Nonlinear Stationary Subdivision

Michael S. Floater
SINTEF
P. O. Box 124 Blindern, 0314 Oslo, Norway
E-mail: michael.floater@math.sintef.no

Charles A. Micchelli
IBM Corporation
T.J. Watson Research Center
Department of Mathematical Sciences
P. O. Box 218
Yorktown Heights, N. Y. U.S.A. 10598
E-mail: cam@watson.ibm.com

Abstract

In this paper we study a concrete interpolatory subdivision scheme based on rational interpolation and show that it preserves convexity.

§1. Introduction

There are two ideas which interest us in this paper. The first is to explore concrete possibilities of *nonlinear* stationary subdivision strategies. The linear case is treated in some detail in [7] where its connection to modeling of curves as well to wavelet construction is highlighted. The second idea that is featured here is that of convexity preserving interpolatory subdivision, as studied, for example, in [6].

One of the most celebrated concrete interpolatory subdivision schemes was introduced in [1]. This scheme is based upon *local polynomial interpolation* and is intimately connected to orthonormal wavelet construction. In [8] local *exponential* interpolation with *real* frequencies was considered and also shown to lead to a multiparameter family of wavelets of minimal support. Further developments of these ideas appear in [5] and [9]. The paper [9] reveals their applicability to wavelet construction in Sobolev spaces while [5] focuses upon the connection to the construction of conjugate filters with prescribed zeros.

In this paper we propose to generate nonlinear interpolatory subdivision by *rational* interpolation. We study one such example and prove that when the data is convex the interpolant to the data generated by the subdivision scheme is likewise convex. A counterexample to higher order convexity preservation is also presented.

§2. Nonlinear Subdivision

In this section we formulate a notion of nonlinear stationary subdivision. All the methods we consider are *stationary* and *homogeneous*. This terminology, borrowed from the theory of Markoff chains, means that we always iterate the *same* operator–*homogeniety* and this operator commutes with shift by an integer—*stationarity*. Stationarity of subdivision must be formulated with care. To be precise, we consider the linear space X of all bi-infinite real sequences $x = (x_j : j \in \mathbb{Z})$. On this space acts the forward shift operator $T : X \to X$ defined by the equation

$$(Tx)_j = x_{j+1}, \qquad j \in \mathbb{Z}. \tag{2.1}$$

Let $F : X \to X$ be *any* mapping from X into itself. We say F is *stationary* provided that it commutes with the shift operator T, in other words

$$TF = FT.$$

Every such mapping is determined by *one scalar-valued* function $f : X \to \mathbb{R}$ by the formula

$$(Fx)_i = f(T^i x), \qquad i \in \mathbb{Z}.$$

For instance, if $a = (a_i : i \in \mathbb{Z})$ is a bi-infinite vector of compact support, that is $a_j = 0$, $j < l$ or $j > m$ for some integers l and m, and f is given by

$$f(x) = \sum_{j \in \mathbb{Z}} a_{-j} x_j,$$

then the function $F(x)$ becomes the *convolution* of a with x, that is,

$$(Fx)_i = \sum_{j \in \mathbb{Z}} a_j x_{i-j}, \qquad i \in \mathbb{Z}. \tag{2.2}$$

This is the only linear and stationary mapping.

A subdivision operator $S_{\mathbf{h}} : X \to X$ is determined by *two* scalar-valued mappings $\mathbf{h} := (h_0, h_1) : X \to \mathbb{R}^2$ and is defined by the formula

$$(S_{\mathbf{h}}x)_i := \begin{cases} h_0(T^j x), & i = 2j; \\ h_1(T^j x), & i = 2j+1. \end{cases} \tag{2.3}$$

If we introduce the bi-infinite sequence $(y_k(x) : k \in \mathbb{Z}/2)$ defined by

$$y_k(x) := (S_{\mathbf{h}}x)_{2k}, \qquad k \in \mathbb{Z}/2 \tag{2.4}$$

then it easily follows that $y_{k+1}(x) = y_k(Tx)$, that is, $S_{\mathbf{h}}$ is *stationary* when the vector in the range of $S_{\mathbf{h}}$ is indexed over the fine lattice $\mathbb{Z}/2$. When each of the functions h_0 and h_1 are linear, in particular, $h_\ell(x) = \sum_{k\in\mathbb{Z}} a_{2k+\ell}x_{-k}$, for $\ell \in \{0,1\}$, $x = (x_k : k \in \mathbb{Z}) \in X$ and $a = (a_i : i \in \mathbb{Z})$ is a prescribed bi-infinite vector of finite support then $S_{\mathbf{h}}$ has the familiar form

$$(L_a x)_i = \sum_{k\in\mathbb{Z}} a_{i-2k}x_k. \tag{2.5}$$

Similar to the linear case we say that $S_{\mathbf{h}}$ converges with respect to some subspace $Y \subset X$ if for every $x \in Y$ there is a function f_x which is continuous on $\mathbb{R}$ such that

$$\lim_{r\to\infty} \sup\{|(S^r_{\mathbf{h}}x)_j - f_x(j/2^r)| : j \in \mathbb{Z}\} = 0 \tag{2.6}$$

and for some $x \in Y$ we have that $f_x \neq 0$.

We will now give an example of a nonlinear subdivision scheme generated by *local rational interpolation*. To this end, let us consider the following problem. Given points $x_0, x_1, \ldots, x_{n+1}$ we wish to find a rational function R of the form P/Q where P is a polynomial of degree at most n and Q has degree at most one such that

$$R(t_j) = x_j, \qquad j = 0, 1, \ldots, n+1 \tag{2.7}$$

where $t_0 < t_1 < \cdots < t_{n+1}$ are prescribed. There is a special circumstance which should be considered separately. To this end, for $i = 0$ or 1 we let $w_i = [x_i, x_{i+1}, \ldots, x_{i+n}]$, be the divided difference of $x_i, x_{i+1}, \ldots, x_{i+n}$ at $t_i, t_{i+1}, \ldots, t_{i+n}$. Recall that

$$w := \frac{w_1 - w_0}{t_{n+1} - t_0} = [x_0, x_1, \ldots, x_{n+1}]$$

is the leading coefficient of the polynomial of degree at most $n+1$ (one more than we require for P above) which satisfies the interpolation conditions (2.7). Hence, when $w_1 = w_0$ this polynomial has at most degree n and is the solution R to our interpolation problem. There remains the case that $w_1 \neq w_0$. We only discuss the case that $w_1 w_0 > 0$. In this case our rational interpolant is given by the formula

$$R(t) = \frac{w_1(t_{n+1} - t)p_-(t) + w_0(t - t_0)p_+(t)}{w_1(t_{n+1} - t) + w_0(t - t_0)} \tag{2.8}$$

where in this formula p_- and p_+ are polynomials of degree at most n which interpolate the $n+1$ data $x_0, x_1, \ldots, x_n$ at $t_0, t_1, \ldots, t_n$ and $x_1, x_2, \ldots, x_{n+1}$ at $t_1, t_2, \ldots, t_{n+1}$, respectively. It is important to rewrite this function in another form. To this end we let p be the unique polynomial of degree at most $n - 1$ which solves the interpolation problem

$$p(t_j) = x_j, \qquad j = 1, 2, \ldots, n.$$

Proposition 2.1.

$$R(t) = p(t) + (t - t_1) \cdots (t - t_n) \frac{w_0 w_1}{w_0 \lambda_0(t) + w_1 \lambda_1(t)}$$

where

$$\lambda_1(t) = \frac{t_{n+1} - t}{t_{n+1} - t_0}, \qquad \lambda_0(t) = \frac{t - t_0}{t_{n+1} - t_0}$$

are the barycentric coordinates of t relative to t_0 and t_{n+1}.

Proof: Using the Newton form of polynomial interpolation we have that

$$p_+(t) = p(t) + w_1(t - t_1) \cdots (t - t_n)$$

and

$$p_-(t) = p(t) + w_0(t - t_1) \cdots (t - t_n).$$

Substituting these two formulas into equation (2.8) and simplifying the resulting expression proves the formula. $\square$

To motivate the use of this result for subdivision we restrict ourselves to the case that $n = 2$. We choose $t_i = i - 1$, $i = 0, 1, 2, 3$, interpolate the data x_{-1}, x_0, x_1, x_2 and evaluate the rational interpolant at $t = 1/2$. This gives us the formula

$$R(1/2) = \frac{x_0 + x_1}{2} - \frac{1}{8} H(x_{-1} - 2x_0 + x_1,\, x_0 - 2x_1 + x_2) \tag{2.9}$$

where $H(a,b)$ is the *harmonic mean* defined for $a,b \in \mathbb{R}_+$ by the formula

$$H(a,b) = \begin{cases} \frac{2ab}{a+b}, & (a,b) \neq (0,0); \\ 0, & (a,b) = (0,0). \end{cases} \tag{2.10}$$

To ensure the validity of formula (2.9) we must demand that both $x_1 - 2x_0 + x_{-1}$ and $x_0 - 2x_1 + x_2$ are positive. Note the important fact that

$$H(a,b) \leq 2 \min(a,b), \qquad a,b \in R_+.$$

In formula (2.9) we think of the nonlinear term as a *perturbation* of the linear term. This suggests to us to multiply the nonlinear term by a relaxation parameter and also use this formula to generate the following nonlinear subdivision scheme.

Let $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given continuous function. For any bi-infinite vector $x = (x_i : i \in \mathbb{Z})$ we set for $l \in \mathbb{Z}$

$$\Delta^2 x_l = x_l - 2x_{l+1} + x_{l+2}.$$

Choose $w \in \mathbb{R}$ and define the bi-infinite vector $y = (y_i : i \in \mathbb{Z})$ by the formulas

$$y_{2i} = x_i, \qquad i \in \mathbb{Z}$$

$$y_{2i+1} = \frac{x_i + x_{i+1}}{2} - \lambda E(\Delta^2 x_{i-1}, \Delta^2 x_i), \qquad i \in \mathbb{Z}. \tag{2.11}$$

This defines a nonlinear subdivision scheme. The special case above would correspond to $\lambda = 1/8$ and

$$E(a,b) := H(|a|,|b|). \tag{2.12}$$

Hence, when the vector x is convex, that is $\Delta^2 x_i \geq 0, i \in \mathbb{Z}$, and $\lambda = 1/8$, (2.11) reduces to (2.9). In this case the scheme reproduces rational polynomials whose numerators are of degree at most two and whose denominators have degree at most one.

We also remark that when

$$E(a,b) = \frac{1}{2}(a+b), \qquad a,b \in \mathbb{R}$$

and $\lambda = 1/8$ the linear subdivision scheme (2.11) is obtained by interpolating the data $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ at the points $i-1$, i, $i+1$, $i+2$, respectively by a *cubic* polynomial and evaluating the resulting polynomial at the point $i+1/2$. This is the special case of the interpolatory subdivision scheme introduced in [1]

Returning to (2.11) we shall rewrite it in another form. To this end, we introduce the sequence $m=(m_j : j\in \mathbb{Z})$ defined by setting $m_0=1, m_1=m_{-1}=1/2$ and $m_j=0$, $j\notin\{-1,0,1\}$. Then the subdivision scheme

$$(L_m x)_i = \sum_{j\in\mathbb{Z}} m_{i-2j}x_j \tag{2.13}$$

corresponds to the linear term in (2.11) and L_m converges with limit

$$f_x(t) = \sum_{j\in\mathbb{Z}} x_j M(t-j), \qquad t\in\mathbb{R} \tag{2.14}$$

where $M(t)=\max(1-|t|,0)$, $t\in\mathbb{R}$. The function f_x is the piecewise linear function with breakpoints at integers such that $f_x(j)=x_j$, $j\in\mathbb{Z}$. Also, we set for $x\in X$

$$\mathbf{e}(x) = (0, E(x_{-1}-2x_0+x_1, x_0-2x_1+x_2)). \tag{2.15}$$

This mapping $\mathbf{e}: X\to\mathbb{R}^2$ determines a nonlinear subdivision scheme and thus our scheme (2.11) has the form

$$F_m(x) = L_m(x) - \lambda S_{\mathbf{e}}(x), \qquad x\in X.$$

When E is *nonnegative* and $\lambda\geq 0$ then formula (2.11) shows for $i\in\mathbb{Z}$ that

$$F_m(x)\leq L_m(x).$$

For λ and E further constrained the subdivision scheme (2.11) has the useful property of being *convexity preserving*. This fact is proved next. We let

$$C_2 = \{x : x\in X, \Delta^2 x_i\geq 0, i\in\mathbb{Z}\}.$$

Proposition 2.2. Suppose there is a constant $\mu>0$ such that for $a,b\in\mathbb{R}_+$

$$0\leq E(a,b)\leq \mu\min(a,b). \tag{2.16}$$

Then for $0\leq\lambda\leq\frac{1}{4\mu}$,

$$F_m(C_2)\subset C_2.$$

Proof. First, we observe for all $j \in \mathbb{Z}$ that

$$\begin{aligned}(\Delta^2 F_m(x))_{2j} &= (F_m(x))_{2j} - 2(F_m(x))_{2j+1} + (F_m(x))_{2j+2} \\ &= x_j - 2\{\frac{x_j + x_{j+1}}{2} - \lambda E(\Delta^2 x_{j-1}, \Delta^2 x_j)\} + x_{j+1} \\ &= 2\lambda E(\Delta^2 x_{j-1}, \Delta^2 x_j) \geq 0\end{aligned}$$

and then we verify that

$$\begin{aligned}&(\Delta^2 F_m(x))_{2j-1} \\ &= (F_m(x))_{2j-1} - 2(F_m(x))_{2j} + (F_m(x))_{2j+1} \\ &= \frac{x_{j-1} + x_j}{2} - 2x_j + \frac{x_j + x_{j+1}}{2} \\ &\quad -\lambda\{E(\Delta^2 x_{j-2}, \Delta^2 x_{j-1}) + E(\Delta^2 x_{j-1}, \Delta^2 x_j)\} \\ &= \frac{1}{2}\Delta^2 x_{j-1} - \lambda\{E(\Delta^2 x_{j-2}, \Delta^2 x_{j-1}) + E(\Delta^2 x_{j-1}, \Delta^2 x_j)\} \\ &\geq \frac{1}{2}\Delta^2 x_{j-1} - 2\lambda\mu\Delta^2 x_{j-1} = \frac{1}{2}(1 - 4\lambda\mu)\Delta^2 x_{j-1}.\end{aligned}$$

□

Remark 2.3. When in addition E has the property that $E(a, b) > 0$ whenever a and b are positive then strictly convex data, $\Delta^2 x_i > 0$, $i \in \mathbb{Z}$ is mapped into such by the the nonlinear scheme (2.11).

From Proposition 2.2 follows the next result.

Theorem 2.4. Let $x^0 \in C_2$, define $x^r := F_m^r(x^0), r \in \mathbb{Z}_+$ and suppose the hypothesis of Proposition 2.2 holds. Then the sequence of polygonal lines

$$f^r(t) := \sum_{j \in \mathbb{Z}} x_j^r\, M(2^r t - j), \qquad t \in \mathbb{R}$$

for $r \in \mathbb{Z}_+$ satisfies the following properties.

(i) For all $t \in \mathbb{R}$ and $r \in \mathbb{Z}_+$

$$f^r(t) \geq f^{r+1}(t).$$

(ii) For all $r \in \mathbb{Z}_+$, f^r is convex on $\mathbb{R}$.

(iii) $\lim_{r \to \infty} f^r(t) = f_x(t)$, $t \in \mathbb{R}$, f_x is continuous on $\mathbb{R}$ and $f_x(j) = x_j$, $j \in \mathbb{Z}$.

Proof : Since M is the refinable function associated with the linear subdivision scheme L_m we have for $t \in \mathbb{R}$ and $r \in \mathbb{Z}_+$ that

$$f^{r+1}(t) - f^r(t) = \sum_{j \in \mathbb{Z}} (F_m^{r+1}(x^0) - L_m F_m^r(x^0))_j M(2^{r+1}t - j). \tag{2.17}$$

But

$$F_m^{r+1}(x^0) = F_m(F_m^r(x^0)) \leq L_m(F_m^r(x^0)) \tag{2.18}$$

and so (i) follows.

The function f^r is convex if and only if $x^r \in C_2$ and that is assured by Proposition 2.2 which takes care of (ii). Finally, considering (iii), for $i \in \mathbb{Z}$ let $l_i : \mathbb{R} \to \mathbb{R}$ be the unique linear interpolant satisfying $l_i(i) = x_i$ and $l_i(i+1) = x_{i+1}$ and let $b : \mathbb{R} \to \mathbb{R}$ be the piecewise linear function defined for $t \in [j, j+1]$, $j \in \mathbb{Z}$, as $b(t) = \max(l_{j-1}(t), l_{j+1}(t))$. Due to standard properties of convex functions, all convex interpolants $g : \mathbb{R} \to \mathbb{R}$, $g(j) = x_j$, are bounded below by b and therefore the sequence of functions f^r is bounded below by b. Hence the limit in (iii) exists and must be continuous since it is convex. □

Remark 2.5. For any $p > 0$ we define

$$E_p(a,b) = \phi^{-1}\left(\frac{\phi(|a|) + \phi(|b|)}{2}\right) = \frac{2^{1/p}\,|a|\,|b|}{(|a|^p + |b|^p)^{1/p}} \tag{2.19}$$

where $\phi(t) := t^{-p}$, $t \in \mathbb{R}_+$. Then for any $a, b \in \mathbb{R}$

$$0 \leq E_p(a,b) \leq 2^{1/p} \min(|a|, |b|)$$

so that this family of means satisfies the hypothesis of Proposition 2.2 with $\mu = 2^{1/p}$ and so the corresponding scheme produces a convex interpolant to convex data $\{x_j : j \in \mathbb{Z}\}$ for $0 \leq \lambda \leq 2^{-(2+1/p)}$. When

$$E(a,b) = \frac{1}{2}(a+b), \qquad a, b \in \mathbb{R} \tag{2.20}$$

the hypothesis of Proposition 2.2 is *not* satisfied. This *linear* subdivision scheme is precisely the one studied in [2] in which $w = \lambda/2$ plays the role of tension parameter. This scheme does not preserve convexity even when it converges.

The case of the nonlinear means above for $p = 1$ corresponding to the harmonic mean was independently considered in [3]. We fell upon it through the process of rational interpolation. This fact seems to have not been noticed in [3] as well as the monotonicity of the polygonal lines embodied in (i) of Theorem 2.4. Since then we received [4] where similar issues are studied further.

We remark that the proof of Proposition 2.2 shows that for arbitrary $\lambda \geq 0$, property (i) still holds. Hence the subdivision scheme converges for every $x \in X$ in the sense that

$$\lim_{r\to\infty} f^r(t)$$

exists for each $t \in \mathbb{R}$ but we cannot rule out that the limit function may be $-\infty$ at some point and also not continuous. For a more restrictive range of λ we prove convergence in terms of our definition (2.6). To this end we define the subspace of X

$$Y_2 = \{x : x \in X, ||\Delta^2 x||_\infty < \infty\},$$

where $\|x\|_\infty := \sup\{|x_i| : i \in \mathbb{Z}\}$ for a bi-infinite sequence $x = \{x_i : i \in \mathbb{Z}\}$.

Theorem 2.6. Suppose there exists a constant $\rho > 0$ such that for all $a, b \in \mathbb{R}$

$$|E(a,b)| \leq \rho \max(|a|, |b|). \tag{2.21}$$

Then for $|\lambda| < (4\rho)^{-1}$, the subdivision scheme (2.11) converges with respect to Y_2 and the limit function f_x interpolates x_j at $t = j$, $j \in \mathbb{Z}$, that is,

$$f_x(j) = x_j, \qquad j \in \mathbb{Z}.$$

Remark 2.7. When ϕ is a monotonic function on $\mathbb{R}_+$ (either increasing or decreasing) then

$$E(a,b) = \phi^{-1}\left(\frac{\phi(|a|) + \phi(|b|)}{2}\right)$$

has the property that

$$E(a,b) \leq \max\{|a|, |b|\}.$$

In fact, if $|a| \leq |b|$ and ϕ increases then ϕ^{-1} also does so that

$$\phi^{-1}\left\{\frac{\phi(|a|) + \phi(|b|)}{2}\right\} \leq \phi^{-1}\left\{\frac{2\phi(|b|)}{2}\right\} = |b|.$$

Likewise, if $|a| \leq |b|$ and ϕ decreases then so does ϕ^{-1} and so since $\phi(|a|) \geq \phi(|b|)$,

$$\frac{\phi(|a|) + \phi(|b|)}{2} \geq \phi(|b|)$$

from which we get again

$$E(a,b) \le |b|.$$

Hence (2.11) converges for $|\lambda| < \frac{1}{4}$.

When $E(a,b) = \frac{1}{2}(a+b)$, (2.11) is the scheme in [2]. In this case the equivalent result that the scheme converges for $|w| < \frac{1}{8}$ is proved there.

Alternatively, this linear scheme can be written in the standard form (2.5) where $a_j = 0$, $|j| > 3$ and $a_{-3} = a_3 = -w$, $a_{-2} = a_2 = 0$, $a_{-1} = a_1 = \frac{1}{2} + w$, $a_0 = 1$. In this case, the symbol of the scheme is given by

$$a(z) = \sum_{j \in \mathbb{Z}} a_j z^j = 1 + (w + \frac{1}{2})(z^1 + z^{-1}) - w(z^3 + z^{-3}).$$

When $z = e^{i\theta}$ and $x = \cos\theta$ we have that

$$a(z) = (1+x)(1 + 8wx(1-x)).$$

A direct computation confirms that when $-\frac{1}{2} < w \le \frac{1}{16}$, $a(e^{i\theta}) \ge 0$ for $|\theta| \le \pi$ with equality if and only if $x = -1$. Hence it follows that the scheme converges for this range, [8]. Moreover, the refinable function f corresponding to the vector $x_j = 0$, $j \in \mathbb{Z}\backslash\{0\}$, $x_0 = 1$ which is supported on $(-3,3)$ is the autocorrelation of a refinable function ϕ_w which yields an orthonormal wavelet of finite support (for an explanation of wavelet construction see Chapter 2 of [7]) . When $w = \frac{1}{16}$ this wavelet is one of the family constructed by Daubechies, [8]. The special case $w = \frac{1}{16}$ corresponds to local cubic interpolation and is a special case of the schemes appearing in [1].

Proof: According to formula (2.17) we have that

$$\|f^{r+1} - f^r\|_\infty \le |\lambda|\,\rho\,\|\Delta^2 x^r\|_\infty$$

where $x^r = F_m^r(x)$ and $\|f\|_\infty := \sup\{|f(t)| : t \in \mathbb{R}\}$. Also, returning to the proof of Proposition 2.2 we see for $j \in \mathbb{Z}$ and $r \in \mathbb{N}$ that

$$|(\Delta^2 x^r)_{2j}| \le 2|\lambda|\,\rho\,\|\Delta^2 x^{r-1}\|_\infty \tag{2.22}$$

and

$$|(\Delta^2 x^r)_{2j+1}| \le (\frac{1}{2} + 2|\lambda|\rho)\,\|\Delta^2 x^{r-1}\|_\infty. \tag{2.23}$$

Combining the inequalities (2.22) and (2.23) we obtain for $r \in \mathbb{Z}_+$ the inequality

$$\|f^{r+1} - f^r\|_\infty \le |\lambda|\rho\,\gamma^r\,\|\Delta^2 x^0\|_\infty \tag{2.24}$$

where $\gamma := \frac{1}{2} + 2|\lambda|\rho$. Since by hypothesis, $\gamma < 1$, this proves $\lim_{r\to\infty} f^r = f_x$ uniformly on $\mathbb{R}$.

Next, we shall show f_x is Hölder continuous and afterwards that it is the limit of the subdivision scheme in the sense of (2.6). For the first claim we note that since $f^r(\frac{t}{2^r})$ is a piecewise linear function which interpolates x_j^r at $t = j$, $j \in \mathbb{Z}$ it follows for $r \in \mathbb{Z}_+$, $t, s \in \mathbb{R}$ that

$$|f^r(t) - f^r(s)| \leq 2^r \|\Delta x^r\|_\infty |t - s|.$$

From the formula (2.11) and our hypothesis it follows for $r \in \mathbb{N}$ that

$$\|\Delta x^r\|_\infty \leq \frac{1}{2}\|\Delta x^{r-1}\|_\infty + |\lambda|\rho\|\Delta^2 x^{r-1}\|_\infty.$$

Consequently, we have that

$$\begin{aligned}\|\Delta x^r\|_\infty &\leq (\frac{1}{2} + 2|\lambda|\rho)\|\Delta x^{r-1}\|_\infty \\ &\leq (\frac{1}{2} + 2|\lambda|\rho)^r\|\Delta x^0\|_\infty\end{aligned}$$

and so for $r \in \mathbb{Z}_+$, $t, s \in \mathbb{R}$ we have that

$$|f^r(t) - f^r(s)| \leq (2\gamma)^r \|\Delta x^0\|_\infty |t - s|. \tag{2.25}$$

Combining this inequality with (2.24) we conclude with the help of Lemma 2.1 of [7], p.82, that f_x is Hölder continuous with exponent $\mu = -\log_2 \gamma$ which is positive since $\gamma < 1$. Thus there is a constant $C > 0$ such that

$$|f_x(t) - f_x(s)| \leq C|t - s|^\mu,$$

for all $t, s \in \mathbb{R}$. To finish the proof we note that the function

$$g^r(t) = \sum_{j\in\mathbb{Z}} f_x(\frac{j}{2^r})M(2^r t - j), \qquad t \in \mathbb{R}$$

has the property that

$$|g^r(t) - f_x(t)| \leq 2^{-\mu r}C, \qquad t \in \mathbb{R}.$$

Hence, we have the bound

$$\begin{aligned}\sup\{|x_j^r &- f_x(\frac{j}{2^r})| : j \in \mathbb{Z}\} \\ &\leq \sup\{|\sum_{j\in\mathbb{Z}}(x_j^r - f_x(\frac{j}{2^r}))M(2^r t - j)| : t \in \mathbb{R}\} \\ &\leq \|f^r - f_x\|_\infty + 2^{-\mu r}C, \qquad r \in \mathbb{Z},\end{aligned}$$

and sending $r \to \infty$ proves the result. □

One may be optimistic that interpolatory subdivision based on rational interpolation with one pole as appears in Proposition 2.1 with $n \geq 3$ would preserve higher order convexity. Unfortunately, this conjecture fails as the next observation indicates. Specifically, instead of the nonlinear scheme of the form (2.11) we consider, for a given function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is *symmetric*, i.e. $F(s,t) = F(t,s)$, $s,t \in \mathbb{R}$, and a constant $\mu \in \mathbb{R}$, the nonlinear subdivision scheme

$$y_{2i} = x_i, \qquad i \in \mathbb{Z},$$

$$y_{2i+1} = A(x_i, x_{i+1}) - \frac{1}{8}A(\Delta^2 x_{i-1}, \Delta^2 x_i) + \mu F(\Delta^4 x_{i-2}, \Delta^4 x_{i-1}), \quad i \in \mathbb{Z}. \tag{2.26}$$

where

$$A(s,t) = \frac{1}{2}(t+s), \qquad t,s \in \mathbb{R}.$$

For the choice $\mu = 3/128$ and $F(s,t) = A(s,t)$ this becomes the linear interpolatory subdivision scheme in [1] corresponding to quintic interpolation. For the choice $\mu = 3/128$ and $F(s,t) = H(s,t)$, with H given in (2.10), the scheme corresponds to rational interpolation with rational polynomials P/Q where P is quartic and Q linear. When $\mu = 0$, the scheme reduces to that of [1] corresponding to cubic interpolation, or the scheme (2.11) with $\lambda = 1/8$.

For ease of notation we set $\alpha_i = \Delta^4 x_i$ and $F_i = F(\alpha_{i-2}, \alpha_{i-1}), i \in \mathbb{Z}$ in the computations we perform next.

Let us now investigate the higher order convexity preservation of this scheme. To this end, we recall that

$$\Delta^4 y_\ell = y_\ell - 4y_{\ell+1} + 6y_{\ell+2} - 4y_{\ell+3} + y_{\ell+4}, \qquad \ell \in \mathbb{Z}$$

and therefore we obtain

$$\Delta^4 y_{2i} = \frac{1}{4}\alpha_{i-1} - 4\mu(F_i + F_{i+1}), \qquad i \in \mathbb{Z} \tag{2.27}$$

and

$$\Delta^4 y_{2i+1} = -\frac{1}{8}A(\alpha_{i-1}, \alpha_i) + \mu(F_i + 6F_{i+1} + F_{i+2}), \qquad i \in \mathbb{Z} \tag{2.28}$$

We note for the case $\mu = 3/128$ and $F(t,t) = t$, $t \in \mathbb{R}$ that whenever $x_i = p(i), i \in \mathbb{Z}$, $p \in \pi_4$, and $\alpha_i = k$, we have that

$$\Delta^4 y_i = \frac{1}{16}k, \qquad i \in \mathbb{Z}$$

which means the scheme reproduces quartic polynomials.

If the subdivision procedure (2.26) preserves four-convexity, that is, whenever $\Delta^4 x_i \geq 0$, $i \in \mathbb{Z}$ it follows that $\Delta^4 y_i \geq 0$, $i \in \mathbb{Z}$ we conclude first from (2.27) that for any sequence $\alpha_i \geq 0$, $i \in \mathbb{Z}$ we have

$$\alpha_{i-1} \geq 16\mu(F_i + F_{i+1}), \qquad i \in \mathbb{Z} \tag{2.29}$$

and then from (2.28) that

$$\alpha_{i-1} + \alpha_i \leq 16\mu(F_i + 6F_{i+1} + F_{i+2}), \qquad i \in \mathbb{Z}. \tag{2.30}$$

For any $a, b \geq 0$, letting $\alpha_{i-1} = a$, $\alpha_{i-2} = \alpha_i = b$ in (2.29) yields

$$a \geq 32\mu F(a, b) \tag{2.31}$$

and letting $\alpha_{i-2} = \alpha_i = a$, $\alpha_{i-1} = \alpha_{i+1} = b$ in (2.30) yields

$$a + b \leq 128\mu F(a, b). \tag{2.32}$$

Combining (2.31) and (2.32) we conclude for $a, b \geq 0$ that

$$b \leq 3a,$$

an apparent contradiction.

§3. Numerical Examples

The subdivision scheme (2.11) was applied to two data sets (i) and (ii), tabulated in Tables 1 and 2 respectively

t_i	-2.0	-1.0	0.0	1.0	2.0	3.0	4.0	5.0	6.0
x_i	0.00	0.00	0.00	0.15	0.50	2.00	5.00	13.0	21.0

Table 1. Data set (i)

t_i	-2.0	-1.0	0.0	1.0	2.0	3.0	4.0	5.0	6.0
x_i	0.00	0.00	0.00	0.15	1.00	2.00	5.00	13.0	21.0

Table 2. Data set (ii)

Figures 1, 2, 3, 4, and 5 show the result of applying the scheme to data set (i) while Figures 6 and 7 depict the ouput when starting with data set (ii). In Figures 1 and 6 we used the linear scheme in which $E(a, b)$ is the arithmetic mean (2.20) and $\lambda = 1/8$. Figures 2 and 7 show the result of the rational convexity preserving scheme defined by setting $E(a,b)$ to be the harmonic mean (2.12) and $\lambda = 1/8$. In Figure 3 the harmonic mean was used again, this time setting $\lambda = 1/16$, thereby increasing the tension. This scheme also preserves convexity. The harmonic mean (2.12) is equivalent to the function E_1 where E_p is defined in (2.19). Figure 4 shows the result of applying instead E_2 and this time $\lambda = 1/(4\sqrt{2})$ which is the upper limit of the range of λ for which Proposition 2.2 guarantees convexity preservation. Figure 5 on the hand shows the output using $E_{1/2}$ and $\lambda = 1/16$ which again is the limit of the range in Proposition 2.2.

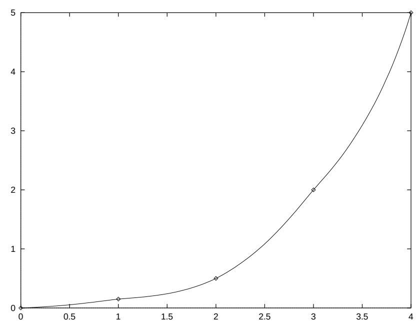

Fig. 1. Linear, $\lambda = 1/8$, data (i).

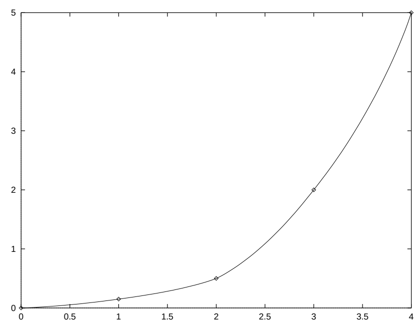

Fig. 2. Harmonic, $\lambda = 1/8$, data (i).

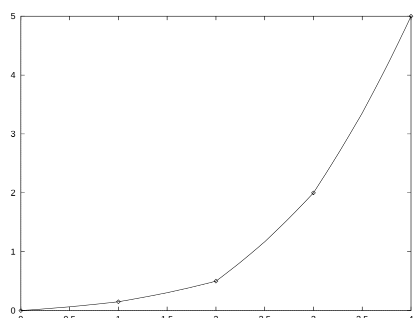

Fig. 3. Harmonic, $\lambda = 1/16$, data (i).

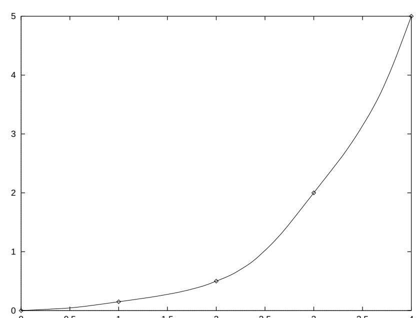

Fig. 4. E_2, $\lambda = 1/(4\sqrt{2})$, data (i).

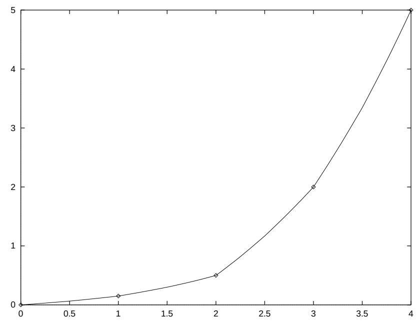

Fig. 5. $E_{1/2}$, $\lambda = 1/16$, data (i).

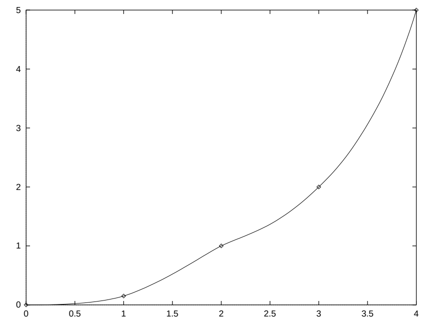

Fig. 6. Linear, $\lambda = 1/8$, data (ii).

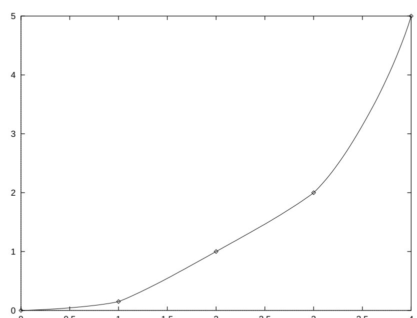

Fig. 7. Harmonic, $\lambda = 1/8$, data (ii).

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