Linear Convexity Conditions for Rectangular and Triangular Bernstein–Bézier Surfaces

J. M. Carnicer [a,1,2], M. S. Floater [b,1] and J. M. Peña [a,1,2]

[a] *Departamento de Matemática Aplicada, Universidad de Zaragoza, Spain*

[b] *SINTEF, P. O. Box 124 Blindern, 0314 Oslo, Norway*

Abstract

The goal of this paper is to derive linear convexity conditions for Bernstein–Bézier surfaces defined on rectangles and triangles. Previously known linear conditions are improved on, in the sense that the new conditions are weaker. Geometric interpretations are provided.

1 Introduction

One of the most important properties of surfaces occurring in Computer-Aided Geometric Design is convexity. Several authors [1–4,6,7] have derived sufficient conditions to ensure the convexity of triangular and tensor-product Bernstein–Bézier surfaces. Further references on this topic can be found in [5].

Convexity conditions occur in surface fairing under convexity constraints and in convexity-preserving approximation or interpolation of scattered data. Typically, a nonlinear (usually quadratic) programming problem arises, where the objective function takes care of the fairness, approximation or interpolation properties of the surface under construction, and the constraints include the convexity conditions. In this context, it is an advantage if the convexity constraints are linear and so we concentrate on linear convexity conditions in this paper.

Historically, convexity conditions were first derived for triangular Bézier surfaces. Chang and Davis [2] showed that a sufficient condition for such a surface to be convex is that the control net is convex. This is a linear condition on

[1] Partially supported by the project CHRX–CT94–0522, European Union.
[2] Partially supported by DGICYT PB93–0310 Research Grant, Spain.

the control points and was also studied in [4]. Chang and Feng [3] derived weaker but nonlinear conditions. Lai [7] derived linear conditions weaker than those of Chang and Davis. In Section 4, we present linear conditions which are weaker than all linear conditions mentioned above. Our conditions have the advantage that they are symmetric with respect to the three barycentric coordinates. Moreover, we can interpret them geometrically in terms of the convexity of quadrilaterals defined by control points.

Later, convexity conditions were derived for bivariate tensor-product Bézier surfaces. Cavaretta and Sharma [1] showed that when the (bilinear) control net of such a surface is convex then so is the surface. Though the conditions are linear, they are very restrictive since they imply that the surface $S(x,y)$ is translational, that is $S(x,y) = f(x) + g(y)$, for some some univariate functions f and g. Weaker but nonlinear conditions were later obtained by Floater [6]. Starting from these latter conditions we derive, in Section 3, linear conditions for convexity which do not demand that S is translational. Even though these conditions are stronger than those in [6], they have the advantage of being linear and moreover we are able to interpret them geometrically, in a similar way to the triangular case.

In Sections 3 and 4 the domains of definition for the surfaces are respectively the unit square $[0,1] \times [0,1]$ and the standard 2-simplex, i.e. the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. However our results are equally valid when the surfaces are defined on rectangles with sides parallel to the x and y axes and on arbitrary triangles. This is because the convexity conditions discussed in this paper are invariant under any affine change of the domain.

All the conditions in this paper involve the positive semidefiniteness of 2×2 symmetric matrices. For this reason, we have chosen to collect together all auxiliary matrix results in Section 2.

2 Auxiliary matrix properties

Let A be a symmetric 2×2 matrix

$$A = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \tag{1}$$

with real coefficients. Then A is positive semidefinite if and only if $x \geq 0$, $z \geq 0$ and $xz \geq y^2$. Since positive semidefiniteness is quadratic in terms of the matrix elements, it cannot be expressed equivalently by a finite number of linear inequalities. However we will study finite sets of linear sufficient conditions for positive semidefiniteness.

A first simple condition consists of imposing diagonal dominance to A, that is

$$x \geq |y|, \quad z \geq |y|, \tag{2}$$

or equivalently

$$x + y \geq 0, \quad x - y \geq 0, \quad x + z \geq 0, \quad x - z \geq 0.$$

Now we derive a one-parameter family of weaker linear conditions which will be applied to the convexity of tensor-product Bernstein–Bézier surfaces in Section 3.

Proposition 1 *Let $\lambda \geq 1$ and let x, y, z be real numbers such that*

$$x \geq |y|/\lambda,\ z \geq |y|/\lambda,\ x + \lambda z \geq (\lambda + 1)|y|,\ \lambda x + z \geq (\lambda + 1)|y|. \tag{3}$$

Then the matrix A in (1) is positive semidefinite.

Proof. Without loss of generality we may assume that $x \geq z$. If $z \geq |y|$ then $x \geq z \geq |y| \geq 0$ and $xz \geq y^2$. Therefore we may assume that $z < |y|$. Using (3) we know that $x + \lambda z - (\lambda + 1)|y| \geq 0$, $z - |y|/\lambda \geq 0$, $z \geq 0$ and since $|y| - z > 0$ we have

$$xz - y^2 = (x + \lambda z - (\lambda + 1)|y|)z + \lambda(|y| - z)(z - |y|/\lambda) \geq 0.$$

□

Remark 2 *Let us observe that when $\lambda = 1$ the conditions (2) and (3) are equivalent. Furthermore, when $\lambda > 1$ conditions (2) imply (3). Therefore the linear conditions*

$$x \geq y/\lambda, \quad x \geq -y/\lambda, \quad z \geq y/\lambda, \quad z \geq -y/\lambda,$$

$$x + \lambda z \geq (\lambda + 1)y, \quad x + \lambda z \geq -(\lambda + 1)y,$$

$$\lambda x + z \geq (\lambda + 1)y, \quad \lambda x + z \geq -(\lambda + 1)y$$

are weaker than (2). For $\lambda = 2$ we obtain

$$x \geq |y|/2, \quad z \geq |y|/2, \quad x + 2z \geq 3|y|, \quad 2x + z \geq 3|y|. \tag{4}$$

Now we shall give a set of linear conditions which are stronger than (4) but which will play an important role in the study of convexity conditions for triangular Bernstein–Bézier surfaces in Section 4.

Proposition 3 *If*

$$\begin{array}{llll} \text{(i)} & x \geq -y, & \text{(iv)} & z \geq y/2, \\ \text{(ii)} & z \geq -y, & \text{(v)} & z + 2x \geq 3y, \\ \text{(iii)} & x \geq y/2, & \text{(vi)} & x + 2z \geq 3y, \end{array} \tag{5}$$

then (4) follows.

Proof. If $y \geq 0$ then (4) follows from (iii) to (vi). If $y < 0$ then (4) follows from (i) and (ii). □

In order to look more closely at conditions which can be applied to the convexity of triangular Bernstein–Bézier surfaces, let us assume that the matrix A in (1) is given in the form

$$A = \begin{pmatrix} \Delta_a + \Delta_c & \Delta_c \\ \Delta_c & \Delta_b + \Delta_c \end{pmatrix}. \tag{6}$$

Chang and Feng (formula (13) of [3]) showed that A is positive semidefinite if and only if the following conditions, which are symmetric in a, b, c, hold:

$$\begin{array}{l} \Delta_a + \Delta_b \geq 0, \quad \Delta_b + \Delta_c \geq 0, \quad \Delta_c + \Delta_a \geq 0, \\ \Delta_b\Delta_c + \Delta_c\Delta_a + \Delta_a\Delta_b \geq 0. \end{array} \tag{7}$$

Now, if we substitute $x = \Delta_a + \Delta_c$, $y = \Delta_c$ and $z = \Delta_b + \Delta_c$ in conditions (5) we obtain the symmetric linear conditions

$$\begin{array}{lll} \Delta_a + 2\Delta_c \geq 0, & \Delta_b + 2\Delta_c \geq 0, & \Delta_c + 2\Delta_a \geq 0, \\ \Delta_c + 2\Delta_b \geq 0, & \Delta_b + 2\Delta_a \geq 0, & \Delta_a + 2\Delta_b \geq 0. \end{array} \tag{8}$$

On the other hand, diagonal dominance in (6) is equivalent to

$$\Delta_a \geq 0, \quad \Delta_b \geq 0, \quad \Delta_a + 2\Delta_c \geq 0, \quad \Delta_b + 2\Delta_c \geq 0, \tag{9}$$

which is in turn a reformulation of (2) and stronger than (8). Finally the conditions

$$\Delta_a \geq 0, \quad \Delta_b \geq 0, \quad \Delta_c \geq 0, \tag{10}$$

which were used by Chang and Davis [2], are clearly even stronger than (9).

Let us now interpret conditions (7)–(10) geometrically. Observe that (7), (8) and (10) are symmetric in $\Delta_a, \Delta_b, \Delta_c$ and that (7), (8), (9) and (10) are homogeneous. Further note that, from (7), the three conditions

$$\Delta_a + \Delta_b \geq 0, \quad \Delta_b + \Delta_c \geq 0, \quad \Delta_c + \Delta_a \geq 0$$

are necessary for positive semidefiniteness.

Let us assume that $\Delta_a + \Delta_b > 0$ and define $r := (\Delta_b + \Delta_c)/(\Delta_a + \Delta_b)$ and $s := (\Delta_c + \Delta_a)/(\Delta_a + \Delta_b)$. Then conditions (7) can be expressed as a single inequality

$$(r - s)^2 - 2(r + s) + 1 \leq 0. \tag{11}$$

This represents the region R enclosed by a parabola (see Figure 1).

On the other hand, conditions (8) become

$$\begin{array}{ll} r - s \leq 3, & s - r \leq 3, \\ s - 1 \leq 3r, & r - 1 \leq 3s, \\ 1 - r \leq 3s, & 1 - s \leq 3r, \end{array} \tag{12}$$

These conditions represent a six-sided convex unbounded polygon R_1 contained within the parabola. Now conditions (9) become

$$r + 1 \geq s, \quad s + 1 \geq r, \quad 3r + s \geq 1, \quad 3s + r \geq 1, \tag{13}$$

representing a four sided figure R_2 contained in the previous one. Finally, conditions (10) can be represented as the half strip R_3 given by

$$r + 1 \geq s, \quad s + 1 \geq r, \quad r + s \geq 1. \tag{14}$$

The regions R, R_1, R_2, R_3 defined by (11)–(14) are displayed in Figure 1. A similar figure could be drawn in order to represent inequalites (2), (4), (5) or even (3).

Let us remark that the linear conditions corresponding to (12), (13), (14) define a convex polygon inside the parabola (11). By considering larger polygons inscribed in the parabola weaker and weaker sufficient conditions could be obtained. However the number of inequalities involved increases, which could be less practical in numerical computations.

We now introduce a set of three one-parameter families of linear conditions which are equivalent to positive semidefiniteness. As we have mentioned, no

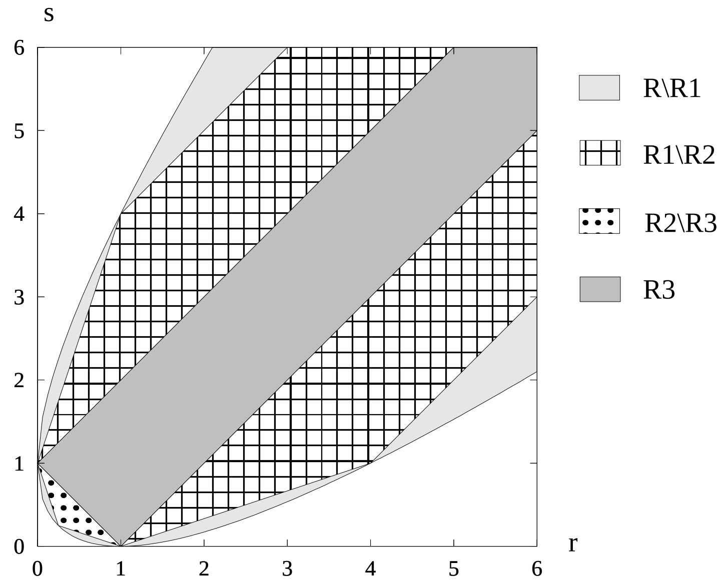

Fig. 1. Graphical representation of the regions R, R_1, R_2, R_3

finite set of linear inequalities can be equivalent to positive semidefiniteness. But, since they form a set of linear conditions, they will allow us to give a geometric interpretation in Section 4 of the Chang and Feng conditions (7).

Theorem 4 *The matrix A in (6) is positive semidefinite if and only if the following conditions hold for all* $t \in [0, 1]$*:*

$$(1-t)^2(\Delta_a + \Delta_c) + 2t(1-t)\Delta_c + t^2(\Delta_b + \Delta_c) \geq 0, \tag{15}$$
$$(1-t)^2(\Delta_b + \Delta_a) + 2t(1-t)\Delta_a + t^2(\Delta_c + \Delta_a) \geq 0, \tag{16}$$
$$(1-t)^2(\Delta_c + \Delta_b) + 2t(1-t)\Delta_b + t^2(\Delta_a + \Delta_b) \geq 0. \tag{17}$$

Proof. Let us assume that A is positive semidefinite. Then (15) holds. Since positive semidefiniteness of A is equivalent to conditions (7) and these are symmetric, (16) and (17) readily follow.

Now assume that (15)–(17) is satisfied for all $t \in [0, 1]$ and let us prove that A is positive semidefinite, i.e. (7) holds. By the symmetry of (7) and (15)–(17) we may assume that

$$\Delta_a \geq \Delta_b \geq \Delta_c. \tag{18}$$

If (15)–(17) holds for $t = 0$ we obtain

$$\Delta_a + \Delta_b \geq 0, \quad \Delta_b + \Delta_c \geq 0, \quad \Delta_c + \Delta_a \geq 0. \tag{19}$$

Using (18) and (19) we deduce that Δ_a and Δ_b are nonnegative. Now, if $\Delta_a + \Delta_b = 0$ then it follows that $\Delta_a = \Delta_b = \Delta_c = 0$ and so (7) is satisfied. In consequence, from now on we suppose that $\Delta_a + \Delta_b > 0$ and therefore $\Delta_a/(\Delta_a + \Delta_b) \in [0, 1]$. Applying (15) with $t = \Delta_a/(\Delta_a + \Delta_b)$ we obtain

$$\frac{\Delta_b^2}{(\Delta_a + \Delta_b)^2}(\Delta_a + \Delta_c) + 2\frac{\Delta_a\Delta_b}{(\Delta_a + \Delta_b)^2}\Delta_c + \frac{\Delta_a^2}{(\Delta_a + \Delta_b)^2}(\Delta_b + \Delta_c) \geq 0,$$

which reduces to

$$\frac{\Delta_b\Delta_a + \Delta_b\Delta_c + \Delta_a\Delta_c}{\Delta_a + \Delta_b} \geq 0.$$

This combined with (19) yield (7). □

Let us observe that, even if $\Delta_a + \Delta_b \geq 0$, in Theorem 4 it is not sufficient that only (15) holds for all $t \in [0, 1]$. In fact, taking $\Delta_a = 5$, $\Delta_b = -2$, $\Delta_c = 3$, (15) becomes

$$8(1 - t)^2 + 6t(1 - t) + t^2 \geq 0, \quad \forall\, t \in [0, 1],$$

which clearly holds. However the matrix

$$\begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$$

is not positive semidefinite. On the other hand, the condition that (15) holds for all t in the extended range $\mathbf{R}$ is trivially equivalent to A being positive semidefinite.

3 Convexity conditions for tensor-product surfaces

In this section we apply some results from Section 2 to obtain linear sufficient conditions for the convexity of tensor-product Bernstein–Bézier surfaces.

Let

$$S(x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij} B_{i,m}(x) B_{j,n}(y) \tag{20}$$

where $p_{ij} \in \mathbf{R}$, $B_{i,m}(x) = \binom{m}{i} x^i(1-x)^{m-i}$, $i = 0\ldots, m$, and we assume that $m \geq 2$, $n \geq 2$. Let us define the finite difference operator

$$\Delta_{kl}p_{ij} = \sum_{r=0}^{k}\sum_{s=0}^{l}(-1)^{k+l-r-s}\binom{k}{r}\binom{l}{s}p_{i+r,j+s}.$$

It was shown in Theorem 2′ of [6] that if

$$\Delta_{20}p_{ij} \geq 0 \quad \text{for } i = 0,\ldots,m-2,\ j = 0,\ldots,n, \tag{21}$$
$$\Delta_{02}p_{ij} \geq 0 \quad \text{for } i = 0,\ldots,m,\ j = 0,\ldots,n-2, \tag{22}$$

and

$$\begin{array}{l}\Delta_{20}p_{i,j+l+s}\Delta_{02}p_{i+k+r,j} - \kappa(\Delta_{11}p_{i+k,j+l})^2 \geq 0 \\ \text{for } i = 0,\ldots,m-2,\ j = 0,\ldots,n-2,\ k,l,r,s \in \{0,1\},\end{array} \tag{23}$$

where $\kappa = mn/(m-1)(n-1)$, then S is convex. Let us observe that if condition (23) holds for $\kappa = 4$, then it also holds for any $\kappa < 4$. Now conditions (21)–(23) with $\kappa = 4$ are equivalent to the matrix

$$A = \begin{pmatrix} \Delta_{20}p_{i,j+l+s} & 2\Delta_{11}p_{i+k,j+l} \\ 2\Delta_{11}p_{i+k,j+l} & \Delta_{02}p_{i+k+r,j} \end{pmatrix} \tag{24}$$

being positive semidefinite for all $i = 0,\ldots,m-2$, $j = 0,\ldots,n-2$, $k,l,r,s \in \{0,1\}$. Thus applying the sufficient conditions (2) we obtain the following linear convexity conditions in terms of the control points p_{ij}:

Proposition 5 *If*

$$\begin{array}{l}\Delta_{20}p_{i,j+s} \geq 2|\Delta_{11}p_{i+k,j}| \\ \textit{for } i = 0,\ldots,m-2,\ j = 0,\ldots,n-1,\ k,s \in \{0,1\},\end{array} \tag{25}$$

$$\begin{array}{l}\Delta_{02}p_{i+r,j} \geq 2|\Delta_{11}p_{i,j+l}| \\ \textit{for } i = 0,\ldots,m-1,\ j = 0,\ldots,n-2,\ l,r \in \{0,1\},\end{array} \tag{26}$$

then S in (20) is convex.

Proof. From (2) it is sufficient that

$$\Delta_{20}p_{i,j+l+s} \geq 2|\Delta_{11}p_{i+k,j+l}|, \quad \Delta_{02}p_{i+k+r,j} \geq 2|\Delta_{11}p_{i+k,j+l}|,$$

for $i = 0,\ldots,m-2$, $j = 0,\ldots,n-2$, $k,l,r,s \in \{0,1\}$. Now some of these inequalities are repeated but all of them are included in (25), (26). □

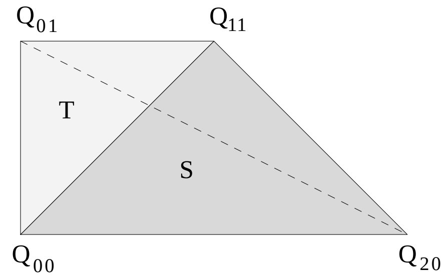

Fig. 2. Geometric interpretation of the first inequality

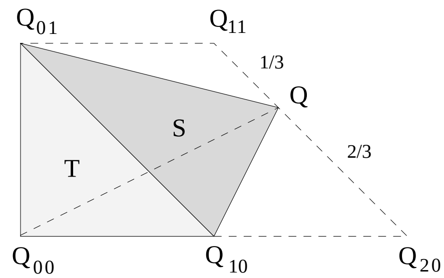

Fig. 3. Geometric interpretation of the second inequality

We can give a geometric interpretation to (25), (26) by noting that there is only one case up to symmetry. By interchanging the roles of the variables x and y, the interpretation of (26) is reduced to that of (25). Furthermore, the cases $k, s \in \{0, 1\}$ in (25) can be reduced to the cases $k = s = 0$ by reflection about lines parallel to the x and y axes. By translating, we may also assume that $i = j = 0$. Thus we now consider

$$\Delta_{20}p_{00} \geq 2\Delta_{11}p_{00} \geq -\Delta_{20}p_{00}. \tag{27}$$

This is equivalent to the two following inequalities

$$\frac{2p_{01} + p_{20}}{3} \geq \frac{p_{00} + 2p_{11}}{3}, \tag{28}$$

$$\frac{3p_{00} + p_{20} + 2p_{11}}{6} \geq \frac{2p_{10} + p_{01}}{3}. \tag{29}$$

The points $Q_{ij} = (i/m, j/n, p_{ij})$ are known as control points. The relevant control points in (28), (29) are Q_{00}, Q_{10}, Q_{20}, Q_{01}, Q_{11}.

For the interpretation of (28), let T be the triangle $\triangle Q_{00}Q_{11}Q_{01}$ and S the triangle $\triangle Q_{00}Q_{20}Q_{11}$, depicted in Figure 2. Now (28) holds if and only if $T \cup S$ is the graph of a convex function. A similar interpretation for (29) can be given in terms of the triangles shown in Figure 3 where $Q = (Q_{20} + 2Q_{11})/3$.

The following example shows that conditions (25), (26) do not require that $S(x, y)$ be translational, that is the sum of a function of x and a function of

y; see [6,5].

Example 6 *Let* $m = n = 2$ *and* $p_{00} = p_{20} = p_{02} = p_{22} = 2$, $p_{10} = p_{01} = p_{21} = p_{12} = 1$, *and* $p_{11} = \alpha$. *So*

$$S(x, y) = 2(1 - X - Y - 2\alpha XY), \tag{30}$$

where $X = x(1 - x)$ *and* $Y = y(1 - y)$. *Then we find*

$$\Delta_{02}p_{00} = \Delta_{20}p_{00} = 2,\ \Delta_{02}p_{10} = \Delta_{20}p_{01} = 2(1-\alpha),\ \Delta_{02}p_{20} = \Delta_{20}p_{02} = 2,$$

and

$$|\Delta_{11}p_{ij}| = |\alpha|, \quad \text{for } i, j \in \{0, 1\}.$$

Then all conditions (25), (26) reduce to either

$$2 \geq 2|\alpha|, \qquad \text{or} \qquad 2(1 - \alpha) \geq 2|\alpha|.$$

Thus for $\alpha \in [-1, 1/2]$, *the biquadratic tensor-product surface* S *given by (30) is convex. We note that when* $\alpha \neq 0$, S *is not translational.*

Let us compute the interval of α*'s such that* $S(x, y)$ *is convex, that is, such that the Hessian matrix*

$$H(x, y) = \begin{pmatrix} 4\big(1 - 2\alpha y(1 - y)\big) & 4\alpha(1 - 2x)(1 - 2y) \\ 4\alpha(1 - 2x)(1 - 2y) & 4\big(1 - 2\alpha x(1 - x)\big) \end{pmatrix}$$

is positive semidefinite for all $x, y \in [0, 1]$. *Since*

$$H(0, 0) = 4\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix},$$

we have that $\alpha \in [-1, 1]$ *is necessary. Let us check that this condition is also sufficient. Clearly when* $\alpha \in [-1, 1]$, *the entries* $h_{11}(x, y)$ *and* $h_{22}(x, y)$ *in* $H(x, y)$ *are greater than or equal to* $4 - 2\alpha$ *and so, non-negative. Now, noting that* $(1 - 2x)^2 = 1 - 4X$, *we find that*

$$\frac{1}{16}\det H(x, y) = 1 - 2\alpha(X + Y) - \alpha^2(1 - 4(X + Y) + 12XY).$$

Then taking $u = 2(X+Y)$, $v = 2(X-Y)$ and noting that $X = (u+v)/4$, $Y = (u-v)/4$ we may write

$$\frac{1}{16}\det H(x,y) = (1-\alpha^2)(1-u) + \left(1-\frac{\alpha}{2}\right)^2 u + \frac{3}{4}\alpha^2 u(1-u) + \frac{3}{4}\alpha^2 v^2,$$

which is nonnegative taking into account that $u \in [0,1]$.

Weaker conditions than (25), (26) which are also linear can be derived by applying (3) instead of (2). Indeed using a similar argument as in Proposition 5, we see that for any $\lambda \geq 1$, if

$$\Delta_{20} p_{i,j+s} \geq \frac{2}{\lambda}|\Delta_{11} p_{i+k,j}|,$$

for $i = 0, \ldots, m-2$, $j = 0, \ldots, n-1$, and

$$\Delta_{02} p_{i+r,j} \geq \frac{2}{\lambda}|\Delta_{11} p_{i,j+l}|,$$

for $i = 0, \ldots, m-1$, $j = 0, \ldots, n-2$, and

$$\Delta_{20} p_{i,j+l+s} + \lambda \Delta_{02} p_{i+k+r,j} \geq 2(\lambda+1)|\Delta_{11} p_{i+k,j+l}|,$$

$$\lambda \Delta_{20} p_{i,j+l+s} + \Delta_{02} p_{i+k+r,j} \geq 2(\lambda+1)|\Delta_{11} p_{i+k,j+l}|,$$

for $i = 0, \ldots, m-2$, $j = 0, \ldots, n-2$, and for $k, l, r, s \in \{0, 1\}$ then S in (20) is convex.

4 Convexity conditions for triangular Bernstein surfaces

A triangular Bernstein–Bézier surface of degree n is defined as

$$S(x,y) = \sum_{i+j+k=n} p_{ijk} B_{ijk}(x,y), \quad x \geq 0,\ y \geq 0,\ x+y \leq 1,$$

where $p_{ijk} \in \mathbf{R}$ and

$$B_{ijk}(x,y) = \frac{(i+j+k)!}{i!j!k!} x^i y^j (1-x-y)^k.$$

Let us define the following difference operators:

$$\Delta_a p_{ijk} = p_{i+2,j,k} + p_{i,j+1,k+1} - p_{i+1,j+1,k} - p_{i+1,j,k+1},$$

$$\Delta_b p_{ijk} = p_{i,j+2,k} + p_{i+1,j,k+1} - p_{i+1,j+1,k} - p_{i,j+1,k+1},$$

$$\Delta_c p_{ijk} = p_{i,j,k+2} + p_{i+1,j+1,k} - p_{i+1,j,k+1} - p_{i,j+1,k+1},$$

for $i + j + k = n - 2$, $i, j, k \geq 0$. It was suggested by Chang and Davis [2] and shown by Chang and Feng [4] that $S(x, y)$ is convex provided that the matrices

$$A = \begin{pmatrix} (\Delta_a + \Delta_c)p_{ijk} & \Delta_c p_{ijk} \\ \Delta_c p_{ijk} & (\Delta_b + \Delta_c)p_{ijk} \end{pmatrix} \tag{31}$$

are positive semidefinite for all $i, j, k \geq 0$ with $i + j + k = n - 2$.

In order to identify the matrices (31) with the matrices (6), one can simply remove the p_{ijk} in our notations. For instance, the nonlinear conditions (7) obtained by Chang and Feng [3] when applied to the problem of describing convexity conditions for a Bernstein surface are

$$\begin{aligned} &(\Delta_a + \Delta_b)p_{ijk} \geq 0, \quad (\Delta_b + \Delta_c)p_{ijk} \geq 0, \quad (\Delta_c + \Delta_a)p_{ijk} \geq 0, \\ &\Delta_b p_{ijk} \Delta_c p_{ijk} + \Delta_c p_{ijk} \Delta_a p_{ijk} + \Delta_a p_{ijk} \Delta_b p_{ijk} \geq 0, \end{aligned} \tag{32}$$

for all $i, j, k \geq 0$, with $i + j + k = n - 2$.

Analogous interpretation of conditions (9) and (10) lead to the linear conditions obtained by Lai [7] and Chang and Davis [2], respectively. Our suggestion is to use the weaker linear conditions obtained from (8) instead. Let us observe that, unlike the inequalities (9), our inequalities (8) are symmetric, that is, they are invariant under exchanging the roles of i, j, k.

Let us provide a geometric interpretation to the set of inequalities obtained from (8). By the symmetry mentioned above, we only need to consider the first one and a translation argument allows us to set $i = j = 0$, $k = n - 2$. In this case, the inequality $\Delta_a + 2\Delta_c \geq 0$ becomes

$$\frac{p_{2,0,n-2} + p_{1,1,n-2} + 2p_{0,0,n}}{4} \geq \frac{p_{0,1,n-1} + 3p_{1,0,n-1}}{4}. \tag{33}$$

The points

$$Q_{ijk} = \left(\frac{i}{i+j+k}, \frac{j}{i+j+k}, p_{ijk} \right) \tag{34}$$

for $i, j = 0, \ldots, n$, $k = n - i - j$, are the control points. In (33) the relevant points are $Q_{0,0,n}, Q_{1,0,n-1}, Q_{2,0,n-2}, Q_{0,1,n-1}, Q_{1,1,n-2}$. In order to interpret

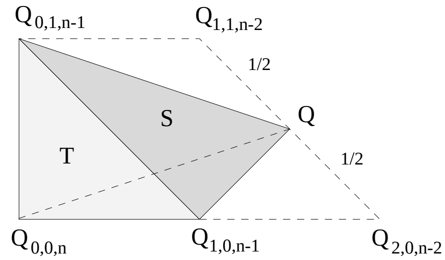

Fig. 4. The graph of $T \cup S$ must be convex

(33), let T be the triangle $\Delta Q_{0,0,n}Q_{1,0,n-1}Q_{0,1,n-1}$ and let S be the triangle $\Delta Q_{1,0,n-1}QQ_{0,1,n-1}$, where $Q = (Q_{2,0,n-2} + Q_{1,1,n-2})/2$. Now (33) holds if and only if $T \cup S$ is the graph of a convex function as shown in Figure 4.

Example 7 *Let $n = 2$ and $P_{002} = P_{200} = P_{020} = 0$, $P_{011} = P_{110} = -1$ and $P_{101} = -\alpha$. Then we find that $\Delta_a = \Delta_c = \alpha$ and $\Delta_b = 2 - \alpha$. Then our conditions (8) hold if and only if $0 \leq \alpha \leq 4$. However, when $2 < \alpha \leq 4$, the conditions (9) and (10) are not satisfied.*

We have now seen that both the conditions of Chang and Davis and our conditions (8) can be described in terms of the convexity of piecewise linear functions over quadrilaterals. This suggests that also the quadratic conditions of Chang and Feng may have a similar interpretation.

To this end, let us apply Theorem 4 to give the following set of sufficient conditions for convexity:

$$((1-t)^2(\Delta_a + \Delta_c) + 2t(1-t)\Delta_c + t^2(\Delta_b + \Delta_c))p_{ijk} \geq 0, \tag{35}$$
$$((1-t)^2(\Delta_b + \Delta_a) + 2t(1-t)\Delta_a + t^2(\Delta_c + \Delta_a))p_{ijk} \geq 0, \tag{36}$$
$$((1-t)^2(\Delta_c + \Delta_b) + 2t(1-t)\Delta_b + t^2(\Delta_a + \Delta_b))p_{ijk} \geq 0 \tag{37}$$

for all $t \in [0, 1]$ and all $i + j + k = n - 2$, $i, j, k \geq 0$.

These conditions are equivalent to all matrices (31) being positive semidefinite and are therefore equivalent to the Chang and Feng conditions (32). Since the latter conditions are also necessary in the case $n = 2$ (see Theorem 1 of [3]), conditions (35)–(37) are equivalent to the convexity of S in the case $n = 2$. We now provide a geometric interpretation of (35)–(37) in terms of the convexity of a certain one-parameter family of quadrilaterals associated with the control net.

First of all we expand the terms in (35) and obtain

$$\begin{aligned}&(1-t)^2(p_{i+2,j,k} - 2p_{i+1,j,k+1} + p_{i,j,k+2})+\\&+2t(1-t)(p_{i,j,k+2} + p_{i+1,j+1,k} - p_{i+1,j,k+1} - p_{i,j+1,k+1})+\\&+t^2(p_{i,j+2,k} - 2p_{i,j+1,k+1} + p_{i,j,k+2}) \geq 0.\end{aligned}$$

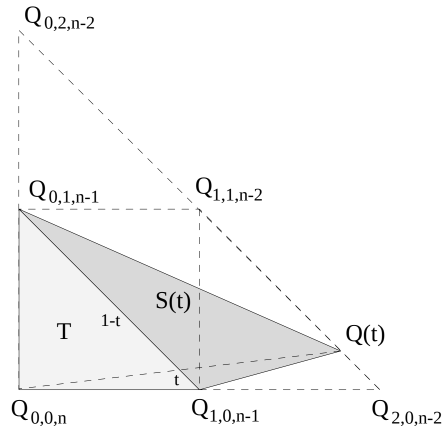

Fig. 5. Interpretation for $i = j = 0$ of one of the inequalities

Now, combining terms we obtain:

$$(1-t)^2 p_{i+2,j,k} + 2t(1-t)p_{i+1,j+1,k} + t^2 p_{i,j+2,k} + p_{i,j,k+2}$$
$$-2((1-t)p_{i+1,j,k+1} + tp_{i,j+1,k+1}) \geq 0,$$

and letting

$$p(t) := (1-t)^2 p_{i+2,j,k} + 2t(1-t)p_{i+1,j+1,k} + t^2 p_{i,j+2,k},$$

we may write

$$\frac{p(t) + p_{i,j,k+2}}{2} \geq (1-t)p_{i+1,j,k+1} + tp_{i,j+1,k+1}. \tag{38}$$

Let $Q(t) = (1-t)^2 Q_{i+2,j,k} + 2t(1-t)Q_{i+1,j+1,k} + t^2 Q_{i,j+2,k}$ and observe that it is the Bézier curve with control points $Q_{i+2-r,j+r,k}$, $r = 0, 1, 2$, evaluated at the parameter value t. Now define the triangles $T = \Delta Q_{i,j,k+2} Q_{i+1,j,k+1} Q_{i,j+1,k+1}$ and, for $t \in [0, 1]$, $S(t) = \Delta Q(t) Q_{i,j+1,k+1} Q_{i+1,j,k+1}$. Then (38) is equivalent to saying that $T \cup S(t)$ is the graph of a convex function for all $t \in [0, 1]$. Figure 5 shows the interpretation of (35) for $i = j = 0$.

References

[1] A. S. Cavaretta and A. Sharma, Variation diminishing properties and convexity for the tensor product Bernstein operator (1990), manuscript.

[2] G. Chang and P. J. Davis, The convexity of Bernstein polynomials over triangles, *J. Approx. Theory* **40** (1984) 11–28.

[3] G. Z. Chang and Y. Y. Feng, An improved condition for the convexity of Bernstein-Bézier surfaces over triangles, *Comp. Aided Geom. Design* **1** (1984) 279–283.

[4] G. Z. Chang and Y. Y. Feng, A new proof for the convexity of Bernstein-Bézier surfaces over triangles, *Chinese Annals of Mathematics* **6** (1985) 171–176.

[5] W. Dahmen, Convexity and Bernstein-Bézier polynomials, in: P. J. Laurent, A. Le Méhauté and L. L. Schumaker, eds., *Curves and Surfaces*, Vol. 1 (Academic Press, Boston, 1991) 107–134.

[6] M. S. Floater, A weak condition for the convexity of tensor-product Bézier and B-spline surfaces, *Adv. in Comp. Math.* **2** (1994) 67–80.

[7] M. J. Lai, Some sufficient conditions for convexity of multivariate Bernstein-Bézier polynomials and box spline surfaces, *Studia Scient. Math. Hung.* **28** (1990) 363–374.