positive. □

Though we do not use it here, we note that if $d\mathbf{y}$ denotes the element of area at $\mathbf{y} \in \partial\Omega$ then $d\mathbf{y} = \|\mathbf{s}^{\perp}(\mathbf{t})\| d\mathbf{t}$, and so a third way representing g is in terms of flux integrals:

$$g(\mathbf{x}) = \int_{\partial\Omega} f(\mathbf{y})\mathbf{F}(\mathbf{y})\cdot\mathbf{N}(\mathbf{y})\, d\mathbf{y} \Big/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \int_{\partial\Omega} \mathbf{F}(\mathbf{y})\cdot\mathbf{N}(\mathbf{y})\, d\mathbf{y},$$

where $\mathbf{F}$ is the vector field

$$\mathbf{F}(\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^{n+1}},$$

and $\mathbf{N}(\mathbf{y})$ is the outward unit normal at $\mathbf{y}$; see Lee [15].

2.3 Non-convex domains

The extension to non-convex domains follows a similar approach to that of planar domains in [3]. The boundary integral formula (2) for g is also well-defined for a non-convex domain Ω at any $\mathbf{x} \in \Omega$ where $\phi(\mathbf{x}) \neq 0$. We will show that in fact $\phi(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$ by converting the boundary integrals in (2) into integrals over the unit sphere, and by this we extend (1) to the non-convex case.

Let Ω be an arbitrary connected open domain in $\mathbb{R}^n$, not necessarily convex. Recall that the intersection between a line and a surface is said to be *transversal* if the line does not lie in the tangent plane of $\partial\Omega$ at the point of intersection. We call $\mathbf{v} \in S$ transversal with respect to $\mathbf{x}$ if all intersections between $\partial\Omega$ and $\{\mathbf{x}+\lambda\mathbf{v} : \lambda \geq 0\}$ are transversal. In that case, we let $n(\mathbf{x}, \mathbf{v})$ be the (odd) number of such intersections, assumed finite, and we let $\mathbf{p}_j(\mathbf{x}, \mathbf{v})$, $j = 1, 2, \ldots, n(\mathbf{x}, \mathbf{v})$, be the points of intersection, ordered so that their distances $\rho_j(\mathbf{x}, \mathbf{v}) = \|\mathbf{p}_j(\mathbf{x}, \mathbf{v}) - \mathbf{x}\|$ are increasing. We make the assumption that the set $\{\mathbf{v} \in S : \mathbf{v} \text{ is non-transversal}\}$ has measure zero, so that non-transversal $\mathbf{v}$ can be ignored when integrating over S. Then, changing the variable of integration from $\mathbf{t}$ to $\mathbf{v}$, we obtain

$$g(\mathbf{x}) = \int_S \sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x}, \mathbf{v})} f(\mathbf{p}_j(\mathbf{x}, \mathbf{v}))\, d\mathbf{v} \Big/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \int_S \sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x}, \mathbf{v})}\, d\mathbf{v}. \quad (5)$$

Similar to [3], due to the inequalities

$$\frac{1}{\rho_1} - \frac{1}{\rho_2} > 0, \quad \frac{1}{\rho_3} - \frac{1}{\rho_4} > 0, \quad \ldots \quad (6)$$

it follows from (5) that ϕ is positive in Ω.

2.4 Bounds on ϕ

We next generalize the upper and lower bounds on ϕ of [3], in terms of the distance $d(\mathbf{x}) = d(\mathbf{x}, \partial\Omega)$, between $\mathbf{x}$ and the boundary $\partial\Omega$. We start by recalling some basic facts about areas and volumes of spheres in $\mathbb{R}^k$. The volume of the unit sphere in $\mathbb{R}^k$ is

$$V_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)},$$

where Γ is the gamma function. Since

$$\Gamma(\frac{k}{2}+1) = \begin{cases} \left(\frac{k}{2}\right)! & \text{for even } k, \\ \sqrt{\pi} 1 \cdot 3 \cdot 5 \cdots k \cdot 2^{-(k+1)/2} & \text{for odd } k, \end{cases}$$

the first few examples are

$$V_1 = 2, \quad V_2 = \pi, \quad V_3 = 4\pi/3, \quad V_4 = \pi^2/2, \quad V_5 = 8\pi^2/15, \quad V_6 = \pi^3/6, \quad \ldots.$$

The area of the unit sphere in $\mathbb{R}^k$ can be found from its volume by the formula $A_k = kV_k$, and so

$$A_1 = 2, \quad A_2 = 2\pi, \quad A_3 = 4\pi, \quad A_4 = 2\pi^2, \quad A_5 = 8\pi^2/3, \quad A_6 = \pi^3, \quad \ldots.$$

The volume and area of a sphere in $\mathbb{R}^k$ of radius r are

$$V_k(r) = V_k r^k, \qquad \text{and} \qquad A_k(r) = A_k r^{k-1}.$$

Theorem 2 *For arbitrary Ω,*

$$\phi(\mathbf{x}) \le \frac{A_n}{d(\mathbf{x})}, \qquad \mathbf{x} \in \Omega.$$

Proof. Since

$$-\frac{1}{\rho_2} + \frac{1}{\rho_3} < 0, \quad -\frac{1}{\rho_4} + \frac{1}{\rho_5} < 0, \quad \ldots$$

in equation (5), we deduce

$$\phi(\mathbf{x}) \le \int_S \frac{1}{\rho_1(\mathbf{x}, \mathbf{v})} \, d\mathbf{v} \le \int_S \frac{1}{d(\mathbf{x})} \, d\mathbf{v} = \frac{A_n}{d(\mathbf{x})}.$$

□

Similar to [3], to derive a lower bound on ϕ, we assume that $d(M_E, \partial\Omega) > 0$. Here, $M_E \subset \mathbb{R}^n$ is the exterior medial axis [2] of $\partial\Omega$, i.e., the set of all points outside Ω whose minimal distance to $\partial\Omega$ is attained at least twice. If Ω is convex then $M_E = \emptyset$ and we define $d(M_E, \partial\Omega) = \infty$.

Theorem 3 *If* $K = \mathrm{d}(M_E, \partial\Omega) > 0$*,*

$$\phi(\mathbf{x}) \geq \frac{A_{n-1}}{d(\mathbf{x})} G_n(\alpha(\mathbf{x})), \qquad \mathbf{x} \in \Omega,$$

where $\alpha(\mathbf{x}) = \sin^{-1}(K/(d(\mathbf{x}) + K))$ *with* $0 < \alpha(\mathbf{x}) < \pi/2$*, and*

$$G_n(\alpha) = \int_0^\alpha \sin(\alpha - \beta) \sin^{n-2} \beta \, d\beta. \tag{7}$$

Since the case $K = \infty$ gives $\alpha(\mathbf{x}) = \pi/2$, and since $G_n(\pi/2) = 1/(n-1)$ we obtain a simpler bound when Ω is convex:

Corollary 1 *For convex* Ω*,*

$$\phi(\mathbf{x}) \geq \frac{V_{n-1}}{d(\mathbf{x})}, \qquad \mathbf{x} \in \Omega.$$

In the case of general Ω, note that $G_n(\alpha) > 0$ for $0 < \alpha < \pi/2$, and because the derivative of $G_n(\alpha)$ is

$$G_n'(\alpha) = \int_0^\alpha \cos(\alpha - \beta) \sin^{n-2} \beta \, d\beta > 0,$$

we see that G_n is monotonically increasing in $(0, \pi/2)$. Using these facts, we obtain a simpler lower bound on ϕ in terms of the diameter of Ω,

$$D = \sup_{\mathbf{y}_1, \mathbf{y}_2 \in \partial\Omega} \|\mathbf{y}_1 - \mathbf{y}_2\|.$$

Since $d(\mathbf{x}) \leq D$ for all $\mathbf{x} \in \Omega$, we deduce

Corollary 2 *If* $K = \mathrm{d}(M_E, \partial\Omega) > 0$*,*

$$\phi(\mathbf{x}) \geq \frac{C}{d(\mathbf{x})}, \qquad \mathbf{x} \in \Omega,$$

where $C = A_{n-1} G_n(\sin^{-1}(K/(D + K))) > 0$*.*

Proof of Theorem 3. Fix $\mathbf{x} \in \Omega$. Let $\mathbf{y}$ be some boundary point such that $d(\mathbf{x}) = \|\mathbf{y} - \mathbf{x}\|$, and let $\mathbf{v}_\mathbf{y} \in S$ be such that $\mathbf{y} = \mathbf{p}_1(\mathbf{x}, \mathbf{v}_\mathbf{y})$. Since $\mathbf{x}$ is fixed, we let $d = d(\mathbf{x})$. Then the open ball $B_1 = B(\mathbf{x}, d)$, centred at $\mathbf{x}$ with radius d, is contained in Ω. By the assumption that $K > 0$, let $\mathbf{x}^c$ be the point in Ω^c on the line through $\mathbf{x}$ and $\mathbf{y}$ whose distance from $\mathbf{y}$ is K; see Figure 2a. Then the open ball $B_2 = B(\mathbf{x}^c, K)$ is contained in Ω^c.

Let $\Gamma \subset \mathbb{R}^3$ be the set of points in ∂B_2 where lines from $\mathbf{x}$ meet ∂B_2 tangentially. When $n = 3$, Γ is a circle. In general Γ is a sphere of dimension $n - 2$. Let $\mathbf{c}_1$ be the open-ended cone with apex $\mathbf{y}$ and base Γ and let $\mathbf{c}_2$ be the open-ended cone with apex $\mathbf{x}^c$ and base Γ. By construction, if $\mathbf{z}$ is any point in Γ, the distance between $\mathbf{x}$ and $\mathbf{z}$ is

$$r = \sqrt{(d + K)^2 - K^2} = \sqrt{d^2 + 2dK},$$

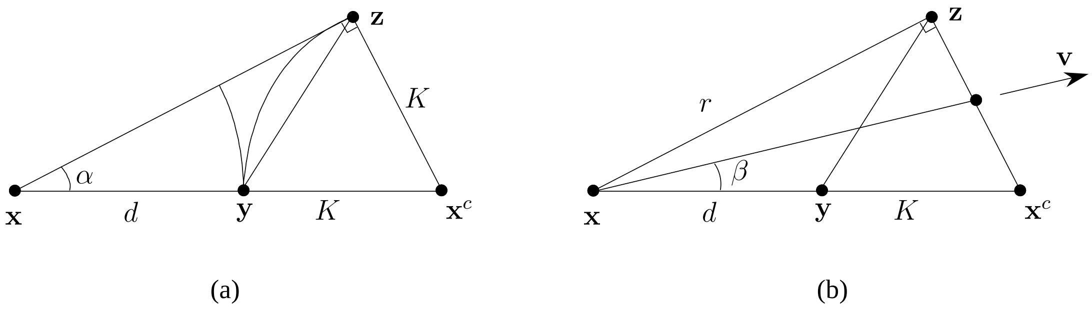

Figure 2: Notation in Theorem 3.

and denoting by α, $0 < \alpha < \pi/2$, the angle between $\mathbf{v_y}$ and the line from $\mathbf{x}$ to $\mathbf{z}$, we have

$$\sin\alpha = \frac{K}{d+K}. \tag{8}$$

Let $C_\alpha \subset S$ be the spherical disc

$$C_\alpha = \{\mathbf{v} \in S : \arccos(\mathbf{v}\cdot\mathbf{v_y}) \leq \alpha\},$$

and let $\mathbf{v}$ be any vector in C_α, transversal w.r.t. $\partial\Omega$; see Figure 2b. Then, considering Figure 2, we see that there is some odd number k, say with $k \leq n(\mathbf{x},\mathbf{v})$, such that the intersection points $\mathbf{p}_1(\mathbf{x},\mathbf{v}),\ldots,\mathbf{p}_k(\mathbf{x},\mathbf{v})$ lie between B_1 and B_2 while $\mathbf{p}_{k+1}(\mathbf{x},\mathbf{v}),\ldots,\mathbf{p}_{n(\mathbf{x},\mathbf{v})}(\mathbf{x},\mathbf{v})$ lie beyond B_2. If $k = n(\mathbf{x},\mathbf{v})$, the sum in ϕ in (5) satisfies the inequality

$$\sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})} \geq \frac{1}{\rho_k(\mathbf{x},\mathbf{v})},$$

while if $k < n(\mathbf{x},\mathbf{v})$, it satisfies

$$\sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})} \geq \frac{1}{\rho_k(\mathbf{x},\mathbf{v})} - \frac{1}{\rho_{k+1}(\mathbf{x},\mathbf{v})}.$$

Consequently, in either case, if $\rho(\mathbf{x},\mathbf{v};\mathbf{q})$ denotes the distance from $\mathbf{x}$ to the point of intersection between the line $\{\mathbf{x}+\lambda\mathbf{v} : \lambda \geq 0\}$ and any surface $\mathbf{q}$,

$$\sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})} \geq \frac{1}{\rho(\mathbf{x},\mathbf{v};\mathbf{c}_1)} - \frac{1}{\rho(\mathbf{x},\mathbf{v};\mathbf{c}_2)},$$

and therefore, from (5),

$$\phi(\mathbf{x}) \geq \int_{C_\alpha} \left(\frac{1}{\rho(\mathbf{x},\mathbf{v};\mathbf{c}_1)} - \frac{1}{\rho(\mathbf{x},\mathbf{v};\mathbf{c}_2)} \right) d\mathbf{v}. \tag{9}$$

Now, for each $\mathbf{v} \in C_\alpha$, if β denotes the angle between $\mathbf{v}$ and $\mathbf{v_y}$, $0 \leq \beta \leq \alpha$, then

$$\frac{1}{\rho(\mathbf{x}, \mathbf{v}; \mathbf{c}_1)} = \frac{\sin(\alpha - \beta)}{\sin \alpha} \frac{1}{d} + \frac{\sin \beta}{\sin \alpha} \frac{1}{r},$$
$$\frac{1}{\rho(\mathbf{x}, \mathbf{v}; \mathbf{c}_2)} = \frac{\sin(\alpha - \beta)}{\sin \alpha} \frac{1}{d + K} + \frac{\sin \beta}{\sin \alpha} \frac{1}{r},$$

and so the integral in (9) reduces to

$$\int_{C_\alpha} F(\beta) \, d\mathbf{v}$$

where

$$F(\beta) = \frac{\sin(\alpha - \beta)}{\sin \alpha} \frac{K}{d(d + K)} = \frac{\sin(\alpha - \beta)}{d},$$

the second step being due to (8). We now change the variable of integration to β by viewing the spherical disc C_α as sliced up into spheres of dimension $n - 2$ (circles in the case $n = 3$):

$$\phi(\mathbf{x}) \geq \int_{C_\alpha} F(\beta) \, d\mathbf{v} = \int_0^\alpha A_{n-1}(\sin \beta) F(\beta) \, d\beta.$$

Since $A_{n-1}(\sin \beta) = A_{n-1} \sin^{n-2} \beta$, we thus find

$$\phi(\mathbf{x}) \geq \frac{A_{n-1}}{d(\mathbf{x})} G_n(\alpha(\mathbf{x})),$$

with G_n as in (7). □

2.5 Proof of interpolation

We can now prove that g really interpolates f under the medial axis condition of Theorem 3. We also make the mild assumption that the maximum number of intersections between any straight line and the surface $\partial\Omega$ is bounded, in which case,

$$N := \sup_{\mathbf{x} \in \Omega} \sup_{\mathbf{v} \in T(\mathbf{x})} n(\mathbf{x}, \mathbf{v}) < \infty, \tag{10}$$

where $T(\mathbf{x}) = \{\mathbf{v} \in S : \mathbf{v} \text{ is transversal w.r.t. } \mathbf{x}\}$. Note that this holds for convex Ω, in which case $N = 1$.

Theorem 4 *If f is continuous on $\partial\Omega$ and $\mathrm{d}(M_E, \partial\Omega) > 0$ then g interpolates f.*

Proof. Similar to [3], if $\mathbf{s}(\mathbf{u})$ is a boundary point and $\mathbf{x} \in \Omega$, then by (2),

$$g(\mathbf{x}) - f(\mathbf{s}(\mathbf{u})) = \frac{1}{\phi(\mathbf{x})} \int_D w(\mathbf{x}, \mathbf{t}) \big(f(\mathbf{s}(\mathbf{t})) - f(\mathbf{s}(\mathbf{u}))\big) \, d\mathbf{t}.$$

For some small $\gamma > 0$ we then split the integral into two parts, $\int_D = \int_I + \int_J$, where $I = B(\mathbf{u}, \gamma)$, a ball in $\mathbb{R}^{n-1}$, and $J = D \setminus I$. Then, with $M := \sup_{\mathbf{y}\in\partial\Omega} |f(\mathbf{y})|$,

$$|g(\mathbf{x}) - f(\mathbf{s}(\mathbf{u}))| \leq \max_{\mathbf{t}\in I} |f(\mathbf{s}(\mathbf{t})) - f(\mathbf{s}(\mathbf{u}))| \frac{1}{\phi(\mathbf{x})} \int_I |w(\mathbf{x}, \mathbf{t})| \, d\mathbf{t} + 2M \frac{1}{\phi(\mathbf{x})} \int_J |w(\mathbf{x}, \mathbf{t})| \, d\mathbf{t}.$$

Then, since by Corollary 2,

$$\frac{C}{d(\mathbf{x})} \leq \phi(\mathbf{x}) \leq \int_D |w(\mathbf{x}, \mathbf{t})| \, d\mathbf{t} \leq \phi(\mathbf{x}) + \frac{2A_n(N-1)}{d(\mathbf{x})},$$

with N as in (10). the rest of the proof follows that of Theorem 4 of [3]. □

3 Hermite interpolation

We next extend the construction of the Hermite interpolant of [3] to $\mathbb{R}^n$. Given the values and inward normal derivatives of a function f defined on $\partial\Omega$, we seek a function $p : \overline{\Omega} \to \mathbb{R}$ satisfying

$$p(\mathbf{y}) = f(\mathbf{y}) \qquad \text{and} \qquad \frac{\partial p}{\partial \mathbf{n}}(\mathbf{y}) = \frac{\partial f}{\partial \mathbf{n}}(\mathbf{y}), \qquad \mathbf{y} \in \partial\Omega, \tag{11}$$

in the form

$$p(\mathbf{x}) = g(\mathbf{x}) + \psi(\mathbf{x})\hat{g}(\mathbf{x}), \tag{12}$$

where g is the Lagrange mean value interpolant to f, ψ is the weight function $\psi(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})}$, and $\hat{g}$ is a second Lagrange mean value interpolant whose data is yet to be decided. We need to show that $\psi(\mathbf{y}) = 0$ and $\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) \neq 0$ for $\mathbf{y} \in \partial\Omega$, and we then obtain (11) by setting

$$\hat{g}(\mathbf{y}) = \left(\frac{\partial f}{\partial \mathbf{n}}(\mathbf{y}) - \frac{\partial g}{\partial \mathbf{n}}(\mathbf{y}) \right) \Big/ \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}), \qquad \mathbf{y} \in \partial\Omega. \tag{13}$$

Thus, we also need to determine $\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y})$ and $\frac{\partial g}{\partial \mathbf{n}}(\mathbf{y})$. We treat each of these requirements in turn.

First, observe that Theorem 2 and Corollary 2 give, under the condition of the latter, the upper and lower bounds

$$\frac{1}{A_n} d(\mathbf{x}) \leq \psi(\mathbf{x}) \leq \frac{1}{C} d(\mathbf{x}), \qquad \mathbf{x} \in \Omega,$$

and so $\psi(\mathbf{x}) \to 0$ as $\mathbf{x} \to \partial\Omega$, and so ψ extends continuously to $\partial\Omega$ with value zero there.

Next we show that ψ has a non-zero normal derivative.

Theorem 5 *If* $\mathrm{d}(M_E, \partial\Omega) > 0$ *and* $\mathrm{d}(M_I, \partial\Omega) > 0$ *and* $\mathbf{y} \in \partial\Omega$ *then*

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{V_{n-1}}.$$

Hence, in $\mathbb{R}^2$ the normal derivative is $1/2$, as found in [3], while in $\mathbb{R}^3$ it is $1/\pi$.

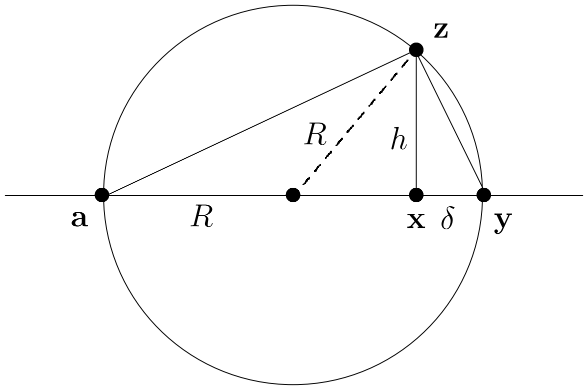

Figure 3: Symbols used in proof of Theorem 5.

Proof. Let $R = \mathrm{d}(M_I, \partial\Omega)$ and let $B \subset \Omega$ be the open ball of radius R that is tangential to $\partial\Omega$ at $\mathbf{y}$. Let $\mathbf{a}$ be the point in ∂B antipodal to $\mathbf{y}$. For small enough $\delta > 0$, the point $\mathbf{x} = \mathbf{y} + \delta\mathbf{n}$ is in B. Let $\Gamma \subset \mathbb{R}^3$ be the $(n-2)$-dimensional sphere (a circle when $n = 3$) in ∂B_2 with centre $\mathbf{x}$. Let $\mathbf{c}$ be the union of the two cones, both with base Γ, and one with apex $\mathbf{y}$ and the other with apex $\mathbf{a}$. Figure 3 illustrates the construction, where $\mathbf{z}$ is any point in Γ.

Then

$$\phi(\mathbf{x}) \leq \int_S \frac{1}{\rho(\mathbf{x}, \mathbf{v}; \mathbf{c})}\, d\mathbf{v}.$$

Like in the proof of Theorem 3, consider some $\mathbf{v} \in S$ and let β be the angle it makes with $-\mathbf{n}$, $0 \leq \beta \leq \pi$. Then

$$\frac{1}{\rho(\mathbf{x}, \mathbf{v}; \mathbf{c})} = F(\beta) = \begin{cases} \sin(\pi/2 - \beta)\dfrac{1}{\delta} + \sin\beta\dfrac{1}{h} & 0 \leq \beta \leq \pi/2, \\ \sin(\pi - \beta)\dfrac{1}{h} + \sin(\beta - \pi/2)\dfrac{1}{2R - \delta} & \pi/2 \leq \beta \leq \pi, \end{cases}$$

where $\delta = \|\mathbf{y} - \mathbf{x}\|$ and h is the radius of Γ. Therefore,

$$\int_S \frac{1}{\rho(\mathbf{x}, \mathbf{v}; \mathbf{c})}\, d\mathbf{v} = \int_0^\pi A_{n-1}(\sin\beta)F(\beta)\, d\beta,$$

and so

$$\phi(\mathbf{x}) \leq A_{n-1}\left(\frac{J_1}{\delta} + \frac{J_2}{h} + \frac{J_3}{2R - \delta}\right),$$

where

$$J_1 = \int_0^{\pi/2} \sin(\pi/2 - \beta) \sin^{n-2} \beta \, d\beta,$$
$$J_2 = \int_0^{\pi/2} \sin^{n-1} \beta \, d\beta + \int_{\pi/2}^{\pi} \sin(\pi - \beta) \sin^{n-2} \beta \, d\beta,$$
$$J_3 = \int_{\pi/2}^{\pi} \sin(\beta - \pi/2) \sin^{n-2} \beta \, d\beta.$$

Since for small δ, $h = \sqrt{(2R - \delta)\delta} \approx \sqrt{2R\delta}$, and because $J_1 = 1/(n-1)$, and recalling that $\mathbf{x} = \mathbf{y} + \delta\mathbf{n}$, we deduce that

$$\limsup_{\delta \to 0} \delta\phi(\mathbf{y} + \delta\mathbf{n}) \le A_{n-1}/(n-1) = V_{n-1}.$$

On the other hand, since $G_n(\pi/2) = 1/(n-1)$ in Theorem 3, we have

$$\liminf_{\delta \to 0} \delta\phi(\mathbf{y} + \delta\mathbf{n}) \ge A_{n-1} G_n(\pi/2) = A_{n-1}/(n-1) = V_{n-1},$$

and so

$$\delta\phi(\mathbf{y} + \delta\mathbf{n}) \to V_{n-1} \quad \text{as } \delta \to 0.$$

Thus,

$$\frac{\partial\psi}{\partial\mathbf{n}}(\mathbf{y}) = \lim_{\delta \to 0} \frac{\psi(\mathbf{y} + \delta\mathbf{n}) - \psi(\mathbf{y})}{\delta} = \lim_{\delta \to 0} \frac{1}{\delta\phi(\mathbf{y} + \delta\mathbf{n})} = \frac{1}{V_{n-1}}.$$

□

We have now shown that the Hermite construction (12) works, and that the normal derivative of ψ is $1/V_{n-1}$. To be able to apply (13) it remains to compute the normal derivative of g.

Theorem 6 *Let g be as in* (2)*. If* $\mathrm{d}(M_E, \partial\Omega) > 0$ *and* $\mathrm{d}(M_I, \partial\Omega) > 0$*, and* $\mathbf{y} \in \partial\Omega$ *then*

$$\frac{\partial g}{\partial\mathbf{n}}(\mathbf{y}) = \frac{1}{V_{n-1}} \int_D w(\mathbf{y}, \mathbf{t}) \big(f(\mathbf{s}(\mathbf{t})) - f(\mathbf{y})\big) \, d\mathbf{t}.$$

Proof. For small $\delta > 0$, let $\mathbf{x} = \mathbf{y} + \delta\mathbf{n}$. Then

$$g(\mathbf{x}) - f(\mathbf{y}) = \frac{1}{\phi(\mathbf{x})} \int_D w(\mathbf{x}, \mathbf{t}) \big(f(\mathbf{s}(\mathbf{t})) - f(\mathbf{y})\big) \, d\mathbf{t},$$

and dividing both sides by δ and letting $\delta \to 0$ gives the result since $\delta\phi(\mathbf{x}) \to V_{n-1}$. □

4 A minimum principle

It was shown in [3] that in $\mathbb{R}^2$ the weight function $\psi = 1/\phi$ has no local minima in Ω due to a simple formula for the Laplacian of ϕ. As we will see, this property extends to $\mathbb{R}^n$.

Lemma 1 *For arbitrary Ω, with ϕ given by (2) or (5),*

$$\Delta\phi(\mathbf{x}) = (n+1)\int_D \frac{w(\mathbf{x},\mathbf{t})}{\|\mathbf{s}(\mathbf{t})-\mathbf{x}\|^2}\,d\mathbf{t} = (n+1)\int_S \sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j^3(\mathbf{x},\mathbf{v})}\,d\mathbf{v}.$$

Proof. It is sufficient to prove that

$$\Delta w = (n+1)w/r^2,$$

where $r(\mathbf{t}) = \|\mathbf{s}(\mathbf{t}) - \mathbf{x}\|$, and since the vector $\mathbf{s}^\perp$ in (3) are constant with respect to $\mathbf{x}$, it is enough to prove that if $w_i = r_i/r^{n+1}$, $i = 1,\ldots,n$, where $r_i(\mathbf{t}) = s_i(\mathbf{t}) - x_i$, then

$$\Delta w_i = (n+1)w_i/r^2. \tag{14}$$

Using the fact that $(\partial/\partial x_i)r = -r_i/r$ and $(\partial/\partial x_i)r_i = -1$, direct differentiation of w_i gives

$$\frac{\partial}{\partial x_i}w_i = \frac{(n+1)r_i^2}{r^{n+2}} - \frac{1}{r^{n+1}}, \qquad \frac{\partial^2}{\partial x_i^2}w_i = \frac{(n+1)(n+3)r_i^3}{r^{n+5}} - \frac{3(n+1)r_i}{r^{n+3}},$$

and for $j \neq i$,

$$\frac{\partial}{\partial x_j}w_i = \frac{(n+1)r_ir_j}{r^{n+3}}, \qquad \frac{\partial^2}{\partial x_j^2}w_i = \frac{(n+1)(n+3)r_ir_j^2}{r^{n+5}} - \frac{(n+1)r_i}{r^{n+3}}.$$

Therefore, after some cancellation, we find

$$\Delta w_i = \frac{\partial^2}{\partial x_i^2}w_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\partial^2}{\partial x_j^2}w_i = \frac{(n+1)r_1}{r^{n+3}},$$

which proves (14). □

Due to the inequalities (6), Lemma 1 shows that $\Delta\phi > 0$ in Ω. Since the weight function $\psi = 1/\phi$ has the Laplacian

$$\Delta\psi = -\frac{\Delta\phi}{\phi^2} + 2\frac{|\nabla\phi|^2}{\phi^3},$$

we deduce, as in the $\mathbb{R}^2$ case of [3], that the weight function ψ has no local minima.

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