The inverse of a rational bilinear mapping

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Abstract

We study the problem of inverting rational bilinear mappings, which leads to a one-parameter family of generalized barycentric coordinates for quadrilaterals, including Wachspress coordinates as a special case.

Keywords: Bilinear mappings, inverse mappings, rational coordinates, barycentric coordinates.

1 Introduction

In a recent paper, Sederberg and Zheng [5] studied rational bilinear mappings from the unit square to a convex quadrilateral, and their inverses. Such mappings have played an important role in computer graphics and geometric design [7]. If $P \subset \mathbb{R}^2$ is the quadrilateral, with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ in $\mathbb{R}^2$, as in Figure 1, and weights $w_1, w_2, w_3, w_4 > 0$ are chosen, the mapping

$$\mathbf{r}(s,t) := \frac{(1-s)(1-t)w_1\mathbf{v}_1 + s(1-t)w_2\mathbf{v}_2 + stw_3\mathbf{v}_3 + (1-s)tw_4\mathbf{v}_4}{(1-s)(1-t)w_1 + s(1-t)w_2 + stw_3 + (1-s)tw_4},$$

is a bijection $\mathbf{r} : [0,1]\times[0,1] \to P$. This can be seen from the fact that for a fixed s in $[0,1]$, $\mathbf{r}$ is a line segment connecting a point on the edge $[\mathbf{v}_1, \mathbf{v}_2]$ to a point on $[\mathbf{v}_4, \mathbf{v}_3]$, and as s increases from 0 to 1, the convexity of P ensures that these line segments cover every point of P once only. Thus $\mathbf{r}$ has an inverse and for any point $\mathbf{x} \in P$, we can solve

$$\mathbf{r}(s,t) = \mathbf{x}, \tag{1}$$

uniquely for s and t in $[0,1]$. Sederberg and Zheng [5] derived a condition on the weights that makes the inversion particularly simple. For $i = 1,2,3,4$, let C_i denote the triangle area,

$$C_i = A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}),$$

where vertices are indexed cyclically, $\mathbf{v}_{i+4} = \mathbf{v}_i$, $i \in \mathbb{Z}$. They showed that if

$$\frac{w_1 w_3}{w_2 w_4} = \frac{C_1 C_3}{C_2 C_4}, \tag{2}$$

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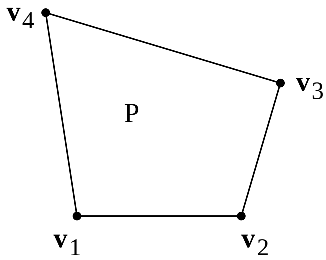

Figure 1: Convex quadrilateral

then s and t are rational functions of $\mathbf{x}$.

In this short note we firstly point out that under condition (2), s and t can be obtained from the area form of Wachspress' rational coordinates for convex polygons. Secondly, we derive a solution for s and t for arbitrary weights w_i. From this general solution we obtain generalized barycentric coordinates for P that do not seem to have been studied previously. They form a one-parameter family of coordinates, including Wachspress coordinates as a special case.

2 Wachspress coordinates

Wachspress developed rational barycentric coordinates for convex polygons in [6]. The quadrilateral case has been studied in [3] and [1]. Meyer et al [4] found a formula for the coordinates in terms of triangle areas, which for the quadrilateral P means that $\mathbf{x} \in P$ can be expressed as

$$\mathbf{x} = \frac{C_1 A_2 A_3 \mathbf{v}_1 + C_2 A_3 A_4 \mathbf{v}_2 + C_3 A_4 A_1 \mathbf{v}_3 + C_4 A_1 A_2 \mathbf{v}_4}{C_1 A_2 A_3 + C_2 A_3 A_4 + C_3 A_4 A_1 + C_4 A_1 A_2}, \tag{3}$$

where, in addition to the triangle areas C_i, the A_i are also triangle areas,

$$A_i = A_i(\mathbf{x}) = A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}).$$

An example of weights w_i that satisfy condition (2) is $w_i = C_i$. In this case, we see that by dividing numerator and denominator of (3) by $(A_4 + A_2)(A_1 + A_3)$ the solution to (1) is

$$s = \frac{A_4}{A_4 + A_2}, \qquad t = \frac{A_1}{A_1 + A_3}.$$

For general weights satisfying (2), we can first multiply the numerator and denominator of (3) by w_1/C_1 to express it as

$$\mathbf{x} = \frac{A_2 A_3 w_1 \mathbf{v}_1 + \rho_1 A_3 A_4 w_2 \mathbf{v}_2 + \rho_1 \rho_2 A_4 A_1 w_3 \mathbf{v}_3 + \rho_2 A_1 A_2 w_4 \mathbf{v}_4}{A_2 A_3 w_1 + \rho_1 A_3 A_4 w_2 + \rho_1 \rho_2 A_4 A_1 w_3 + \rho_2 A_1 A_2 w_4},$$

where

$$\rho_1 := \frac{w_1 C_2}{C_1 w_2}, \qquad \rho_2 := \frac{w_1 C_4}{C_1 w_4},$$

and then we see that the solution to (1) under condition (2) is

$$s = \frac{\rho_1 A_4}{\rho_1 A_4 + A_2}, \qquad t = \frac{\rho_2 A_1}{\rho_2 A_1 + A_3}.$$

3 General solution

Consider now solving the inversion for arbitary weights w_i. We can adapt the formula derived recently in [2] for the inverse of the bilinear mapping

$$\mathbf{p}(s,t) := (1-s)(1-t)\mathbf{v}_1 + s(1-t)\mathbf{v}_2 + st\mathbf{v}_3 + (1-s)t\mathbf{v}_4. \tag{4}$$

For any point $\mathbf{x} \in P$ the solution (s,t) to

$$\mathbf{p}(s,t) = \mathbf{x} \tag{5}$$

can be expressed as follows. For $i = 1,2,3,4$, define the vectors $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$. Then (5) can be expressed as

$$(1-s)(1-t)\mathbf{d}_1 + s(1-t)\mathbf{d}_2 + st\mathbf{d}_3 + (1-s)t\mathbf{d}_4 = 0.$$

Then, using the fact that

$$A_i = \frac{1}{2}\mathbf{d}_i \times \mathbf{d}_{i+1},$$

and further defining

$$B_i = B_i(\mathbf{x}) = A(\mathbf{x}, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}) = \frac{1}{2}\mathbf{d}_{i-1} \times \mathbf{d}_{i+1},$$

(and noting that $B_3 = -B_1$ and $B_4 = -B_2$), it was shown in [2] that

$$s = \frac{2A_4}{2A_4 - B_1 + B_2 + \sqrt{D}}, \qquad t = \frac{2A_1}{2A_1 - B_1 - B_2 + \sqrt{D}}, \tag{6}$$

where

$$D = B_1^2 + B_2^2 + 2A_1A_3 + 2A_2A_4.$$

Consider now the rational case. If we define the scaled vectors $\tilde{\mathbf{d}}_i = w_i(\mathbf{v}_i - \mathbf{x})$ we can express (1) in the similar form

$$(1-s)(1-t)\tilde{\mathbf{d}}_1 + s(1-t)\tilde{\mathbf{d}}_2 + st\tilde{\mathbf{d}}_3 + (1-s)t\tilde{\mathbf{d}}_4 = 0.$$

Thus we can find s and t from the formulas (6) if we replace the A_i and B_i by the scaled versions,

$$\tilde{A}_i = \frac{1}{2}\tilde{\mathbf{d}}_i \times \tilde{\mathbf{d}}_{i+1}, \qquad \tilde{B}_i = \frac{1}{2}\tilde{\mathbf{d}}_{i-1} \times \tilde{\mathbf{d}}_{i+1}.$$

This shows

Theorem 1 *The solution to (1) is*

$$s = \frac{2\tilde{A}_4}{2\tilde{A}_4 - \tilde{B}_1 + \tilde{B}_2 + \sqrt{\tilde{D}}}, \qquad t = \frac{2\tilde{A}_1}{2\tilde{A}_1 - \tilde{B}_1 - \tilde{B}_2 + \sqrt{\tilde{D}}}, \tag{7}$$

with

$$\tilde{D} = \tilde{B}_1^2 + \tilde{B}_2^2 + 2\tilde{A}_1\tilde{A}_3 + 2\tilde{A}_2\tilde{A}_4.$$

4 Dependence on the weights

To find out how s and t depend on the weights w_i, we can use the fact that

$$\tilde{\mathbf{d}}_i = w_i \mathbf{d}_i, \qquad \tilde{A}_i = w_i w_{i+1} A_i, \qquad \tilde{B}_i = w_{i-1} w_{i+1} B_i,$$

and with these substitutions we find that

$$\tilde{D} = w_2 w_4 \sqrt{D'},$$

where

$$D' = B_1^2 + \rho^2 B_2^2 + 2\rho(A_1 A_3 + A_2 A_4),$$

and

$$\rho = \frac{w_1 w_3}{w_2 w_4}.$$

Putting these expressions into (7) gives

$$s = \frac{2(w_1/w_2)A_4}{2(w_1/w_2)A_4 + F_2}, \qquad t = \frac{2(w_1/w_4)A_1}{2(w_1/w_4)A_1 + F_3}. \tag{8}$$

with

$$F_2 := -B_1 + \rho B_2 + \sqrt{D'}, \qquad F_3 := -B_1 - \rho B_2 + \sqrt{D'}.$$

This shows that s depends only on ρ and the ratio w_1/w_2 (or equivalently on ρ and w_3/w_4), and that t depends only on ρ and w_1/w_4 (or equivalently on ρ and w_3/w_2).

Consider now the generalized barycentric coordinates generated by s and t. Defining

$$\phi_1 = \frac{(1-s)(1-t)w_1}{W}, \quad \phi_2 = \frac{s(1-t)w_2}{W}, \quad \phi_3 = \frac{stw_3}{W}, \quad \phi_4 = \frac{(1-s)tw_4}{W}, \tag{9}$$

where

$$W = (1-s)(1-t)w_1 + s(1-t)w_2 + stw_3 + (1-s)tw_4,$$

we have

$$\mathbf{x} = \sum_{i=1}^{4} \phi_i \mathbf{v}_i, \qquad 1 = \sum_{i=1}^{4} \phi_i.$$

To consider how the coordinates ϕ_i depend on the w_i, we can substitute the formulas (8) into (9), and after cancelling common factors and using the definition of ρ we find that $\phi_i = \lambda_i / \sum_{j=1}^4 \lambda_j$, where

$$\lambda_1 = F_2 F_3, \quad \lambda_2 = 2F_3 A_4, \quad \lambda_3 = 4\rho A_4 A_1, \quad \lambda_4 = 2A_1 F_2.$$

This shows that the coordinates ϕ_i depend only on ρ and thus form a one-parameter family that includes Wachspress coordinates in the case $\rho = \rho_* := C_1C_3/(C_2C_4)$.

Figure 2 shows contour plots of ϕ_1, ϕ_2, ϕ_3, ϕ_4 from left to right, for the four values $\rho = 0.1, \rho_*, 1, 10$, where $\rho_* \approx 0.8842$. Thus the coordinates in the second row are Wachspress coordinates, and those in the third row come from inverting $\mathbf{p}$ in (4). It is interesting to observe that for the small and large values of ρ the coordinates appear to approach the piecewise linear coordinates obtained by triangulating P using one or other of the diagonals. This could be a topic for future research. Another interesting question is whether any of the results of this paper extend to three dimensions, for example, to rational trilinear mappings from the unit cube to a convex hexahedron.

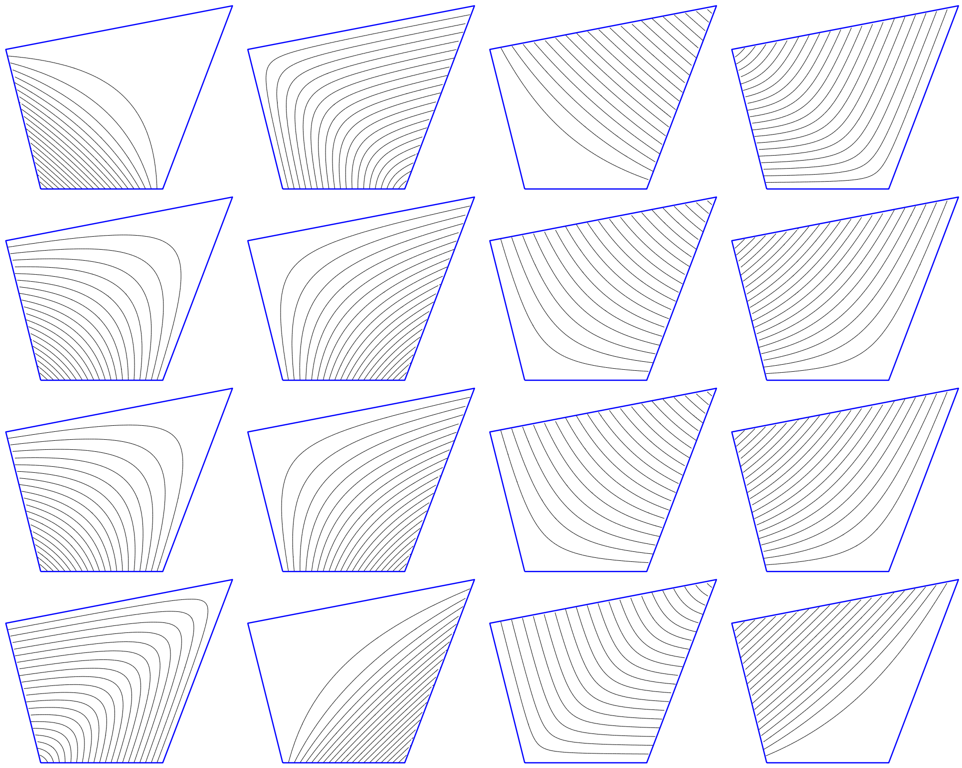

Figure 2: From top to bottom, $\rho = 0.1, \rho_*, 1, 10$.

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