High order approximation of rational curves by polynomial curves

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Abstract

We show that many rational parametric curves can be interpolated, in a Hermite sense, by polynomial curves whose degree, relative to the number of data being interpolated, is lower than usual. The construction unifies and generalizes the families of circle and conic approximations of Lyche and Mørken and the author in which the approximation order is twice the degree of the polynomial.

Keywords: Rational curves, polynomial interpolation, high order approximation, Euclid's algorithm.

1 Introduction

A function can be interpolated uniquely at $m + 1$ points by a polynomial of degree at most m, and the same is true when some of the function values are replaced by derivatives, in the sense of Hermite. When interpolating curves, however, it has been shown by several authors that in certain cases a parametric polynomial of degree m can match more than $m + 1$ 'geometric data' (points, tangents, curvatures, etc.) [1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14]. For planar curves, it is sometimes possible to match $2m$ data. All these results require some kind of assumption on the curve being approximated. For example, non-vanishing curvature of the curve is needed for the cubic interpolant of [1]. The curve is restricted to a circle in [3, 6, 12] and to a conic section in [4, 5].

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For general m, little seems to be known about the existence of such interpolants apart from the two families of interpolants of odd degree m to circles and conic sections found in [12] and [5], each having a total of $2m$ contacts.

The purpose of this paper is to gain further insight into the general problem by showing that a large class of rational parametric curves can be interpolated, in a Hermite sense, by a polynomial of degree m matching $2m - 2k + 4$ data, where k is the total degree of the rational curve. Specifically, let $\mathbf{r} : [a, b] \to \mathbb{R}^d$, $d \geq 2$, be the rational curve

$$\mathbf{r}(t) = \mathbf{f}(t)/g(t),$$

where $\mathbf{f}$ and g are polynomials of degrees at most M and N and let $k = M + N$. For each sample of parameter values $a \leq t_1 < t_2 < \cdots < t_n \leq b$ we find a polynomial $\mathbf{p}$ of degree at most $n + k - 2$ and scalar values $\mu_1, \ldots, \mu_n$ satisfying the $2n$ interpolation conditions

$$\mathbf{p}(t_i) = \mathbf{r}(t_i), \quad \mathbf{p}'(t_i) = \mu_i \mathbf{r}'(t_i), \qquad i = 1, 2, \ldots, n. \tag{1}$$

We make two assumptions on the denominator g for the construction to work:

(A1) g has no roots in $[a, b]$,

(A2) g has no double roots (real or complex).

Assumption (A1) is hardly a restriction because it merely prevents $\mathbf{r}$ having poles in $[a, b]$. Assumption (A2) is not very strong either. For example it holds for the well-known rational representation of a circular arc (15). There are no restrictions on the numerator $\mathbf{f}$. We show that the approximation has order $O(h^{2n})$ as $h \to 0$ where $h = t_n - t_1$. By choosing the points t_i symmetrically and letting some of them coalesce we recover the circle and conic approximations of odd degree n of [12] and [5].

2 The interpolant

The basic idea is to let

$$\mathbf{p}(t) = \mathbf{r}(t) + \lambda(t)\omega_n(t)\mathbf{r}'(t), \tag{2}$$

where

$$\omega_n(t) = (t - t_1)(t - t_2) \cdots (t - t_n),$$

and λ is a polynomial to be determined. This makes $\mathbf{p}$ in general a rational curve but we will show that certain choices of λ force $\mathbf{p}$ to be a polynomial. Consider first the interpolation properties of $\mathbf{p}$. Since $\omega_n(t_i) = 0$, we have $\mathbf{p}(t_i) = \mathbf{r}(t_i)$, and differentiating $\mathbf{p}$ gives

$$\mathbf{p}' = \mathbf{r}' + \lambda' \omega_n \mathbf{r}' + \lambda \omega_n' \mathbf{r}' + \lambda \omega_n \mathbf{r}'',$$

which means that

$$\mathbf{p}'(t_i) = \big(1 + \lambda(t_i)\omega_n'(t_i)\big)\mathbf{r}'(t_i), \tag{3}$$

showing that condition (1) is satisfied with $\mu_i = 1 + \lambda(t_i)\omega_n'(t_i)$. Considering how to find a suitable polynomial λ, observe that

$$\mathbf{p}(t) = \frac{g(t) - \lambda(t)\omega_n(t)g'(t)}{g^2(t)}\mathbf{f}(t) + \frac{\lambda(t)\omega_n(t)}{g(t)}\mathbf{f}'(t).$$

Thus to force $\mathbf{p}$ to be a polynomial it is sufficient to force the coefficients of $\mathbf{f}$ and $\mathbf{f}'$ to be polynomials. This can easily be arranged for the coefficient of $\mathbf{f}'$ by letting

$$\lambda(t) = g(t)X(t),$$

for some polynomial X to be determined. With this substitution, we now have

$$\mathbf{p}(t) = \frac{1 - X(t)\omega_n(t)g'(t)}{g(t)}\mathbf{f}(t) + X(t)\omega_n(t)\mathbf{f}'(t),$$

and it remains to find a polynomial X such that the polynomial $1 - X\omega_n g'$ divides by the polynomial g. Put another way, if we can find two polynomials X and Y such that

$$\omega_n(t)g'(t)X(t) + g(t)Y(t) = 1, \tag{4}$$

then

$$\mathbf{p}(t) = Y(t)\mathbf{f}(t) + X(t)\omega_n(t)\mathbf{f}'(t) \tag{5}$$

is a polynomial satisfying (1).

Consider then equation (4), which can be written as

$$a_0(t)X(t) + a_1(t)Y(t) = 1, \tag{6}$$

where

$$a_0 = \omega_n g', \qquad a_1 = g.$$

It is known from algebra that if a_0 and a_1 are relatively prime, i.e., have no roots in common, then Euclid's g.c.d. algorithm can be used to find unique real polynomial solutions X and Y of degrees at most $d(a_1)-1$ and $d(a_0)-1$ respectively, where $d(p)$ denotes the degree of a polynomial p. A proof of this is given for example in ([2], Chap. 6). This is where we need the assumptions (A1) and (A2): they ensure that a_0 and a_1 are relatively prime. Since a_0 and a_1 have degrees $n+N-1$ and N respectively, and $\mathbf{f}$ has degree M and $\mathbf{f}'$ degree $M-1$, and $k = M+N$, we deduce

Theorem 1 *There are unique polynomials X and Y of degrees at most $N-1$ and $n+N-2$ respectively that solve (4). With these X and Y, $\mathbf{p}$ in (5) is a polynomial of degree at most $n+k-2$ that solves (1).*

We now describe how Euclid's algorithm can be used to find the solutions X and Y. Since $n \geq 1$ note that $d(a_0) \geq d(a_1)$. Then for each $k = 0, 1, 2, \ldots$, we divide a_k by a_{k+1} and find the remainder, which defines the polynomials q_k and a_{k+2} in

$$a_k = q_k a_{k+1} + a_{k+2}, \tag{7}$$

where $d(q_k) = d(a_k) - d(a_{k+1})$ and $d(a_{k+2}) < d(a_{k+1})$. The algorithm stops when the remainder a_{k+2} is a constant polynomial, at which point we let $r = k$. If the remainder a_{r+2} is zero then a_0 and a_1 have the common denominator a_{r+1} and are not coprime. Under Assumptions (A1) and (A2) though, the constant a_{r+2} will be non-zero, in which case we work backwards to obtain the solutions X and Y to (6). We start by rewriting (7) with $k = r$ as

$$a_{r+2} = b_0 a_r + b_1 a_{r+1}, \tag{8}$$

where $b_0 = 1$ and $b_1 = -q_r$. Then (7) with $k = r-1$ gives

$$a_{r+2} = b_0 a_r + b_1 (a_{r-1} - q_{r-1} a_r) = b_1 a_{r-1} + b_2 a_r,$$

where $b_2 = b_0 - q_{r-1} b_1$. Continuing in this way, we end up with

$$a_{r+2} = b_r a_0 + b_{r+1} a_1. \tag{9}$$

where

$$b_j = b_{j-2} - q_{r-j+1} b_{j-1}, \qquad j = 2, 3, \ldots, r+1. \tag{10}$$

Finally, since a_{r+2} is a non-zero constant, we can divide (9) by a_{r+2} to get

$$1 = \frac{b_r}{a_{r+2}} a_0 + \frac{b_{r+1}}{a_{r+2}} a_1,$$

and this shows that (6) has the solutions

$$X(t) = \frac{b_r(t)}{a_{r+2}}, \qquad Y(t) = \frac{b_{r+1}(t)}{a_{r+2}}.$$

Now consider the degrees of X and Y. We have $d(b_0) = 0$ and $d(b_1) = d(q_r)$, and from (10) we find

$$d(b_r) = d(q_1) + \cdots + d(q_r) = d(a_1) - d(a_{r+1}) < d(a_1),$$

and similarly,

$$d(b_{r+1}) = d(a_0) - d(a_{r+1}) < d(a_0).$$

Thus $d(X) < d(a_1)$ and $d(Y) < d(a_0)$, as claimed. The uniqueness of X and Y is easily deduced by supposing their are two solution pairs and taking their differences.

3 Approximation order

The fact that $\mathbf{p}$ has $2n$ geometric contacts with $\mathbf{r}$, counting multiplicities, suggests that the error between $\mathbf{p}$ and $\mathbf{r}$ might be $O(h^{2n})$ as $h \to 0$ where $h = t_n - t_1$. The approach used to obtain the approximation order in [3, 6, 12, 5] was to use the algebraic form of the circle or conic section. Using the algebraic form of an arbitrary parametric rational curve might however create difficulties. Fortunately, it turns out that we do not need the implicit form at all. We can instead use the reparameterization

$$\phi(t) = t + \lambda(t)\omega_n(t). \tag{11}$$

Theorem 2 *There are constants $h_0 > 0$ and $C > 0$ depending only on $\mathbf{r}$, a, b, and n such that*

$$\max_{t_1 \le t \le t_n} |\mathbf{r}(\phi(t)) - \mathbf{p}(t)| \le Ch^{2n} \quad \textit{for } h \le h_0.$$

Proof. If $\mathbf{s}(t) := \mathbf{r}(\phi(t))$ then because

$$\phi(t_i) = t_i, \quad \text{and} \quad \phi'(t_i) = 1 + \lambda(t_i)\omega_n'(t_i), \qquad i = 1, \ldots, n,$$

we have from (3),

$$\mathbf{p}(t_i) = \mathbf{s}(t_i), \quad \text{and} \quad \mathbf{p}'(t_i) = \phi'(t_i)\mathbf{r}'(t_i) = \mathbf{s}'(t_i), \qquad i = 1, \ldots, n.$$

It follows (Chap. 5 of [8]) that for $t \in [a,b]$,

$$\mathbf{s}(t) - \mathbf{p}(t) = (t-t_1)^2 \cdots (t-t_n)^2 [t_1, t_1, t_2, t_2, \ldots, t_n, t_n, t]\mathbf{s}, \qquad (12)$$

the latter term denoting a divided difference of $\mathbf{s}$, and so

$$\max_{t_1 \le t \le t_n} |\mathbf{s}(t) - \mathbf{p}(t)| \le h^{2n} \|\mathbf{s}^{(2n)}\| / (2n)!,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $\|\mathbf{s}^{(2n)}\| = \max_{a \le t \le b} |\mathbf{s}^{(2n)}(t)|$. It thus remains to show that $\|\mathbf{s}^{(2n)}\|$ is bounded by a constant as $h \to 0$. To answer this, observe that by Faà di Bruno's formula [9], the $2n$-th derivative of $\mathbf{s}$ is a linear combination of the derivatives of $\mathbf{r}$ of orders 1 to $2n$, whose coefficients are sums of products of the derivatives of ϕ of orders 1 to $2n$. Since $\mathbf{r}$ is fixed, it is therefore enough to show that all derivatives of ϕ up to order $2n$ are bounded as $h \to 0$. By equation (11), since $\lambda = gX$ and g is fixed and all derivatives of ω_n are bounded as $h \to 0$, it is sufficient to show that all the derivatives of X of order up to $2n$ are bounded as $h \to 0$. Since X has at most degree $N-1$ it remains to show that X has bounded derivatives up to order $N-1$. To this end we make the following observation. If $z_1, \ldots, z_N$ are the roots (real or complex) of g, which are distinct by (A2), then since $X(z_i) = 1/(g'(z_i)\omega_n(z_i))$ for $i = 1, \ldots, N$ in (4), and since X has degree at most $N-1$, it follows that X can be expressed as the Lagrange interpolant

$$X(t) = \sum_{i=1}^{N} \frac{L_i(t)}{g'(z_i)\omega_n(z_i)}, \qquad L_i(t) = \prod_{k=1, k \ne i}^{N} \frac{t - z_k}{z_i - z_k}.$$

Note that even though some of the roots $z_1, \ldots, z_N$ may be complex we already know that X is real. In fact since g is real its complex roots come in conjugate pairs. Thus for each i, there is some j such that $\overline{z_i} = z_j$, where $j \ne i$ if z_i is complex and $j = i$ if z_i is real. It follows that $\overline{L_i(t)} = L_j(t)$ and consequently $\overline{X(t)} = X(t)$.

Thus for $k = 1, \ldots, N-1$, the k-th derivative of X is bounded as

$$|X^{(k)}(t)| \le \sum_{i=1}^{N} \frac{|L_i^{(k)}(t)|}{|g'(z_i)||\omega_n(z_i)|},$$

and it is clearly enough to show that $|\omega_n(z_i)|$ is bounded away from zero. Well by (A1), the minimum distance in $\mathbb{C}$ between the roots $z_1, \ldots, z_N$ and the real interval $[a,b]$ is some $\alpha > 0$, which implies that

$$|\omega_n(z_i)| \geq \alpha^n, \qquad i = 1, \ldots, N.$$

□

4 Interpolating higher order derivatives

All the theory (Theorems 1 and 2) extends in a natural way if we allow some of the parameter values $t_1, \ldots, t_n$ in $[a,b]$ to coalesce, i.e., allow $t_1 \leq \cdots \leq t_n$. We again define $\mathbf{p}$ by equation (2) where $\lambda = Xg$, and X and Y are the unique solutions of degrees $\leq N-1$ and $\leq n+N-2$ to equation (4). As the next theorem shows, $\mathbf{p}$ then interpolates $\mathbf{r}$ in the usual Hermite sense, fulfilling a total of n interpolation conditions, but $\mathbf{p}$ is also a Hermite interpolant to the reparameterized curve $\mathbf{s} = \mathbf{r} \circ \phi$, satisfying a total of $2n$ interpolation conditions; thus $\mathbf{p}$ has $2n$ geometric contacts with $\mathbf{r}$.

Theorem 3 *If the point t_α has multiplicity ℓ then*

$$\mathbf{p}^{(i)}(t_\alpha) = \mathbf{r}^{(i)}(t_\alpha), \qquad 0 \leq i \leq \ell - 1, \tag{13}$$

and

$$\mathbf{p}^{(i)}(t_\alpha) = \mathbf{s}^{(i)}(t_\alpha), \qquad 0 \leq i \leq 2\ell - 1. \tag{14}$$

Proof. Equation (13) follows from differentiating (2) i times and noticing that $\omega_n(t)$ contains the factor $(t - t_\alpha)^\ell$.

Regarding equation (14), observe that the solutions X and Y to equation (4) depend continuously on the t_i, whether the t_i are distinct or not. Thus the same holds for λ, $\mathbf{p}$, ϕ, and $\mathbf{s}$, and it follows that equation (12) extends to non-distinct t_i. Therefore, noticing that we can differentiate $\mathbf{s}$ as often as we like, we can differentiate (12) i times and equation (14) follows from the fact that the polynomial on the right hand side of (12) contains the factor $(t - t_\alpha)^{2\ell}$. □

5 Circle case

Consider applying the theory to a circular arc. A typical representation of the unit circle centred at the origin is the quadratic rational,

$$\frac{(1-t^2, 2t)}{1+t^2}, \tag{15}$$

which, with $-\infty < t < \infty$, covers all points on the circle except $(-1,0)$. With Theorem 1 in mind though, we want to keep the degrees of the numerator and denominator as low as possible. Thus we reduce the degree of the numerator by 1 by adding the vector $(1,0)$ to (15), and we will interpolate the rational curve

$$\mathbf{r}(t) = \frac{\mathbf{f}(t)}{g(t)} = \frac{(2,2t)}{1+t^2}, \tag{16}$$

which represents the circle of unit radius centred at $(1,0)$. There is no loss in applying this shift for if $\mathbf{p}$ is a polynomial interpolant to (16) satisfying (1) then the shift of $\mathbf{p}$ by $(-1,0)$ will be a similar interpolant to (15).

Considering $\mathbf{r}$ in (16), note that since g has the roots $\mathbf{i}$ and $-\mathbf{i}$, where $\mathbf{i} = \sqrt{-1}$, it has no roots in any real interval $[a,b]$ and no double roots in $\mathbb{C}$ and thus satisfies Assumptions (A1) and (A2) in any interval $[a,b]$. Thus, since the numerator $\mathbf{f}$ and denominator g have degrees $M = 1$ and $N = 2$ so that $k = 3$, applying Theorem 1 proves

Theorem 4 *Let $t_1 < \cdots < t_n$ be arbitrary increasing values in $\mathbb{R}$. If $\mathbf{r}$ is the circle in (16), a solution $\mathbf{p}$ to (1) is*

$$\mathbf{p}(t) = Y(t)(2,2t) + X(t)\omega_n(t)(0,2), \tag{17}$$

where X and Y are the unique solutions of degrees at most 1 and n to

$$2t\omega_n(t)X(t) + (1+t^2)Y(t) = 1, \tag{18}$$

and the degree of $\mathbf{p}$ is at most $n+1$.

For most choices of interpolation points $t_1, \ldots, t_n$, two steps of Euclid's algorithm will be required to compute the polynomials $X(t)$ and $Y(t)$ in (18). There are, however, certain choices of the t_i for which only one step is needed, and the degrees of X, Y, and $\mathbf{p}$ are then reduced by one. This happens if we restrict n to be odd and place the parameter values $t_1, \ldots, t_n$ symmetrically around $t = 0$.

Theorem 5 *Suppose $n = 2s+1$ for some $s \geq 0$ and that*

$$(t_1, \ldots, t_n) = (-u_s, \ldots, -u_1, 0, u_1, \ldots, u_s) \tag{19}$$

for some values $0 < u_1 < \cdots < u_s$. Then $\mathbf{p}$ in (17) has degree n and

$$X(t) = \frac{1}{A_0(-1)}, \qquad Y(t) = -\frac{A_0(t^2) - A_0(-1)}{A_0(-1)(1+t^2)}, \tag{20}$$

where
$$A_0(u) = 2u(u-u_1^2)\cdots(u-u_s^2).$$

Proof. The first step of Euclid's algorithm requires finding $q_0(t)$ and $a_2(t)$ such that
$$2t\omega_n(t) = (1+t^2)q_0(t) + a_2(t).$$
But due to the choice of the points t_i,
$$2t\omega_n(t) = 2t^2(t^2-u_1^2)\cdots(t^2-u_s^2) = A_0(t^2),$$
which is a polynomial in t^2. Therefore a_2 is a constant and no further steps of the algorithm are necessary. We find
$$X(t) = \frac{1}{a_2}, \qquad \text{and} \qquad Y(t) = -\frac{q_0(t)}{a_2},$$
where
$$q_0(t) = \frac{A_0(t^2) - A_0(-1)}{1+t^2}, \qquad \text{and} \qquad a_2(t) = A_0(-1).$$
□

If required, one can express $Y(t)$ in (20) explicitly as a polynomial. One way is to define $u_0 = 0$ and
$$B_i(u) = 2(u-u_0^2)\cdots(u-u_{i-1}^2)(-1-u_i^2)\cdots(-1-u_s^2),$$
so that
$$A_0(u) - A_0(-1) = B_{s+1}(u) - B_0(u) = \sum_{i=0}^{s}(B_{i+1}(u) - B_i(u)),$$
giving
$$\frac{A_0(u) - A_0(-1)}{1+u} = 2\sum_{i=0}^{s}(u-u_0^2)\cdots(u-u_{i-1}^2)(-1-u_{i+1}^2)\cdots(-1-u_s^2). \quad (21)$$

Alternatively, we can express $\mathbf{p}$ in Theorem 5 directly as a polynomial by noticing that since $\mathbf{p}$ has degree n, there must be some vector $\mathbf{d} \in \mathbb{R}^2$ such that
$$\mathbf{p}(t) = \sum_{j=1}^{n} L_j(t)\mathbf{r}(t_j) + \mathbf{d}\omega_n(t), \qquad L_j(t) = \prod_{k=1,k\neq j}^{n} \frac{t-t_k}{t_j-t_k}. \quad (22)$$

We can find $\mathbf{d}$ by equating the coefficients of the highest power t^n in (17) and (22). Since the leading term of $Y(t)$ is $-2t^{n-1}/A_0(-1)$ and the leading term of $\omega_n(t)$ is t^n, the leading term of $\mathbf{p}$ is $(0, -2t^n/A_0(-1))$, hence

$$\mathbf{d} = \left(0, \frac{-2}{A_0(-1)}\right) = \left(0, \frac{1}{(-1-u_1^2)\cdots(-1-u_s^2)}\right).$$

On the other hand, if all that is needed is to evaluate $\mathbf{p}$ at some t then the simplest approach is to evaluate X and Y using (20) and to substitute into (17). Such numerical evaluations could be used to represent $\mathbf{p}$ with respect to some other polynomial basis or spline basis using a quasi-interpolant approach.

Two limiting cases of $\mathbf{p}$ in Theorem 5 have been found before. If we let $u_1 = \cdots = u_s = 0$, one obtains, using (21) and (17),

$$X(t) = -(-1)^s/2, \qquad Y(t) = \sum_{i=0}^{s} (-t^2)^i,$$

$$\mathbf{p}(t) = \left(2\sum_{i=0}^{s}(-t^2)^i, 2t\sum_{i=0}^{s-1}(-t^2)^i + t(-t^2)^s\right).$$

This is the Taylor-like approximation found by Lyche and Mørken [12] having $2n$ contacts at $t = 0$. Another limiting case is $u_1 = \cdots = u_s = v > 0$, which gives

$$X(t) = \frac{-1}{2(-1-v^2)^s}, \qquad Y(t) = 1 + t^2\sum_{i=1}^{s}\frac{(t^2-v^2)^{i-1}}{(-1-v^2)^i},$$

and when these are substituted into (17), one finds, after a lengthy calculation, that $\mathbf{p}$ is the Hermite interpolant of [5] applied to the circular arc (16). This approximation has $n-1$ contacts at $t = -v$ and $t = v$ and two at $t = 0$, giving again a total of $2n$. The cubic case $n = 3$ was found earlier by Dokken et al. [3] and Goldapp [6].

Figures 1a to 1f show the interpolant $\mathbf{p}$ of Theorem 5 for various choices of s and $\mathbf{u} = (u_1, \ldots, u_s)$. The data and the error e in the interval $[0.5, 0.5]$ are respectively: (a) $n = 3$, $\mathbf{u} = (0.0)$, $e = 7.8 * 10^{-3}$; (b) $n = 3$, $\mathbf{u} = (0.5)$, $e = 7.4 * 10^{-4}$; (c) $n = 5$, $\mathbf{u} = (0.0, 0.0)$, $e = 4.9 * 10^{-4}$; (d) $n = 5$, $\mathbf{u} = (0.5, 0.5)$, $e = 1.6 * 10^{-5}$; (e) $n = 5$, $\mathbf{u} = (0.25, 0.5)$, $e = 3.6 * 10^{-6}$; (f) $n = 9$, $\mathbf{u} = (0125, 0.25, 0.375, 0.5)$, $e = 2.8 * 10^{-10}$.

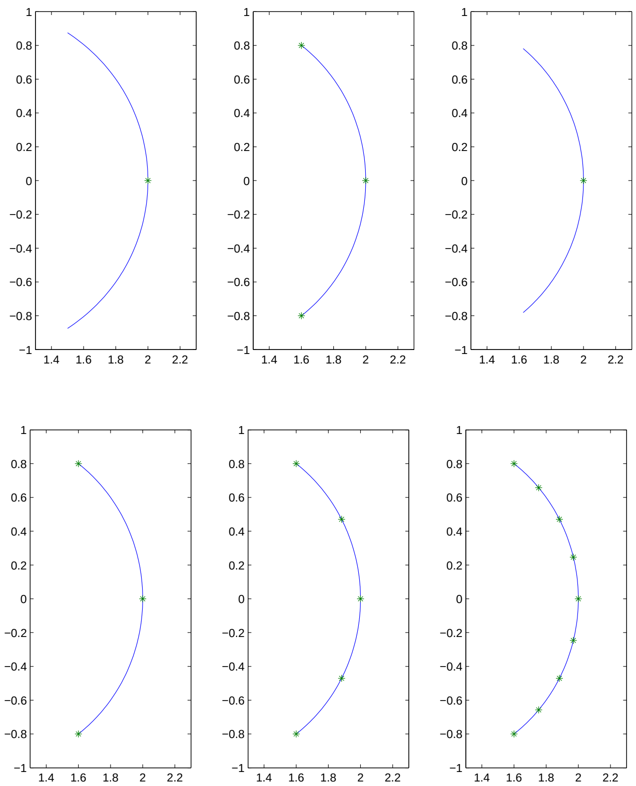

Figure 1: (a)–(c) and (d)–(f). Plots of $\mathbf{p}$ in $[-0.5, 0.5]$.

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