The approximation order of four-point interpolatory curve subdivision

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Abstract

In this paper we derive an approximation property of four-point interpolatory curve subdivision, based on local cubic polynomial fitting. We show that when the scheme is used to generate a limit curve that interpolates given irregularly spaced points, sampled from a curve in any space dimension with a bounded fourth derivative, and the chosen parameterization is chordal, the accuracy is fourth order as the mesh size goes to zero. In contrast, uniform and centripetal parameterizations yield only second order.

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1 Introduction

The subdivision scheme of Dubuc [3], which is based on cubic Lagrange interpolation, is a simple and elegant method of interpolating a sequence of regularly spaced data values with a smooth-looking function. The interpolating limit function has smoothness C^1 and its first derivative is Holder continuous with exponent $1-\epsilon$ for any small $\epsilon > 0$, and in this sense the interpolant is close to C^2; see [3, 2, 5, 6, 15, 1].

The scheme can be applied componentwise to a sequence of points in any space dimension $\mathbb{R}^d$, $d \geq 2$, yielding a parametric curve that passes through

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the points at regularly spaced parameter values. However, while the curves produced by this scheme are typically well behaved when the Euclidean distances between successive points do not vary too much, unwanted effects typically occur when the points are unevenly distributed.

One remedy is to extend Dubuc's scheme to a subdivision scheme whose limit function interpolates data values at irregularly spaced locations, using the fact that Lagrange interpolation can also be applied to non-uniformly spaced data [17, 1, 14, 10]. Warren [17] studied the extension in which the new values at each level of subdivision are placed halfway between each pair of existing function values. Later, Daubechies, Guskov, and Sweldens [1] referred to the latter scheme as *semi-regular* to distinguish it from Dubuc's scheme, which they called *regular*, and from what they called the *irregular* case in which each new function value can be located arbitrarily between two existing data values.

The advantage of the semi-regular and irregular schemes in the parametric case is that they offer the freedom to choose parameter values that reflect the geometry of the data points. It is known from numerical tests on other interpolation methods, such as polynomial and spline fitting, that choosing the length of each parameter interval to be equal to the Euclidean distance between the two corresponding data points: *chordal parameterization*, or the square root of the distance: *centripetal parameterization*, often leads to better behaved curves when the points are unevenly distributed. Moreover, it has recently been established mathematically that when the curve fitting is based on cubic polynomials, centripetal interpolation is 'stable' in the sense that the curve tends to stay close to the data polygon. This is true of C^2 cubic spline interpolation [9], four-point subdivision [4], and Catmull-Rom interpolation [18]. On the other hand, chordal interpolation, for cubic-based curve fitting, often has higher approximation order than other kinds (such as uniform and centripetal). This has been observed and established mathematically for polynomials [7], cubic spline interpolation [8], and quasi-interpolation based on cubic splines [16]. For a summary of the main results of [7] and [8], and several numerical examples, see also [12].

The goal of this paper is to show that for any 'dyadically balanced' irregular extension of Dubuc's four-point subdivision scheme, chordal interpolation has fourth order approximation power. In contrast, uniform and centripetal parameterizations only yield second order, which is no better than piecewise linear interpolation.

2 The scheme

Suppose we are given an ordered sample of points $\mathbf{f}(s_i)$, where

$$\cdots < s_{-1} < s_0 < s_1 < \cdots,$$

and $\mathbf{f}$ is some parametric curve $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$, $d \geq 2$, parameterized with respect to arc length, i.e., $|\mathbf{f}'(s)| = 1$ for all s, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The parameter values s_i are not, however, available to us. We interpolate the points $\mathbf{f}(s_i)$ with a parametric curve $\mathbf{g} : \mathbb{R} \to \mathbb{R}^d$, the limit of the subdivision scheme we will describe in a moment. First, though, we must estimate parameter values t_i for the points $\mathbf{f}(s_i)$, $i \in \mathbb{Z}$. We will use 'chordal' parameter values, i.e., we set $t_0 = 0$ and define t_i sequentially for both $i > 0$ and $i < 0$ by the equation $t_{i+1} - t_i = |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)|$.

We next initialize the scheme by setting $t_{0,k} = t_k$ and $\mathbf{g}_{0,k} = \mathbf{f}(s_k)$ for $k \in \mathbb{Z}$. Then, for each subdivision level $j \geq 0$, we choose new parameter values $t_{j+1,k}$ from the old ones, $t_{j,k}$, by the rules

$$t_{j+1,2k} = t_{j,k}, \tag{1}$$

$$t_{j+1,2k+1} \in (t_{j,k}, t_{j,k+1}). \tag{2}$$

We compute new points $\mathbf{g}_{j+1,k}$ from the old ones, $\mathbf{g}_{j,k}$, by cubic polynomial interpolation. Let $\mathbf{p}_{j,k}$ denote the parametric cubic polynomial that interpolates the points $\mathbf{g}_{j,k}, \ldots, \mathbf{g}_{j,k+3}$ at the values $t_{j,k}, \ldots, t_{j,k+3}$ respectively. We then set

$$\mathbf{g}_{j+1,2k} = \mathbf{g}_{j,k}, \tag{3}$$

$$\mathbf{g}_{j+1,2k+1} = \mathbf{p}_{j,k-1}(t_{j+1,2k+1}). \tag{4}$$

Let $h_{j,k} = t_{j,k+1} - t_{j,k}$ and

$$\lambda = \sup_{j,k} \max \left(\frac{h_{j+1,2k}}{h_{j,k}}, \frac{h_{j+1,2k+1}}{h_{j,k}} \right).$$

In general $1/2 \leq \lambda \leq 1$. For example, the choice

$$t_{j+1,2k+1} = \frac{1}{2}(t_{j,k} + t_{j,k+1}), \tag{5}$$

is the so-called semi-regular case, in which case $\lambda = 1/2$. We say that the scheme is 'dyadically balanced' [1] if $\lambda < 1$, and under this mild condition, it can be shown that the scheme (1–4) is C^1 in the sense that there is a unique continuous curve $\mathbf{g} : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{g}(t_{j,k}) = \mathbf{g}_{j,k}$ for all $j \geq 0$ and $k \in \mathbb{Z}$, and moreover $\mathbf{g} \in C^1$. This is simply a vector-valued version of a result of Daubechies, Guskov, and Sweldens [1], also proved in [10].

We will assume from now on that for each sample of points $\mathbf{f}(s_i)$, we apply the same dyadically balanced subdivision scheme, with some fixed $\lambda < 1$, such as the semi-regular scheme.

3 Numerical tests

We want to test how good an approximation $\mathbf{g}$ is to $\mathbf{f}$, as the sampling density of $\mathbf{f}$ increases. In practice we will have some finite number of samples, $\mathbf{f}(s_0), \mathbf{f}(s_1), \ldots, \mathbf{f}(s_n)$, and since the curve piece $\mathbf{g}|_{[t_k,t_{k+1}]}$ depends on the six points $\mathbf{f}(s_{k-2}), \ldots, \mathbf{f}(s_{k+3})$, we require $n \geq 5$, and we measure the approximation by the Hausdorff distance

$$d_H(\mathbf{f}|_{[s_2,s_{n-2}]}, \mathbf{g}|_{[t_2,t_{n-2}]}).$$

Defining

$$h := \max_{0 \leq i \leq n-1} (s_{i+1} - s_i),$$

we ask: at what rate does this Hausdorff distance go to zero as $h \to 0$?

We investigated this question numerically by sampling various parametric curves $\mathbf{f}$ with initial parameter values $s_{0,0} < s_{0,1} < \cdots < s_{0,n}$ and then refining these by the semi-regular scheme

$$s_{\ell+1,2k} = s_{\ell,k},$$
$$s_{\ell+1,2k+1} = \frac{1}{2}(s_{\ell,k} + s_{\ell,k+1}),$$

and setting $s_k = s_{\ell,k}$ for some ℓ, and computing numerically the error between $\mathbf{f}$ and the subsequent subdivision curve $\mathbf{g}$, in the semi-regular case (5). Even if $\mathbf{f}$ is not parameterized with respect to arc length, this test gives a good indication of the approximation order. If the order is $O(h^p)$ we should expect the error to behave like $2^{-\ell p}$ as ℓ increases.

Specifically, with the ellipse, $\mathbf{f}(s) = (2\cos s, \sin s)$, the error can be approximated from the algebraic representation

$$f(x, y) := x^2/4 + y^2 - 1 = 0.$$

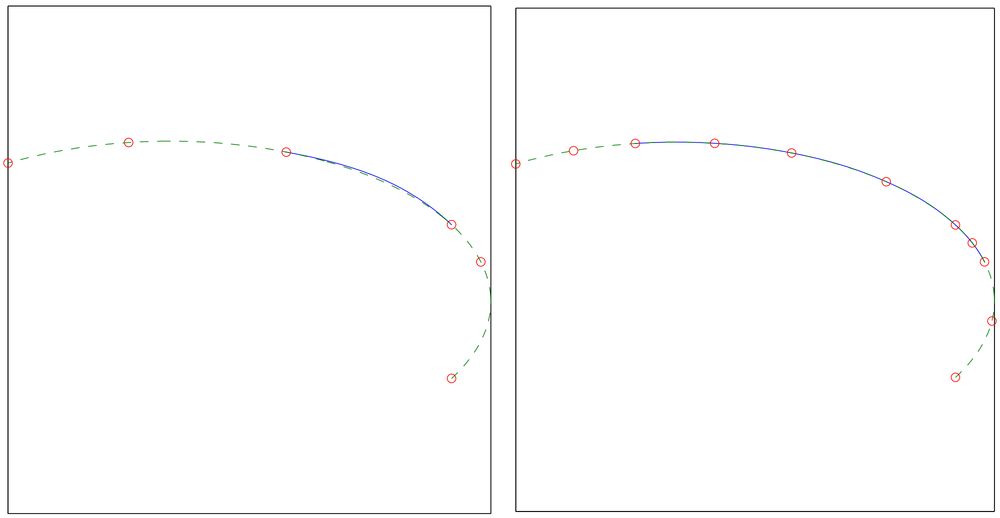

Figure 1: Ellipse and first two subdivision curves $\mathbf{g}$

The distance from this ellipse of a point (x, y) that is close to it is approximately $|f(x, y)|/\|\nabla f(x, y)\|$. Figure 1 shows the ellipse $\mathbf{f}$ and the first two approximations ($\ell = 0$ and $\ell = 1$), when starting with the initial six parameter values

$$(s_{0,0}, s_{0,1}, \ldots, s_{0,5}) = (-0.5, 0.25, 0.5, 1.2, 1.7, 2.1).$$

The second and third columns of Table 1 show the estimated Hausdorff error e_ℓ for each ℓ, and the estimated order $p_\ell = \log(e_\ell/e_{\ell-1})/\log(2)$. We clearly see a fourth order approximation emerging. In contrast, the fourth to seventh columns show the same result when using uniform and centripetal parameterizations, and in these two cases the numerics indicate second order.

4 Approximation order

Our goal is to prove that the chordal parameterization does indeed yield a fourth order approximation, if $\mathbf{f}$ is smooth enough. Similar to the analysis in [7] and [8], we compare the two curves by reparameterizing one of them, so that they have matching parameter values at the interpolation points. For the subdivision scheme we will do this initially using cubic polynomials.

ℓ	Chordal e_ℓ	p_ℓ	Uniform e_ℓ	p_ℓ	Centripetal e_ℓ	p_ℓ
0	1.26e-02		4.59e-02		2.88e-02	
1	1.42e-03	3.14	8.75e-03	2.39	6.72e-03	2.10
2	6.75e-04	1.07	3.26e-03	1.42	2.49e-03	1.43
3	4.57e-05	3.88	7.92e-04	2.04	5.88e-04	2.08
4	2.92e-06	3.97	1.95e-04	2.02	1.46e-04	2.01
5	1.84e-07	3.99	4.85e-05	2.01	3.67e-05	2.00
6	1.15e-08	4.00	1.21e-05	2.00	9.20e-06	2.00
7	7.18e-10	4.00	3.02e-06	2.00	2.30e-06	2.00

Table 1: Error and approximation order

For each k, let ϕ_k be the cubic polynomial such that $\phi_k(t_i) = s_i$ for $i = k, k+1, k+2, k+3$.

The analysis then starts with Proposition 3.1 of [7]: if $\mathbf{f} \in C^2$ then

$$0 \leq (s_{i+1} - s_i) - |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)| \leq \frac{1}{12}(s_{i+1} - s_i)^3 \max_{s_i \leq s \leq s_{i+1}} |\mathbf{f}''(s)|.$$

Thus for the chordal values t_i, we obtain

$$|(s_{i+1} - s_i) - (t_{i+1} - t_i)| \leq C(s_{i+1} - s_i)^3,$$

for some constant C, and therefore, for $s_{i+1} - s_i$ small enough,

$$\left| \frac{s_{i+1} - s_i}{t_{i+1} - t_i} - 1 \right| \leq C'(s_{i+1} - s_i)^2,$$

for some new constant C'. As shown in the proof of Theorem 2.1 of [7] (see also Theorem 1 of [12]), it follows that for any k,

$$[t_i, t_{i+1}]\phi_k - 1 = O(h^2), \quad i = k, k+1, k+2, \tag{6}$$

$$[t_i, t_{i+1}, t_{i+2}]\phi_k = O(h), \quad i = k, k+1, \tag{7}$$

$$[t_k, t_{k+1}, t_{k+2}, t_{k+3}]\phi_k = \phi_k''' = O(1), \tag{8}$$

as $h \to 0$, where

$$h = \max_{i=k,k+1,k+2} (s_{i+1} - s_i).$$

Thus all divided differences of ϕ_k are bounded as $h \to 0$. Using these cubics as reparameterizations of $\mathbf{f}$ we can establish third order approximation.

Theorem 1 *If* $\mathbf{f} \in C^4$*, then*

$$\max_{t_k \le t \le t_{k+1}} |\mathbf{f}(\phi_{k-1}(t)) - \mathbf{g}(t)| = O(h^3), \quad \textit{as } h \to 0,$$

where

$$h = \max_{k-2 \le i \le k+2} (s_{i+1} - s_i). \tag{9}$$

Proof. For each $t \in [t_k, t_{k+1}]$, we use the inequality

$$|\mathbf{f}(\phi_{k-1}(t)) - \mathbf{g}(t)| \le |\mathbf{f}(\phi_{k-1}(t)) - \mathbf{p}_{0,k-1}(t)| + |\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t)|. \tag{10}$$

Since $\mathbf{p}_{0,k-1}$ is the cubic chordal interpolant to $\mathbf{f}$, at the parameter values $t_{k-1}, \ldots, t_{k+2}$, it follows from Theorem 2.1 and Proposition 3.1 of [7] that

$$\max_{t_k \le t \le t_{k+1}} |\mathbf{f}(\phi_{k-1}(t)) - \mathbf{p}_{0,k-1}(t)| = O(h^4), \quad \text{as } h \to 0. \tag{11}$$

This deals with the first term on the right of (10). To treat the second term, we use Lemma 7 and Theorem 1 of [10], adapted to vector-valued functions. Putting $j = 0$ in the first inequality of that theorem, and restricting the inequality to the interval $[t_k, t_{k+1}]$, we deduce that there is some constant C_λ, depending only on λ, such that

$$\max_{t_k \le t \le t_{k+1}} |\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t)| \le C_\lambda h_t^3 M, \tag{12}$$

where

$$h_t = \max_{k-2 \le i \le k+2} (t_{i+1} - t_i),$$

$$M = \max_{i=k-2,k-1} |\Delta \mathbf{g}_i^{[3]}|, \qquad \Delta \mathbf{g}_i^{[3]} := \mathbf{g}_{i+1}^{[3]} - \mathbf{g}_i^{[3]},$$

and

$$\mathbf{g}_i^{[3]} := [t_i, t_{i+1}, t_{i+2}, t_{i+3}]\mathbf{g}, \qquad i = k-2, k-1, k.$$

Since $h_t \le h$, the proof will be complete if we can show that M is bounded as $h \to 0$, and for this it is sufficient to show that $|\mathbf{g}_i^{[3]}|$ is bounded as $h \to 0$ for each $i = k-2, k-1, k$. To this end, observe that

$$\mathbf{g}_i^{[3]} := [t_i, t_{i+1}, t_{i+2}, t_{i+3}](\mathbf{f} \circ \phi_i),$$

and to this expression we can apply the 'inner' divided difference chain rule of [11]. Theorem 2 of [11] gives

$$\begin{aligned}\mathbf{g}_i^{[3]} = {} & [s_i, s_{i+3}]\mathbf{f}\,[t_i, t_{i+1}, t_{i+2}, t_{i+3}]\phi_i \\ & + [s_i, s_{i+1}, s_{i+3}]\mathbf{f}\,[t_i, t_{i+1}]\phi_i\,[t_{i+1}, t_{i+2}, t_{i+3}]\phi_i \\ & + [s_i, s_{i+2}, s_{i+3}]\mathbf{f}\,[t_i, t_{i+1}, t_{i+2}]\phi_i\,[t_{i+2}, t_{i+3}]\phi_i \\ & + [s_i, s_{i+1}, s_{i+2}, s_{i+3}]\mathbf{f}\,[t_i, t_{i+1}]\phi_i\,[t_{i+1}, t_{i+2}]\phi_i\,[t_{i+2}, t_{i+3}]\phi_i. \qquad (13)\end{aligned}$$

Now, using the fact that $\mathbf{f} \in C^3$, by the Genocchi-Hermite formula ([13], Sec. 6.1),

$$|[s_i, s_{i+3}]\mathbf{f}| \le \max_{s_i \le s \le s_{i+3}} |\mathbf{f}'(s)| = 1,$$

$$|[s_i, s_{i+1}, s_{i+3}]\mathbf{f}|, |[s_i, s_{i+2}, s_{i+3}]\mathbf{f}| \le \max_{s_i \le s \le s_{i+3}} |\mathbf{f}''(s)|/2,$$

and

$$|[s_i, s_{i+1}, s_{i+2}, s_{i+3}]\mathbf{f}| \le \max_{s_i \le s \le s_{i+3}} |\mathbf{f}'''(s)|/6,$$

all of which are bounded as $h \to 0$. On the other hand, by (6–8), all the divided differences of ϕ_i in (13) are all bounded as $h \to 0$. □

The numerical tests indicate that the true approximation order is not three but four. With further analysis, and by reparameterizing the polynomial curve $\mathbf{p}_{0,k-1}$ used in inequality (10) we can confirm fourth order.

Theorem 2 *If $\mathbf{f} \in C^4$, then*

$$d_H(\mathbf{f}|_{[s_k, s_{k+1}]}, \mathbf{g}|_{[t_k, t_{k+1}]}) = O(h^4), \quad \textit{as } h \to 0,$$

with h as defined in (9).

Proof. By (11),

$$d_H(\mathbf{f}|_{[s_k, s_{k+1}]}, \mathbf{g}|_{[t_k, t_{k+1}]}) = O(h^4), \quad \text{as } h \to 0,$$

and so by the triangle inequality for the metric d_H, it is sufficient to show that

$$d_H(\mathbf{g}|_{[t_k, t_{k+1}]}, \mathbf{p}_{0,k-1}|_{[t_k, t_{k+1}]}) = O(h^4), \quad \text{as } h \to 0.$$

To prove this, we will show that there is a function $\mu_k : [t_k, t_{k+1}] \to \mathbb{R}$ such that

$$\max_{t_k \le t \le t_{k+1}} |\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t + \mu_k(t))| = O(h^4), \quad \text{as } h \to 0.$$

This requires a refinement of (12) and a more careful use of (13). Consider first (12). The arguments used in the proofs of Lemma 7 and Theorem 1 of [10] show that there must be two functions $a_k : [t_k, t_{k+1}] \to \mathbb{R}$ and $b_k : [t_k, t_{k+1}] \to \mathbb{R}$ such that

$$\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t) = a_k(t)\Delta\mathbf{g}^{[3]}_{k-2} + b_k(t)\Delta\mathbf{g}^{[3]}_{k-1}, \tag{14}$$

and

$$\max_{t_k \le t \le t_{k+1}} (|a_k(t)| + |b_k(t)|) \le C_\lambda h_t^3 \le C_\lambda h^3.$$

Next, consider (13). Applying (6–8) to (13) shows that not only is $|\mathbf{g}^{[3]}_i|$ bounded as $h \to 0$, for $i = k-2, k-1, k$, but moreover,

$$\begin{aligned} \mathbf{g}^{[3]}_i &= \phi'''_i [s_i, s_{i+3}]\mathbf{f} + [s_i, s_{i+1}, s_{i+2}, s_{i+3}]\mathbf{f} + O(h) \\ &= \phi'''_i \mathbf{f}'(s_k) + [s_i, s_{i+1}, s_{i+2}, s_{i+3}]\mathbf{f} + O(h). \end{aligned}$$

This implies that

$$\begin{aligned} \Delta\mathbf{g}^{[3]}_i &= (\phi'''_{i+1} - \phi'''_i)\mathbf{f}'(s_k) + (s_{i+4} - s_i)[s_i, \ldots, s_{i+4}]\mathbf{f} + O(h) \\ &= (\phi'''_{i+1} - \phi'''_i)\mathbf{f}'(s_k) + O(h). \end{aligned}$$

Applying this to (14) gives

$$\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t) = \mu_k(t)\mathbf{f}'(s_k) + O(h^4),$$

where

$$\mu_k(t) = (\phi'''_{k-1} - \phi'''_{k-2})a_k(t) + (\phi'''_k - \phi'''_{k-1})b_k(t),$$

and

$$\max_{t_k \le t \le t_{k+1}} |\mu_k(t)| = O(h^3).$$

Finally, by Theorem 7.1 of [7], the first derivative of $\mathbf{p}_{0,k-1}$ at t approximates the first derivative of $\mathbf{f}$ at $\phi_{k-1}(t)$ to order $O(h^2)$ and therefore

$$\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t) = \mu_k(t)\mathbf{p}'_{0,k-1}(t_k) + O(h^4) = \mu_k(t)\mathbf{p}'_{0,k-1}(t) + O(h^4).$$

We can write this as

$$\mathbf{g}(t) = \mathbf{p}_{0,k-1}(t) + \mu_k(t)\mathbf{p}'_{0,k-1}(t) + O(h^4).$$

But if we let $t_* := t + \mu_k(t)$, then Taylors series gives

$$\mathbf{p}_{0,k-1}(t_*) = \mathbf{p}_{0,k-1}(t) + \mu_k(t)\mathbf{p}'_{0,k-1}(t) + O(h^6),$$

and hence,

$$\mathbf{g}(t) - \mathbf{p}_{0,k-1}(t_*) = O(h^4),$$

as claimed. □

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