Piecewise Linear Prewavelets on Arbitrary Triangulations

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Abstract: This paper studies locally supported piecewise linear prewavelets on bounded triangulations of arbitrary topology. It is shown that a concrete choice of prewavelets form a basis of the wavelet space when the degree of the vertices in the triangulation is not too high.

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1. Introduction

In this paper we investigate piecewise linear prewavelets on a bounded triangulation. Bases of locally supported prewavelets are emerging as a useful tool for data analysis and modelling. They provide a hierarchical approach to storage and modification of data or functions and efficient algorithms for decomposition, reconstruction, editing and compression. Consequently, piecewise linear prewavelets with small support are of interest for various applications ranging from approximation theory and the numerical solution of partial differential equations to computer graphics and practical large-scale data representation. We are particularly concerned with the latter two and we have included at the end of this paper the decomposition of a terrain model as a practical example. An advantage of also allowing the triangulation to be of arbitrary topology is that functions can be decomposed even when they are not defined over regularly shaped domains. Moreover, although our results are formulated for planar triangulations, it is fairly straightforward to extend them to spatial (surface) triangulations with computer graphics in mind.

Traditional wavelet theory, as presented for example in the well-known monographs by Daubechies [5], Meyer [11], and Chui [3], is based on Fourier analysis. More recently, however, multiresolution analysis has been studied in situations where the usual Fourier transform techniques are not available, for example bounded intervals (see e.g. [13]), and bounded domains in higher dimensions.

Working with spatial triangulations of arbitrary topology for applications in computer graphics, the approach taken by Lounsbery, DeRose and Warren [10], and in Chapter 10 of Stollnitz, DeRose and Salesin [16], is to first consider piecewise linear prewavelets with *global* support. They subsequently truncate them to a small region, thus producing functions that are no longer elements of the orthogonal complementary wavelet space.

On the other hand, in the literature on the finite element solution of differential and integral equations, bases of true piecewise linear prewavelets with small support have been constructed by Kotyczka and Oswald [8] and Stevenson [15]. The basis in [8] is over an infinitely extended triangulation of type-1, taking advantage of dilation and translation over a uniform mesh. Meanwhile Stevenson [15] presents a construction of prewavelets over arbitrary triangulations which is also applicable in higher dimensions. On the uniform mesh, Stevenson's prewavelet masks have 23 non-zero coefficients and thus larger support than those of Kotyczka and Oswald which have only 13. On the other hand due to the particular construction in [15] the number of operations involved in the wavelet transform can be reduced. Dahmen and Stevenson [4] discuss extensions to the prewavelets in [15] to dual bases and manifolds.

In the finite element setting, prewavelets are only one possible way to produce Riesz bases. A non-orthogonal scheme, which is more amenable to adaptive refinement, is the well known hierarchical basis studied by Yserentant [17]. A more recent alternative is the three-point hierarchical basis proposed by Stevenson in [14] which is based on (discrete) ℓ_2 orthogonality rather than L_2 and in this sense is closer to the prewavelet approach. Another approach based on approximate L_2 orthogonality is that of Carnicer, Dahmen, and Peña [1]. See Lorentz and Oswald [9] for a survey and analysis of these and other Riesz basis constructions and the references contained there.

In our paper we investigate the existence and explicit construction of a basis of locally supported piecewise linear prewavelets on general bounded triangulations in $\mathbb{R}^2$, whose masks generalize the 13-point masks of Kotyczka and Oswald.

In Section 2, we list basic definitions and notations for triangulations and we introduce the idea of k-connected edges. Nested spaces over dyadically refined triangulations are generated by nodal or 'hat' functions; see Section 3. Uniform refinement allows us to restrict our attention to one refinement level. In Section 4, we present a sufficient condition for a set of general prewavelets to form a basis of the wavelet space. Section 5 contains technical results on inner products of piecewise linear hat functions.

The main results appear in Sections 6, 7, and 8. First, a small region around an edge in the triangulation is determined, on which a nontrivial prewavelet is guaranteed to exist. Formulating the explicit orthogonality conditions, the possible prewavelet coefficients are identified with the elements of the null space of a linear system. In fact, Theorem 6.2 states the exact dimension of the solution space which depends only on the connectedness of the edge. This solution space is characterized in Section 7 for 'simple' edges. In Section 8 we show that a concrete choice of prewavelets form a basis of the wavelet space when the degree of the vertices of the triangulation is not too high. Section 9 consists of conclusions and a numerical example.

2. Triangulations

Let $[X]$ denote the convex hull of a subset X of $\mathbb{R}^2$. We will refer to the convex hull of three non-collinear points in $\mathbb{R}^2$ as a *triangle*.

Let $\mathcal{T} = \{T_1, \dots, T_M\}$ be a set of triangles and let $\Omega = \bigcup_{i=1}^{M} T_i$ be their union.

Definition 2.1. *$\mathcal{T}$ is a* `triangulation` *if*

(i) $T_i \cap T_j$ is either empty or a common vertex or a common edge, $i \neq j$,
(ii) the number of boundary edges incident on a boundary vertex is two,
(iii) Ω is simply connected.

We denote by V the set of all vertices $v \in \mathbb{R}^2$ of triangles in $\mathcal{T}$ and by E the set of all edges $e = [v, w]$ of triangles in $\mathcal{T}$. By a *boundary vertex* or *boundary edge* we mean a vertex or edge contained in the boundary of Ω. All other vertices and edges will be called *interior* *vertices* and *interior edges*. A boundary edge belongs to only one triangle, and an interior edge to two.

A concept which will play a crucial role when studying prewavelets in later sections is that of *connectedness* which we now introduce. For a vertex $v \in V$, the *set of neighbours of v in V* is

$$V_v = \{w \in V : [v, w] \in E\}.$$

Definition 2.2. *An edge $e = [v_1, v_2]$ in E is k-*`connected` *if k is the cardinality of the intersection $V_{v_1} \cap V_{v_2}$.*

Clearly the connectedness of an interior edge is at least two and the connectedness of a boundary edge is at least one. But higher order connectedness is possible. Figure 1 shows, from left to right, a 2-connected interior edge, a 1-connected boundary edge, a 3-connected interior edge, and a 3-connected boundary edge.

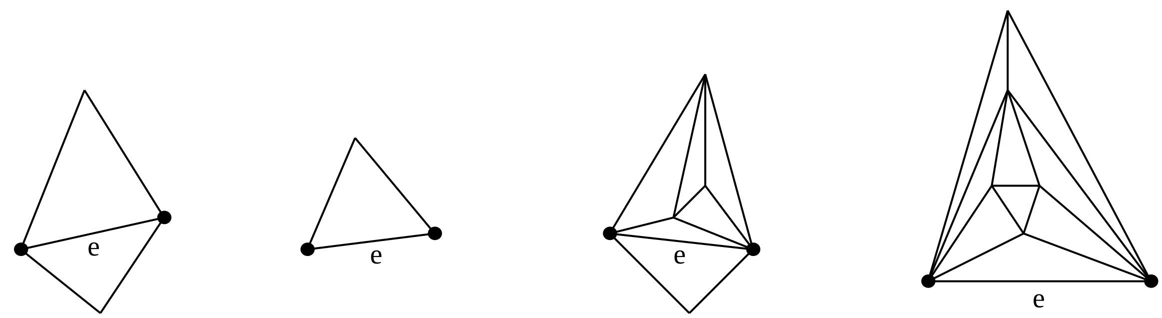

Fig. 1. k-connectedness.

It will sometimes help to distinguish between edges which are minimally connected from those which are not.

Definition 2.3. *An edge $e = [v_1, v_2]$ in E is* `simple` *if for any vertex w in V which is a neighbour of both v_1 and v_2, the triangle $[v_1, v_2, w]$ belongs to $\mathcal{T}$.*

Thus an interior edge is simple if and only if it is 2-connected and a boundary edge is simple if and only if it is 1-connected. It may also help to know whether all edges in the triangulation are simple.

Definition 2.4. *The triangulation $\mathcal{T}$ is* `simple` *if all edges in E are simple.*

It is not difficult to see that $\mathcal{T}$ is simple if and only if whenever three edges $[v_1, v_2]$, $[v_2, v_3]$, $[v_3, v_1]$ belong to E, the triangle $[v_1, v_2, v_3]$ belongs to $\mathcal{T}$.

Many 'structured' triangulations such as a type-1 triangulation (a uniform square grid, where each square is split into two triangles by the diagonal from the lower left to the upper right corner) are simple. However if $\mathcal{T}$ is a triangulation of a set of scattered data then even if it is a 'nice' triangulation, for example a Delaunay triangulation, which maximizes the minimum angle of its triangles (see [12]), or more generally, a triangulation induced by a convex function (see [2]), it may be non-simple. In fact if any interior vertex v in V has exactly three neighbours v_1, v_2, v_3 (that is v has degree three), the triangle

$[v_1, v_2, v_3]$ will not belong to $\mathcal{T}$ and so none of the edges $[v_1, v_2]$, $[v_2, v_3]$, and $[v_3, v_1]$ will be simple.

Suppose next that $\mathcal{T}$ is an arbitrary triangulation. Given data values $f_v \in \mathbb{R}$ for $v \in V$, there is a unique function $f : \Omega \to \mathbb{R}$ which is linear on each triangle in $\mathcal{T}$ and interpolates the data: $f(v) = f_v$, $v \in V$. The function f is piecewise linear and the set of all such f constitute a linear space S with dimension $|V|$.

For each $v \in V$, let $\phi_v : \Omega \to \mathbb{R}$ be the unique ('hat') function in S such that for all $w \in V$,

$$\phi_v(w) = \begin{cases} 1 & w = v; \\ 0 & \text{otherwise.} \end{cases}$$

The set of functions $\Phi = \{\phi_v\}_{v \in V}$ is a basis for the space S and for any function $f \in S$,

$$f(x) = \sum_{v \in V} f(v)\phi_v(x), \qquad x \in \Omega. \tag{2.1}$$

The support of ϕ_v is the union of all triangles which contain v which we denote by $M_v := \bigcup_{v \in T \in \mathcal{T}} T$.

3. Refined Triangulations

Given a triangulation $\mathcal{T}^0 = \{T_1, \ldots, T_M\}$ we next wish to consider a *refined* triangulation, that is a triangulation $\mathcal{T}^1$ such that every triangle in $\mathcal{T}^0$ is the union of triangles in $\mathcal{T}^1$. More precisely we are interested in the following *uniform* or *dyadic* refinement. For a given triangle $T = [x_1, x_2, x_3]$ let $y_1 = (x_2 + x_3)/2$, $y_2 = (x_1 + x_3)/2$, and $y_3 = (x_1 + x_2)/2$ denote the midpoints of its edges. Then the set of four triangles

$$\mathcal{T}_T = \{[x_1, y_2, y_3], [y_1, x_2, y_3], [y_1, y_2, x_3], [y_1, y_2, y_3]\}$$

is a triangulation and a refinement of $\{T\}$. The set of triangles $\mathcal{T}^1 = \bigcup_{T \in \mathcal{T}^0} \mathcal{T}_T$ is evidently a triangulation and a refinement of $\mathcal{T}^0$. Similarly, a whole sequence of triangulations $\mathcal{T}^j$, $j = 0, 1, 2, \ldots$, can be generated by further refinement steps.

In order to discuss some properties of $\mathcal{T}^j$ in relation to $\mathcal{T}^{j-1}$ we let V^j be the set of vertices in $\mathcal{T}^j$, and define E^j, S^j, ϕ_v^j, V_v^j, and M_v^j accordingly. A straightforward calculation shows that

$$\phi_v^{j-1} = \phi_v^j + \frac{1}{2} \sum_{w \in V_v^j} \phi_w^j, \qquad v \in V^{j-1},$$

and therefore we obtain a nested sequence of spaces

$$S^0 \subset S^1 \subset S^2 \subset \cdots .$$

Let $\langle \cdot, \cdot \rangle$ be the following inner product, defined for continuous functions on Ω,

$$\langle f, g \rangle = \sum_{T \in \mathcal{T}^0} \frac{1}{\mathrm{a}(T)} \int_T f(x)g(x)\,dx, \qquad f, g \in C(\Omega), \tag{3.1}$$

where $\mathrm{a}(T)$ is the area of triangle T. We observe that when the area of every triangle in $\mathcal{T}^0$ is some constant a, the inner product reduces to the scaled L_2 inner product

$$\langle f, g\rangle = \frac{1}{\mathrm{a}} \int_\Omega f(x) g(x)\, dx.$$

However in any case the weighted L_2 norm induced by (3.1) is equivalent to the usual L_2 norm because

$$\frac{1}{\max_{T\in\mathcal{T}^0} a(T)} ||f||_{L_2}^2 \leq \langle f, f\rangle \leq \frac{1}{\min_{T\in\mathcal{T}^0} a(T)} ||f||_{L_2}^2.$$

With respect to the inner product (3.1), the spaces S^j become Hilbert spaces. Let W^{j-1} denote the relative orthogonal complement of the coarse space S^{j-1} in the fine space S^j, so that

$$S^j = S^{j-1} \oplus W^{j-1}$$

and the dimension of W^{j-1} is $|V^j| - |V^{j-1}| = |E^{j-1}|$. We remark that the weighted norm (3.1) is standard in some applications of refinable spaces as it can reduce computational costs. This is the approach taken, for example, in computer graphics, see [10] and [16]. Due to the uniform refinement, we can restrict our analysis of W^{j-1} to the first refinement level $j = 1$ in the remainder of this paper.

4. Prewavelets

We will call elements of W^0 *prewavelets*. In this section we wish to investigate *bases* of prewavelets for W^0. We would like the prewavelets to have small support and we will return to this in Section 6. For now let us simply associate a prewavelet ψ_u in W^0 with each vertex u in $V^1 \setminus V^0$ and derive a general sufficient condition for the set $\Psi = \{\psi_u\}_{u\in V^1\setminus V^0}$ to constitute a basis of W^0.

The set Ψ is a basis for W^0 if it is linearly independent. The function ψ_u, being an element of S^1, can be written as a linear combination of the basis functions ϕ_w^1, namely

$$\psi_u(x) = \sum_{w\in V^1} q_{w,u} \phi_w^1(x), \tag{4.1}$$

where, from (2.1), the coefficients of ψ_u are $q_{w,u} = \psi_u(w)$, $w \in V^1$. A sufficient condition on these coefficients for Ψ to form a basis can be derived by evaluating the ψ_u at the vertices in $V^1 \setminus V^0$. Let $u_1, \ldots, u_n$, $n = |E^0|$, be any ordering of the vertices in $V^1 \setminus V^0$.

Lemma 4.1. *A set of prewavelets $\Psi = (\psi_{u_1}, \ldots, \psi_{u_n})$ in W^0 is a basis of W^0 if the matrix $Q = (q_{u_i,u_j})_{i,j}$ is nonsingular.*

Proof: Given a linear combination $\sum_{j=1}^n c_j \psi_{u_j}$ which is identically zero, evaluation at u_i yields

$$\sum_{j=1}^n c_j \psi_{u_j}(u_i) = \sum_{j=1}^n c_j q_{u_i,u_j} = 0,$$

and so $Q\mathbf{c} = 0$ where $\mathbf{c} = (c_1, \ldots, c_n)^T$. Therefore $\mathbf{c} = 0$. ∎

Since a matrix is nonsingular when it is diagonally dominant with respect to its columns, we immediately obtain the following sufficient condition.

Corollary 4.2. *A set of prewavelets* $\Psi = \{\psi_u\}_{u\in V^1\setminus V^0}$ *in* W^0 *is a basis of* W^0 *if*

$$q_{u,u} > \sum_{\bar u\in V^1\setminus V^0, \bar u\neq u} |q_{\bar u,u}|, \qquad u\in V^1\setminus V^0. \tag{4.2}$$

The interesting and useful aspect of condition (4.2) is that each condition depends only on the values of one function ψ_u, even though the linear independence of the set Ψ involves all the functions. Thus the construction of one single prewavelet function ψ_u can be carried out *independently* of the others, using condition (4.2).

5. Inner Products of Hat Functions

Since any element of S^1 has a unique representation as in (4.1), we see that the element ψ_u of S^1 belongs to W^0 if and only if

$$\langle \phi_v^0, \psi_u\rangle = \sum_{w\in V^1} \langle \phi_v^0, \phi_w^1\rangle q_{w,u} = 0, \qquad v\in V^0. \tag{5.1}$$

Therefore, in this section, we determine the inner product of any pair of basis functions ϕ_v^0 in S^0 and ϕ_w^1 in S^1. Due to their local supports many such inner products are zero and in fact $\langle \phi_v^0, \phi_w^1\rangle \neq 0$ if and only if the vertices v and w belong to some common triangle T in $\mathcal{T}^0$. As an aid to calculating the inner product in the latter case, let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be any two linear functions and x_1, x_2, x_3 be the vertices of the triangle T. If $f_i = f(x_i)$ and $g_i = g(x_i)$ for $i = 1, 2, 3$, then recalling that the integral of every quadratic Bernstein polynomial on T is equal, a simple calculation shows that

$$\int_T f(x)g(x)\,dx = \frac{\mathrm{a}(T)}{12}\big(f_1g_1 + f_2g_2 + f_3g_3 + (f_1+f_2+f_3)(g_1+g_2+g_3)\big).$$

Applying this formula to each of the subtriangles of T over which the integral $\int \phi_v^0\phi_w^1$ is non-zero, we find that the whole integral $\int_T \phi_v^0\phi_w^1$ takes on four different values according to the configuration of v and w: (i) $w = v$, (ii) $[v,w] \in E^1$, (iii) $[v,w]\in E^0$, and (iv) $w \notin V^0$ and $[v,w]\notin E^1$, illustrated in Figure 2,

$$96\int_T \phi_v^0(x)\phi_w^1(x)\,dx = \begin{cases} 6\mathrm{a}(T) & \text{(i)},\\ 10\mathrm{a}(T) & \text{(ii)},\\ \mathrm{a}(T) & \text{(iii)},\\ 4\mathrm{a}(T) & \text{(iv)}.\end{cases} \tag{5.2}$$

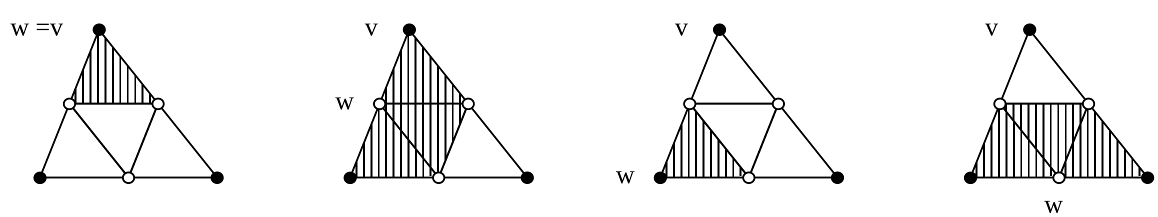

Fig. 2. Intersections of supports within one triangle in cases (i)–(iv).

Combining these integrals over all triangles in the triangulation, we can evaluate the inner product $\langle \phi_v^0, \phi_w^1 \rangle$. Let $t(e)$ denote the number of triangles (one or two) in $\mathcal{T}^0$ containing the edge $e \in E^0$ and $t(v)$ the number of triangles (at least one) containing the vertex $v \in V^0$. Then if $v \in V^0$ and $w \in V^1$ are contained in the same triangle in $\mathcal{T}^0$ we find

$$96\langle \phi_v^0, \phi_w^1 \rangle = \begin{cases} 6t(v) & \text{(i)}, \\ 10t(e) & \text{(ii)}, \\ t(e) & \text{(iii)}, \\ 4 & \text{(iv)}, \end{cases} \tag{5.3}$$

where in cases (ii) and (iii), e is the edge of E^0 containing both v and w. The intersection of the supports of ϕ_v^0 and ϕ_w^1 in the four cases (i)–(iv) are the shaded regions going left to right in Figure 3. In cases (i) and (ii) the topology of the region depends on whether w lies in the interior (top figure) or boundary (bottom figure) of the triangulation $\mathcal{T}^0$. In case (iii) the topology depends on whether the edge $[v, w]$ is an interior (top figure) or boundary edge (bottom figure) of $\mathcal{T}^0$. The topology of the region of intersection is always the same in case (iv).

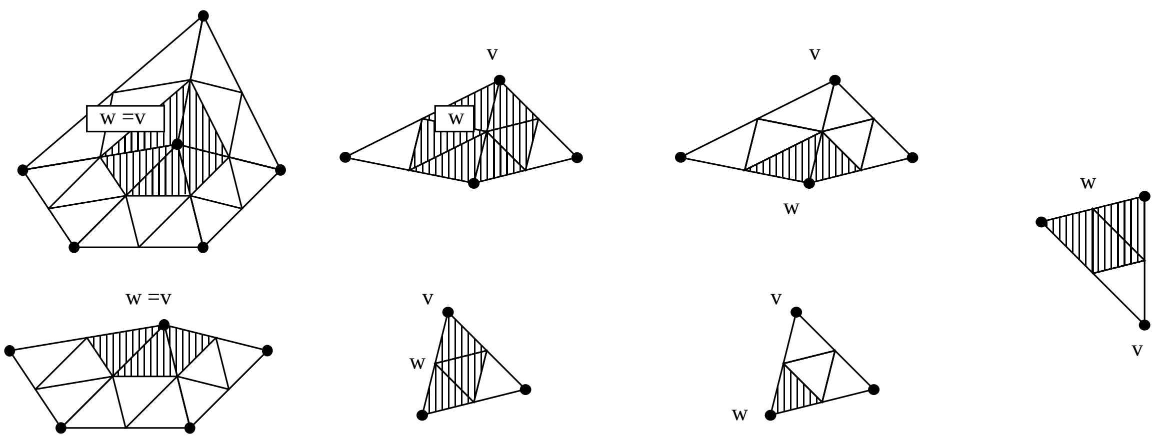

Fig. 3. Intersections of supports in cases (i)–(iv).

6. The Existence of Locally Supported Prewavelets

Let $u \in V^1 \setminus V^0$ be a 'new' vertex. Our aim in this section is to investigate the existence of a nontrivial prewavelet $\psi_u \in W^0$ associated with the vertex u, whose support is *local around* u, *i.e.*, it has the form

$$\psi_u(x) = \sum_{w \in V(u)} q_{w,u} \phi_w^1(x), \tag{6.1}$$

where $V(u) \subset V^1$ is a small set of vertices in $\mathcal{T}^1$ located close to u.

For a given *vertex set* $V(u)$, the largest possible support of a function of the form (6.1) is the union of all triangles in $\mathcal{T}^1$ which contain a vertex in $V(u)$, *i.e.*,

$$M(u) = \bigcup_{w \in V(u)} M_w^1 .$$

In order to lie in W^0, a function ψ_u of the form (6.1) must satisfy the orthogonality conditions (5.1). Nontrivial equations can only occur for coarse hat functions ϕ_v^0 whose support nontrivially overlaps the set $M(u)$. Formally, we collect all such vertices $v \in V^0$ in a *condition set*

$$C(u) = \{v \in V^0 : \exists\, T \in \mathcal{T}^1 \text{s.t. } T \subset M_v^0 \cap M(u)\}\,.$$

With $n(u) = |V(u)|$ denoting the number of vertices in $V(u)$ and thus the number of degrees of freedom of a function of the form (6.1) and $m(u) = |C(u)|$ the number of vertices in the condition set $C(u)$, *i.e.*, the number of 'active' orthogonality conditions, we thus obtain a characterisation of the possible choices of the prewavelet coefficients $q_{w,u}$ in (6.1). Let $\mathbf{q} \in \mathbb{R}^{n(u)}$ be the vector consisting of the coefficients $q_{w,u}$, $w \in V(u)$, in some chosen ordering.

Proposition 6.1. *The function $\sum_{w \in V(u)} q_{w,u}\phi_w^1$, whose support is included in $M(u)$, is an element of the wavelet space W^0 if and only if its coefficient vector $\mathbf{q}$ is an element of the linear space $Q(u) \subset \mathbb{R}^{n(u)}$ defined as*

$$Q(u) = \left\{ \mathbf{q} \mid \sum_{w \in V(u)} q_{w,u} \langle \phi_v^0, \phi_w^1 \rangle = 0 \text{ for all } v \in C(u) \right\}. \tag{6.2}$$

The question of whether a nontrivial prewavelet function with support included in $M(u)$ exists is thus reduced to the question of whether the linear subspace $Q(u)$ contains a nonzero element. The focus of our investigation thus becomes to find a set of $n(u)$ vertices $V(u)$ in the neighbourhood of the new vertex u for which $Q(u)$ contains a nonzero element.

One possible choice of the vertex set is $V(u) = V^1$, for in this case the dimension of $Q(u)$ must be equal to that of W^0. The procedure suggested in [10] and [16] is to compute the best least squares approximant $P\phi_u^1$ from S^0 to the fine hat function ϕ_u^1 (with respect to the inner product (3.1)). Then the difference $\psi_u = \phi_u^1 - P\phi_u^1$ is always an element of the wavelet space W^0 and so its coefficient vector $\mathbf{q}$ provides a non-trivial element of $Q(u)$. Unfortunately, the support of ψ_u will in general be global, that is $M(u) = \Omega$. Although the coefficients in the representation (6.1) decay 'away from the vertex u', simply setting them to zero outside a certain neighbourhood of u typically results in fairly large truncation errors in numerical computations.

On the other hand, the simplest choice of a *local* vertex set is $V(u) = \{u\}$. However, since the only element ψ_u of the form (6.1) is then a scalar multiple of the hat function ϕ_u^1, its support is $M(u) = M_u^1$. This support intersects at least the supports $M_{a_1}^0$ and $M_{a_2}^0$ of the coarse hat functions $\phi_{a_1}^0$ and $\phi_{a_2}^0$ of the two endpoints a_1 and a_2 of the edge in E^0 containing u. Therefore $Q(u) = \{0\}$ and so ψ_u cannot be a prewavelet.

The next logical step is to try to increase the vertex set to

$$V(u) = \{u, a_1, a_2\}$$

in which case

$$M(u) = M_u^1 \cup M_{a_1}^1 \cup M_{a_2}^1$$

and so the condition set is

$$C(u) = V^0_{a_1} \cup V^0_{a_2}. \tag{6.3}$$

This presents us with far too many orthogonality conditions, although taking $q_{a_1,u} = q_{a_2,u} = -1/2$ and $q_{u,u} = 1$ yields an element ψ_u which is *discretely* orthogonal in the sense that $\sum_{w\in V^1} \psi_u(w)\phi^0_v(w) = 0$ for all $v \in V^0$. This element was investigated in [14].

However since $C(u)$ is unchanged if we add to $V(u)$ all neighbours in V^1 of a_1 and a_2, we can greatly increase the degrees of freedom by setting

$$V(u) = \{a_1, a_2\} \cup V^1_{a_1} \cup V^1_{a_2} \tag{6.4}$$

in which case

$$M(u) = M^0_{a_1} \cup M^0_{a_2} \tag{6.5}$$

It will next be shown that for $V(u)$ in (6.4), $Q(u) \neq \{0\}$. In fact we will establish the exact dimension of $Q(u)$. In doing so, we must pay attention to the connectedness of the edge $e = [a_1, a_2]$ in E^0, introduced in Definition 2.2.

Theorem 6.2. *Suppose that for $u \in V^1\backslash V^0$ the edge e in E^0 containing u is k-connected. Then*

$$\dim Q(u) = k + 1.$$

The basic idea in the proof is that the orthogonality conditions in (6.2) can be written as the matrix equation

$$A\mathbf{q} = 0 \tag{6.6}$$

where A is an $m(u) \times n(u)$ matrix and $\mathbf{q}$ is regarded as a column vector of length $n(u)$. Using the identity $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$, for finite sets S_1, S_2, we find from (6.4),

$$n(u) = |V(u)| = |V^1_{a_1}| + |V^1_{a_2}| + 1$$

and from (6.3),

$$m(u) = |C(u)| = |V^0_{a_1}| + |V^0_{a_2}| - k.$$

Using the fact that $|V^1_v| = |V^0_v|$ for vertices v in V^0, the difference between the number of variables and number of conditions is therefore

$$n(u) - m(u) = k + 1.$$

Therefore, in order to prove Theorem 6.2 it remains to show that A has full rank $m(u)$. We do this by first supposing e is a simple edge. We split the case of a simple interior edge into three in Lemmas 6.3, 6.4, and 6.5 and treat simple boundary edges in Lemma 6.6. As we will see, the analysis of non-simple edges can easily be reduced to that of simple ones.

Lemma 6.3. *If e is a simple interior edge and a_1 and a_2 are interior vertices then* $\dim Q(u) = 3$.

Proof: Let $s = |V^0_{a_1}|$ and $t = |V^0_{a_2}|$. This means that the vertex a_1 is connected to $s \geq 3$ neighbouring vertices in $\mathcal{T}^0$ and the vertex a_2 is connected to $t \geq 3$ neighbouring vertices in $\mathcal{T}^0$. Let $a_2, f_1, f_2, \ldots, f_{s-1}$ be the neighbours of a_1 in $\mathcal{T}^0$ in counterclockwise order, while $a_1, g_1, g_2, \ldots, g_{t-1}$ are the neighbours of a_2 in $\mathcal{T}^0$, ordered clockwise. Let $u, b_1, b_2, \ldots, b_{s-1} \in V^1 \setminus V^0$ be the neighbours of a_1 in $\mathcal{T}^1$, counterclockwise, and $u, c_1, c_2, \ldots, c_{t-1} \in V^1 \setminus V^0$ be the neighbours of a_2 in $\mathcal{T}^1$, clockwise.

Thus each vertex b_i is the midpoint of the edge $[a_1, f_i]$, and each vertex c_i the midpoint of the edge $[a_2, g_i]$. Furthermore, $f_1 = g_1$ and $f_{s-1} = g_{t-1}$ and since e is 2-connected, this means that all remaining neighbours $f_2, \ldots, f_{s-2}$ of a_1 are distinct from the remaining neighbours $g_2, \ldots, g_{t-2}$ of a_2. We refer to the edge e as a simple (s, t) edge, exemplified in Figure 4.

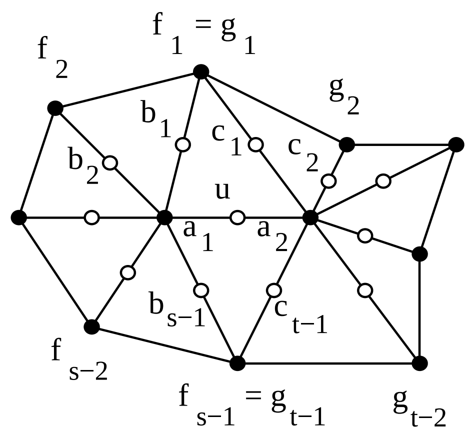

Fig. 4. A simple (s, t) edge.

Thus the vertex set is

$$V(u) = \{a_1, a_2, u, b_1, b_2, \ldots, b_{s-1}, c_1, c_2, \ldots, c_{t-1}\},$$

of size $n(u) = s + t + 1$, and the condition set is

$$C(u) = \{a_1, a_2, f_1 = g_1, f_{s-1} = g_{t-1}, f_2, \ldots, f_{s-2}, g_2, \ldots, g_{t-2}\}$$

of size $m(u) = s + t - 2$.

A prewavelet of the form (6.1) is then

$$\psi_u(x) = U\phi_u^1(x) + A_1\phi_{a_1}^1(x) + A_2\phi_{a_2}^1(x) + \sum_{i=1}^{s-1} B_i\phi_{b_i}^1(x) + \sum_{i=1}^{t-1} C_i\phi_{c_i}^1(x)\,, \tag{6.7}$$

where, for notational convenience, we use the abbreviations

$$U = q_{u,u}, \quad A_1 = q_{a_1,u}, \quad A_2 = q_{a_2,u}, \quad B_i = q_{b_i,u}, \quad C_i = q_{c_i,u}.$$

Non-zero inner products $\langle \phi_{a_1}^0, \phi_w^1 \rangle$ are covered for $w = a_1$ by case (i) in (5.3), for $w = a_2$ by case (iii), for $w = u$ and $w = b_i$ by case (ii), and for $w = c_1$ and $w = c_{t-1}$ by case (iv). Thus the orthogonality condition $\langle\phi^0_{a_1},\psi_u\rangle = 0$, multiplied by 96 to eliminate denominators, results in the linear equation

$$(a_1) \qquad 6sA_1 + 2A_2 + 20U + 20\sum_{i=1}^{s-1} B_i + 4C_1 + 4C_{t-1} = 0\,.$$

Similarly, we obtain for $\phi^0_{a_2}$ the equation

$$(a_2) \qquad 2A_1 + 6tA_2 + 20U + 20\sum_{i=1}^{t-1} C_i + 4B_1 + 4B_{s-1} = 0\,.$$

The only non-zero inner products $\langle\phi^0_{f_1},\phi^1_w\rangle$ arise from $w = a_1, a_2$ which give case (iii), $w = b_1, c_1$ giving case (ii), and $w = b_2, c_2, u$ giving case (iv), yielding (again with normalization by 96) the condition

$$(f_1 = g_1) \qquad 2A_1 + 2A_2 + 4U + 20B_1 + 4B_2 + 20C_1 + 4C_2 = 0\,,$$

and analogously for $\phi^0_{f_{s-1}}$,

$$(f_{s-1} = g_{t-1}) \qquad 2A_1 + 2A_2 + 4U + 20B_{s-1} + 4B_{s-2} + 20C_{t-1} + 4C_{t-2} = 0\,.$$

Finally, $\langle\phi^0_{f_i},\phi^1_w\rangle$, $i = 2,\ldots,s-2$, is non-zero only for four vertices w, namely $w = a_1$ giving case (iii) in (5.3), $w = b_i$ giving case (ii), and $w = b_{i-1}, b_{i+1}$ giving case (iv). Consequently, the resulting orthogonality equations are

$$(f_i) \qquad 2A_1 + 4B_{i-1} + 20B_i + 4B_{i+1} = 0, \quad i = 2,\ldots,s-2\,,$$

and analogously

$$(g_i) \qquad 2A_2 + 4C_{i-1} + 20C_i + 4C_{i+1} = 0, \quad i = 2,\ldots,t-2\,.$$

In order to prove the lemma, we need to show that the matrix A in (6.6) has full rank $m(u)$. First let us delete the three columns of A corresponding to the variables B_1, C_{t-1}, and U, resulting in a square matrix A'. We will show that A' is non-singular. This will be achieved by showing that after reordering its rows and after making two elementary row operations, the resulting matrix A'' is diagonally dominant.

It is immediately clear that the variables B_i are dominant in equations (f_i), $i = 2,\ldots,s-2$, and the variables C_i are dominant in equations (g_i), $i = 2,\ldots,t-2$. With B_1, C_{t-1}, and U deleted, the variable C_1 is also dominant in equation (f_1) and B_{s-1} in equation (f_{s-1}). It just remains to consider the variables A_1 and A_2. Let us replace equation (a_1) by the equation $7(a_1) - 5\sum_{i=2}^{s-2}(f_i) - 5(f_{s-1})$ which (after the same deletions) has the form

$$(32s + 20)A_1 + 4A_2 + 20B_2 + 20B_{s-1} + 28C_1 - 20C_{t-2} = 0\,.$$

With a_1 being an interior vertex, we have $s \geq 3$, and thus A_1 is dominant in this equation. Analogously, A_2 dominates the equation $7(a_2) - 5\sum_{i=2}^{t-2}(g_i) - 5(g_1)$. ■

Lemma 6.4. *If e is a simple interior edge and a_1 is a boundary vertex and a_2 is an interior vertex then* $\dim Q(u) = 3$.

Proof: Suppose $|V^0_{a_1}| = s_1 + s_2 + 1$ where $s_1 \geq 1$ is the number of neighbours of a_1 in V^0 to the left of the directed edge $e = [a_1, a_2]$ and $s_2 \geq 1$ the number to the right. Also let $t = |V^0_{a_2}| \geq 3$.

Let $f^1_1, f^1_2, \ldots, f^1_{s_1}$ be the neighbours of a_1 in $\mathcal{T}^0$ in counterclockwise order on the left of edge e, while $f^2_1, f^2_2, \ldots, f^2_{s_2}$ are the neighbours of a_1 in $\mathcal{T}^0$ in clockwise order on the right. For the other vertex, let $a_1, g_1, g_2, \ldots, g_{t-1}$ be the neighbours of a_2 in $\mathcal{T}^0$, ordered clockwise, such that $f^1_1 = g_1$ and $f^2_1 = g_{t-1}$, and none of the other neighbours of a_1 coincides with a neighbour of a_2 because e is 2-connected. Furthermore, let $b^1_1, b^1_2, \ldots, b^1_{s_1} \in V^1 \setminus V^0$ be the neighbours of a_1 in $\mathcal{T}^1$ on one side of e, $b^2_1, b^2_2, \ldots, b^2_{s_2} \in V^1 \setminus V^0$ the neighbours in $\mathcal{T}^1$ on the other side of e, with $u, c_1, c_2, \ldots, c_{t-1} \in V^1 \setminus V^0$ the neighbours of a_2 in $\mathcal{T}^1$, clockwise. We refer to this kind of edge as a simple $((s_1, s_2), t)$ edge, see Figure 5.

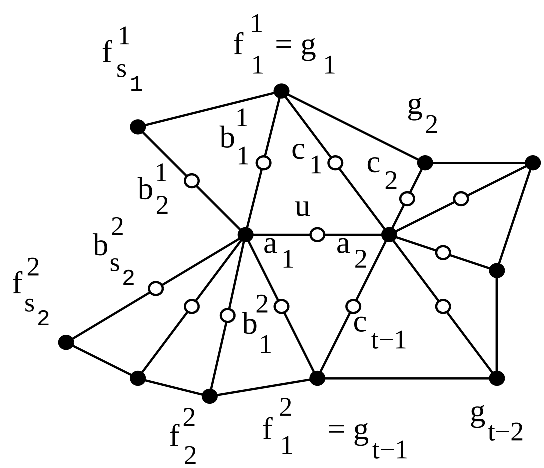

Fig. 5. A simple $((s_1, s_2), t)$ edge.

In this case, the vertex set is

$$V(u) = \{a_1, a_2, u, b^1_1, b^1_2, \ldots, b^1_{s_1}, b^2_1, b^2_2, \ldots, b^2_{s_2}, c_1, c_2, \ldots, c_{t-1}\},$$

of size $n(u) = s_1 + s_2 + t + 2$ and the condition set

$$C(u) = \{a_1, a_2, f^1_1 = g_1, f^2_1 = g_{t-1}, f^1_2, \ldots, f^1_{s_1}, f^2_2, \ldots, f^2_{s_2}, g_2, \ldots, g_{t-2}\},$$

of size $m(u) = s_1 + s_2 + t - 1$.

With similar notations as in the previous case, a prewavelet as in (6.1) now has the form

$$\psi_u(x) = U\phi^1_u(x) + A_1\phi^1_{a_1}(x) + A_2\phi^1_{a_2}(x) + \sum_{i=1}^{s_1} B^1_i \phi^1_{b^1_i}(x) + \sum_{i=1}^{s_2} B^2_i \phi^1_{b^2_i}(x) + \sum_{i=1}^{t-1} C_i \phi^1_{c_i}(x). \tag{6.8}$$

By appealing again to the different cases of (5.3), the explicit orthogonality conditions imposed by $C(u)$ lead to the equations (if $s_1 \geq 2$ and $s_2 \geq 2$)

$$6(s_1+s_2)A_1 + 2A_2 + 20U + 20\sum_{i=1}^{s_1-1} B_i^1 + 10B_{s_1}^1$$

$$(a_1) \qquad + 20\sum_{i=1}^{s_2-1} B_i^2 + 10B_{s_2}^2 + 4C_1 + 4C_{t-1} = 0\,,$$

$$(a_2) \qquad 2A_1 + 6tA_2 + 20U + 20\sum_{i=1}^{t-1} C_i + 4B_1^1 + 4B_1^2 = 0\,,$$

$$(f_1^1 = g_1) \qquad 2A_1 + 2A_2 + 4U + 20B_1^1 + 4B_2^1 + 20C_1 + 4C_2 = 0\,,$$

$$(f_1^2 = g_{t-1}) \qquad 2A_1 + 2A_2 + 4U + 20B_1^2 + 4B_2^2 + 20C_{t-1} + 4C_{t-2} = 0\,,$$

$$(f_i^1) \qquad 2A_1 + 4B_{i-1}^1 + 20B_i^1 + 4B_{i+1}^1 = 0, \quad i = 2,\ldots,s_1-1\,,$$

$$(f_{s_1}^1) \qquad A_1 + 4B_{s_1-1}^1 + 10B_{s_1}^1 = 0\,,$$

$$(f_i^2) \qquad 2A_1 + 4B_{i-1}^2 + 20B_i^2 + 4B_{i+1}^2 = 0, \quad i = 2,\ldots,s_2-1\,,$$

$$(f_{s_2}^2) \qquad A_1 + 4B_{s_2-1}^2 + 10B_{s_2}^2 = 0\,,$$

$$(g_i) \qquad 2A_2 + 4C_{i-1} + 20C_i + 4C_{i+1} = 0, \quad i = 2,\ldots,t-2\,.$$

The dominance argument remains the same as before, deleting here the columns corresponding to B_1^1, C_{t-1}, and U from the matrix A. The equations (f_i^1) are dominated by the variables B_i^1 with $i = 2,\ldots,s_1$, respectively, as well as (f_i^2) by B_i^2, $i = 2,\ldots,s_2$, and (g_i) by C_i, $i = 2,\ldots,t-2$. Also, after the deletions, (g_1) is dominated by C_1, and (g_{t-1}) by B_1^2.

The modified equation for A_1 is, after the deletions, $7(a_1) - 5\sum_{i=2}^{s_1}(f_i^1) - 5\sum_{i=2}^{s_2}(f_i^2) - (g_1) - 5(g_{t-1})$, namely

$$(32(s_1+s_2)+18)A_1 + 2A_2 + 16B_2^1 + 20B_1^2 + 8C_1 - 4C_2 - 20C_{t-2} = 0\,,$$

which is sufficient as $s_1 + s_2 \geq 2$. The equation for A_2 is $7(a_2) - 5\sum_{i=2}^{t-2}(g_i) - 5(g_1)$, namely

$$4A_1 + (32t+20)A_2 - 20B_2^1 + 28B_1^2 + 20C_1 + 20C_{t-2} = 0\,,$$

with $t \geq 3$. In the case $s_1 = 1$ (or $s_2 = 1$), the equation (f_1^1) (or (f_1^2)) is slightly different, but the dominance argument still holds. ■

Lemma 6.5. *If e is a simple interior edge and a_1 and a_2 are boundary vertices then* $\dim Q(u) = 3$.

Proof: Suppose $|V_{a_1}^0| = s_1 + s_2 + 1$ where $s_1 \geq 1$ is the number of neighbours of a_1 in V^0 to the left of the directed edge $e = [a_1, a_2]$ and $s_2 \geq 1$ the number to the right. Similarly suppose $|V_{a_2}^0| = t_1 + t_2 + 1$ where $t_1 \geq 1$ is the number of neighbours of a_2 in V^0 to the left of e and $t_2 \geq 1$ the number to the right.

With similar notations as introduced before, we have the vertex set

$$V(u) = \{a_1, a_2, u, b_1^1, b_2^1, \ldots, b_{s_1}^1, b_1^2, b_2^2, \ldots, b_{s_2}^2, c_1^1, c_2^1, \ldots, c_{t_1}^1, c_1^2, c_2^2, \ldots, c_{t_2}^2\},$$

of size $n(u) = s_1 + s_2 + t_1 + t_2 + 3$ and the condition set

$$C(u) = \{a_1, a_2, f_1^1 = g_1^1, f_1^2 = g_1^2, f_2^1, \ldots, f_{s_1}^1, f_2^2, \ldots, f_{s_2}^2, g_2^1, \ldots, g_{t_1}^1, g_2^2, \ldots, g_{t_2}^2\},$$

of size $m(u) = s_1 + s_2 + t_1 + t_2$. We call this a simple $((s_1, s_2), (t_1, t_2))$ edge, see Figure 6.

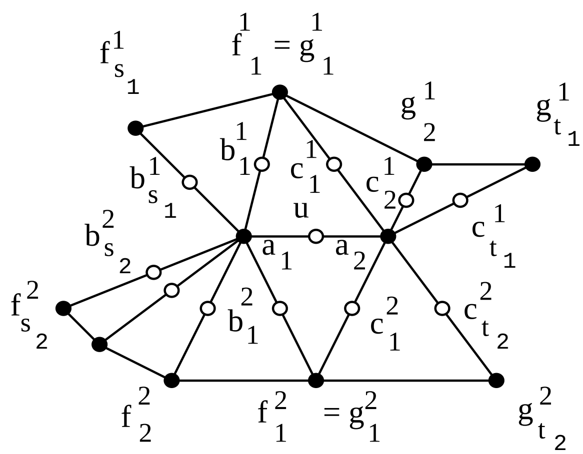

Fig. 6. A simple $((s_1, s_2), (t_1, t_2))$ edge.

A prewavelet thus has the form

$$\begin{aligned} \psi_u(x) &= U\phi_u^1(x) + A_1\phi_{a_1}^1(x) + A_2\phi_{a_2}^1(x) \\ &\quad + \sum_{i=1}^{s_1} B_i^1 \phi_{b_i^1}^1(x) + \sum_{i=1}^{s_2} B_i^2 \phi_{b_i^2}^1(x) + \sum_{i=1}^{t_1} C_i^1 \phi_{c_i^1}^1(x) + \sum_{i=1}^{t_2} C_i^2 \phi_{c_i^2}^1(x) \,. \end{aligned} \tag{6.9}$$

Here, the different cases of (5.3) yield the orthogonality conditions

$$\begin{aligned} &6(s_1+s_2)A_1 + 2A_2 + 20U + 20\sum_{i=1}^{s_1-1} B_i^1 + 10B_{s_1}^1 \\ (a_1) \qquad &+ 20\sum_{i=1}^{s_2-1} B_i^2 + 10B_{s_2}^2 + 4C_1^1 + 4C_1^2 = 0\,, \\ &2A_1 + 6(t_1+t_2)A_2 + 20U + 20\sum_{i=1}^{t_1-1} C_i^1 + 10C_{t_1}^1 \\ (a_2) \qquad &+ 20\sum_{i=1}^{t_2-1} C_i^2 + 10C_{t_2}^2 + 4B_1^1 + 4B_1^2 = 0\,, \end{aligned}$$

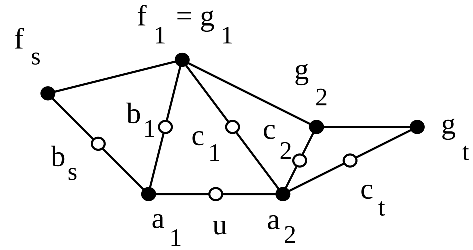

Fig. 7. A simple $((s,0),(t,0))$ edge.

$$
\begin{array}{ll}
(f_1^1 = g_1^1) & 2A_1 + 2A_2 + 4U + 20B_1^1 + 4B_2^1 + 20C_1^1 + 4C_2^1 = 0\,, \\
(f_1^2 = g_1^2) & 2A_1 + 2A_2 + 4U + 20B_1^2 + 4B_2^2 + 20C_1^2 + 4C_2^2 = 0\,, \\
(f_i^1) & 2A_1 + 4B_{i-1}^1 + 20B_i^1 + 4B_{i+1}^1 = 0, \quad i = 2,\dots,s_1-1\,, \\
(f_{s_1}^1) & A_1 + 4B_{s_1-1}^1 + 10B_{s_1}^1 = 0\,, \\
(f_i^2) & 2A_1 + 4B_{i-1}^2 + 20B_i^2 + 4B_{i+1}^2 = 0, \quad i = 2,\dots,s_2-1\,, \\
(f_{s_2}^2) & A_1 + 4B_{s_2-1}^2 + 10B_{s_2}^2 = 0\,, \\
(g_i^1) & 2A_2 + 4C_{i-1}^1 + 20C_i^1 + 4C_{i+1}^1 = 0, \quad i = 2,\dots,t_1-1\,. \\
(g_{t_1}^1) & A_2 + 4C_{t_1-1}^1 + 10C_{t_1}^1 = 0\,, \\
(g_i^2) & 2A_2 + 4C_{i-1}^2 + 20C_i^2 + 4C_{i+1}^2 = 0, \quad i = 2,\dots,t_2-1\,, \\
(g_{t_2}^2) & A_2 + 4C_{t_2-1}^2 + 10C_{t_2}^2 = 0\,.
\end{array}
$$

Deleting B_1^1, C_1^2, and U, again only equations (a_1) and (a_2) need further manipulation. The equation $7(a_1) - 5\sum_{i=2}^{s_1}(f_i^1) - 5\sum_{i=2}^{s_2}(f_i^2) - (f_1^1) - 5(f_1^2)$ then produces

$$(32(s_1+s_2)+18)A_1 + 2A_2 + 16B_2^1 + 20B_1^2 + 8C_1^1 - 4C_2^1 - 20C_2^2 = 0\,.$$

Since $s_1 + s_2 \geq 2$, the variable A_1 is dominant in this equation. Analogously, the equation $7(a_2) - 5\sum_{i=2}^{t_1}(g_i^1) - 5\sum_{i=2}^{t_2}(g_i^2) - 5(g_1^1) - (g_1^2)$ reduces to

$$2A_1 + (32(t_1+t_2)+18)A_2 + 16C_2^2 + 20C_1^1 + 8B_1^2 - 4B_2^2 - 20B_2^1 = 0\,,$$

and since $t_1 + t_2 \geq 2$, A_2 is dominant. Once again, the simplest cases, *i.e.*, $s_1 = 1$, $s_2 = 1$, $t_1 = 1$, $t_2 = 1$, produce slightly different equations, but the same results. ■

Lemma 6.6. *If e is a simple boundary edge then* $\dim Q(u) = 2$.

Proof: This case can more or less be treated as in the previous lemma, with $s_2 = t_2 = 0$. Let $|V_{a_1}^0| = s+1$ and $|V_{a_2}^0| = t+1$. In this case, with $s,t \geq 1$, we have the vertex set

$$V(u) = \{a_1, a_2, u, b_1, b_2, \dots, b_s, c_1, c_2, \dots, c_t\}\,,$$

of size $n(u) = s+t+3$ and the condition set

$$C(u) = \{a_1, a_2, f_1 = g_1, f_2, \dots, f_s, g_2, \dots, g_t\},$$

of size $m(u) = s+t+1$. We call the edge e a simple $((s,0),(t,0))$ edge, see Figure 7.

A prewavelet (6.1) has the form

$$\psi_u(x) = U\phi_u^1(x) + A_1\phi_{a_1}^1(x) + A_2\phi_{a_2}^1(x) + \sum_{i=1}^{s} B_i\phi_{b_i}^1(x) + \sum_{i=1}^{t} C_i\phi_{c_i}^1(x)\,. \tag{6.10}$$

The equations resulting from the orthogonality conditions are in this case, again obtained from the different cases in (5.3),

$$\begin{array}{ll}
(a_1) & 6sA_1 + A_2 + 10U + 20\sum_{i=1}^{s-1} B_i + 10B_s + 4C_1 = 0\,, \\
(a_2) & A_1 + 6tA_2 + 10U + 20\sum_{i=1}^{t-1} C_i + 10C_t + 4B_1 = 0\,, \\
(f_1 = g_1) & 2A_1 + 2A_2 + 4U + 20B_1 + 4B_2 + 20C_1 + 4C_2 = 0\,, \\
(f_i) & 2A_1 + 4B_{i-1} + 20B_i + 4B_{i+1} = 0, \quad i = 2, \ldots, s-1\,, \\
(f_s) & A_1 + 4B_{s-1} + 10B_s = 0\,, \\
(g_i) & 2A_2 + 4C_{i-1} + 20C_i + 4C_{i+1} = 0, \quad i = 2, \ldots, t-1\,, \\
(g_t) & A_2 + 4C_{t-1} + 10C_t = 0\,.
\end{array}$$

In this case, the desired result is $\dim Q(u) = 2$. We only delete two columns from A, namely those associated with the variables B_1 and U, to generate a square matrix A'. As before, only the equations (a_1) and (a_2) warrant further inspection. We replace equation (a_1) by the equation $7(a_1) - 5\sum_{i=2}^{s}(f_i) - (f_1)$, which reduces to

$$(32s + 13)A_1 + 5A_2 + 16B_2 + 8C_1 - 4C_2 = 0\,.$$

For $s \geq 2$, A_1 is therefore dominant in this equation. Analogously A_2 is dominant in the equation $7(a_2) - 5\sum_{i=2}^{t}(g_i) - 5(g_1)$ which reduces to

$$-3A_1 + (32t + 5)A_2 + 20C_1 - 20B_2 = 0\,,$$

when $t \geq 2$. Writing down the specific equations for the remaining case $s = 1$ and $t = 1$ also yields the dominance of A_1 and A_2. ■

Figure 8 shows a simple triangulation in which the four cases treated in Lemmas 6.3–6.6 occur. In each case the set $M(u)$ (which contains the support of ψ_u) is shaded.

Proof of Theorem 6.2: Lemmas 6.3–6.6 cover all the cases in which the edge e containing u is simple. We do not intend to work through all possible cases of higher order connectedness in detail, using induction to establish the desired final results. Instead, we limit ourselves to one specific situation to explain carefully what happens, leaving the analogous deductions in other situations to the reader.

Therefore, we consider only the case of Lemma 6.3 except that the edge e is now 3-connected rather than 2-connected. For example in the third configuration of Figure 1,

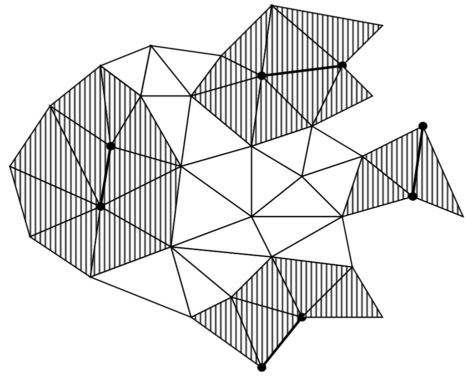

Fig. 8. The four kinds of simple edges in a simple triangulation.

$|V^0_{a_1} \cap V^0_{a_2}| = 3$ and with the notation of Figure 4, we have the identity $f_2 = g_3$ as well as $f_1 = g_1$ and $f_{s-1} = g_{t-1}$.

The vertex set is still

$$V(u) = \{a_1, a_2, u, b_1, b_2, \ldots, b_{s-1}, c_1, c_2, \ldots, c_{t-1}\}\,,$$

of size $n(u) = s+t+1$ but now, apart from $f_1 = g_1$ and $f_{s-1} = g_{t-1}$, there exist in general indices ℓ, $2 \le \ell \le s-1$, and m, $2 \le m \le t-1$, such that $f_\ell = g_m$. All other neighbours of a_1 in $\mathcal{T}^0$ are still different from the neighbours of a_2. Thus the condition set is now

$$\begin{aligned} C(u) = \{a_1, a_2\} &\cup \{f_1 = g_1, f_\ell = g_m, f_{s-1} = g_{t-1}\} \\ &\cup \{f_i \mid i = 2, \ldots, s-2, i \neq \ell\} \cup \{g_i \mid i = 2, \ldots, t-2, i \neq m\}\,, \end{aligned}$$

of size $m(u) = s+t-3$.

It is now not difficult to see that the orthogonality conditions are exactly the same as in Lemma 6.3 except that two equations

$$(f_\ell) \qquad 2A_1 + 4B_{\ell-1} + 20B_\ell + 4B_{\ell+1} = 0\,,$$

and

$$(g_m) \qquad 2A_2 + 4C_{m-1} + 20C_m + 4C_{m+1} = 0\,,$$

are replaced by one, namely their sum

$$(f_\ell = g_m) \qquad 2A_1 + 2A_2 + 4B_{\ell-1} + 20B_\ell + 4B_{\ell+1} + 4C_{m-1} + 20C_m + 4C_{m+1} = 0\,.$$

We have to show that $\dim Q(u) = 4$. Let $\hat{A}$ be the matrix associated with these new orthogonality conditions. Since $\hat{A}$ is the matrix resulting from replacing two rows of A in Lemma 6.3 by their sum, the rows of $\hat{A}$ like those of A must be linearly independent and therefore $\hat{A}$ has full rank.

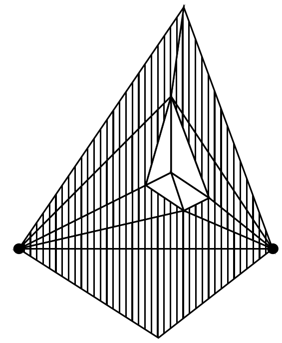

Fig. 9. The set $M(u)$ for a prewavelet on a non-simple edge.

For higher orders of connectedness, there are simply more pairs of similar equations (f_i) and (g_j) which are combined into one, and a similar argument as above establishes the fullness of the rank of $\hat{A}$ in every case. ■

We note that any solution to the orthogonality conditions in Lemma 6.3 is also a solution when we replace equations (f_ℓ) and (g_m) by the composite equation $(f_\ell = g_m)$ in the above proof of Theorem 6.2. Similar considerations hold for all other higher connected edges.

If e is a non-simple edge the set $M(u)$ may not be simply connected. This occurs in the triangulation shown in Figure 9, where the set $M(u)$ is shaded.

Finally, let us remark that Stevenson [15] has constructed prewavelets in arbitrary dimensions which in the bivariate case have larger support than the ones we have constructed. Specifically if $e = [a_1, a_2]$ is an interior edge and f and g are the remaining vertices of the two triangles containing e then the vertex set of Stevenson's element is

$$V(u) = \{a_1, a_2, f, g\} \cup V_{a_1}^1 \cup V_{a_2}^1 \cup V_f^1 \cup V_g^1$$

and its support is

$$M(u) = M_{a_1}^0 \cup M_{a_2}^0 \cup M_f^0 \cup M_g^0.$$

7. Explicit Local Prewavelets

The goal of this section is to provide some explicit elements of the space $Q(u)$, and thus explicit locally supported prewavelets, for the choice of $V(u)$ in (6.4).

We derive three independent solutions for an interior edge, and two independent solutions for a boundary edge. Due to Theorem 6.2, they form a basis of $Q(u)$ when the edge is simple. Thus when the triangulation is simple, we characterize the space $Q(u)$ for all new vertices u.

The majority of the orthogonality equations are covered by the following lemma.

Lemma 7.1. *The bi-infinite system of linear equations*

$$2a + 4x_{i-1} + 20x_i + 4x_{i+1} = 0, \quad i \in \mathbf{Z},$$

has the solutions

$$x_i = -\frac{1}{14}a + K_1\lambda^i + K_2\lambda^{-i}, \quad i \in \mathbf{Z}\,,$$

where K_1 *and* K_2 *are arbitrary real constants, and*

$$\lambda = -\frac{5}{2} + \frac{\sqrt{21}}{2} = -0.208712\ldots.$$

The infinite system

$$2a + 4x_{i-1} + 20x_i + 4x_{i+1} = 0, \quad i \in \mathbb{N}\,,$$

with initial condition

$$a + 4x_0 + 10x_1 = 0\,,$$

has the following solutions, with an arbitrary constant K*,*

$$x_i = -\frac{1}{14}a + K(\lambda^i + \lambda^{-i}), \quad i \in \mathbb{N}_0\,.$$

Proof: One finds that a solution $x_i = \mu^i$ of exponential type for the homogeneous equations $4x_{i-1} + 20x_i + 4x_{i+1} = 0$ leads to the condition $1 + 5\mu + \mu^2 = 0$, which has the two solutions λ and $\lambda^{-1} = -5/2 - \sqrt{21}/2$. Substituting the corresponding terms from the bi-infinite solution into the initial condition establishes the second part. ■

7.1 A simple (s,t) edge

We will specify three different sets of values for the coefficients U, B_i, C_j, and A_1, A_2.

One can readily deduce from Lemma 7.1 that the expressions

$$B_i = -\frac{1}{14}A_1 + K_1\lambda^i + K_2\lambda^{-i}, \quad i = 1,\ldots,s-1\,,$$
$$C_j = -\frac{1}{14}A_2 + K_3\lambda^j + K_4\lambda^{-j}, \quad j = 1,\ldots,t-1\,,$$

satisfy already the equations (f_i), for $i = 2,\ldots,s-2$, and (g_j), for $j = 2,\ldots,t-2$, with arbitrary values for the constants K_1, K_2, K_3 and K_4.

One can easily show that the three choices of the K_i below satisfy the remaining four equations (a_1), (a_2), $(f_1 = g_1)$, and $(f_{s-1} = g_{t-1})$, taking into account the following identities involving λ:

$$\lambda^{i-1} + 5\lambda^i + \lambda^{i+1} = 0, \quad i \in \mathbf{Z}, \qquad \lambda - \lambda^{-1} = \sqrt{21}, \qquad (1+\lambda)/(1-\lambda) = \sqrt{21}/7,$$

and

$$\sum_{i=1}^{s-1}(\lambda^i + \lambda^{s-i}) = 2\sum_{i=1}^{s-1}\lambda^i = 2\lambda\frac{1-\lambda^{s-1}}{1-\lambda}.$$

In each of the three cases, the values of A_1 and A_2 are only relevant in the two equations (a_1) and (a_2).

1) The choice $K_1 = 1/(1-\lambda^s)$, $K_2 = \lambda^s/(1-\lambda^s)$, $K_3 = 1/(1-\lambda^t)$, $K_4 = \lambda^t/(1-\lambda^t)$, yields a prewavelet $\psi^1_{(s,t)}$ given by the coefficients

$$
\begin{aligned}
A_1 &= -\frac{3\sqrt{21}}{2s}, \quad A_2 = -\frac{3\sqrt{21}}{2t}, \\
B_i &= -\frac{1}{14}A_1 + \frac{\lambda^i + \lambda^{s-i}}{1-\lambda^s}, \quad i = 1, \dots, s-1, \\
C_j &= -\frac{1}{14}A_2 + \frac{\lambda^j + \lambda^{t-j}}{1-\lambda^t}, \quad j = 1, \dots, t-1, \\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{1+\lambda^s}{1-\lambda^s} + \frac{1+\lambda^t}{1-\lambda^t}.
\end{aligned}
\tag{7.1}
$$

2) Another choice of suitable coefficients is $K_1 = 1/(1-\lambda^s)$, $K_2 = \lambda^s/(1-\lambda^s)$, $K_3 = -1/(1-\lambda^t)$, $K_4 = -\lambda^t/(1-\lambda^t)$, producing a prewavelet $\psi^2_{(s,t)}$ given by

$$
\begin{aligned}
A_1 &= \frac{\sqrt{21}}{4s}, \quad A_2 = -\frac{\sqrt{21}}{4t}, \\
B_i &= -\frac{1}{14}A_1 + \frac{\lambda^i + \lambda^{s-i}}{1-\lambda^s}, \quad i = 1, \dots, s-1, \\
C_j &= -\frac{1}{14}A_2 - \frac{\lambda^j + \lambda^{t-j}}{1-\lambda^t}, \quad j = 1, \dots, t-1, \\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{1+\lambda^s}{1-\lambda^s} - \frac{1+\lambda^t}{1-\lambda^t}.
\end{aligned}
\tag{7.2}
$$

3) Finally, with $K_1 = 1/(1-\lambda^s)$, $K_2 = -\lambda^s/(1-\lambda^s)$, $K_3 = -1/(1-\lambda^t)$, $K_4 = \lambda^t/(1-\lambda^t)$, a third prewavelet $\psi^3_{(s,t)}$ possesses the coefficients

$$
\begin{aligned}
A_1 &= A_2 = U = 0, \\
B_i &= \frac{\lambda^i - \lambda^{s-i}}{1-\lambda^s}, \quad i = 1, \dots, s-1, \\
C_j &= -\frac{\lambda^j - \lambda^{t-j}}{1-\lambda^t}, \quad j = 1, \dots, t-1.
\end{aligned}
\tag{7.3}
$$

Since the coefficients for A_1, A_2, and B_1 in the three cases form a nonsingular 3×3 matrix, the three coefficient vectors $\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3$ in $Q(u)$ for the three prewavelets are linearly independent.

7.2 A simple $((s_1, s_2), t)$ edge

In this case, Lemma 7.1 implies that any choice of coefficients, with arbitrary values for the constants K_1, K_2, K_3, and K_4,

$$\begin{aligned}
B_i^1 &= -\frac{1}{14}A_1 + K_1(\lambda^{i-s_1} + \lambda^{s_1-i}), \quad i = 1, \ldots, s_1\,, \\
B_i^2 &= -\frac{1}{14}A_1 + K_2(\lambda^{i-s_2} + \lambda^{s_2-i}), \quad i = 1, \ldots, s_2\,, \\
C_j &= -\frac{1}{14}A_2 + K_3\lambda^j + K_4\lambda^{-j}, \quad j = 1, \ldots, t-1\,,
\end{aligned}$$

satisfies all orthogonality equations except (a_1), (a_2), $(f_1^1 = g_1)$, and $(f_1^2 = g_{t-1})$.

The three solutions below can be checked, taking into account, in addition to the identities used in Subsection 7.1, the identity

$$\sum_{i=-K}^{K} \lambda^i = \frac{1}{7}\big(\lambda^K + \lambda^{-K} - (\lambda^{K+1} + \lambda^{-K-1})\big).$$

1) The factors $K_1 = (\lambda^{s_2} + \lambda^{-s_2})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_2 = (\lambda^{s_1} + \lambda^{-s_1})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_3 = 1/(1-\lambda^t)$, $K_4 = \lambda^t/(1-\lambda^t)$, produce a prewavelet $\psi^1_{((s_1,s_2),t)}$, given by the coefficients

$$\begin{aligned}
A_1 &= -\frac{3\sqrt{21}}{2(s_1+s_2)}\,, \quad A_2 = -\frac{3\sqrt{21}}{2t}\,, \\
B_i^1 &= -\frac{1}{14}A_1 + \frac{\lambda^{s_2} + \lambda^{-s_2}}{\lambda^{-s_1-s_2} - \lambda^{s_1+s_2}}(\lambda^{i-s_1} + \lambda^{s_1-i}), \quad i = 1, \ldots, s_1\,, \\
B_i^2 &= -\frac{1}{14}A_1 + \frac{\lambda^{s_1} + \lambda^{-s_1}}{\lambda^{-s_1-s_2} - \lambda^{s_1+s_2}}(\lambda^{i-s_2} + \lambda^{s_2-i}), \quad i = 1, \ldots, s_2\,, \\
C_j &= -\frac{1}{14}A_2 + \frac{\lambda^j + \lambda^{t-j}}{1-\lambda^t}, \quad j = 1, \ldots, t-1\,, \\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{(\lambda^{s_1} + \lambda^{-s_1})(\lambda^{s_2} + \lambda^{-s_2})}{\lambda^{-s_1-s_2} - \lambda^{s_1+s_2}} + \frac{1+\lambda^t}{1-\lambda^t}\,.
\end{aligned} \tag{7.4}$$

2) Another valid choice is $K_1 = (\lambda^{s_2} + \lambda^{-s_2})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_2 = (\lambda^{s_1} + \lambda^{-s_1})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_3 = -1/(1-\lambda^t)$, $K_4 = -\lambda^t/(1-\lambda^t)$, where the prewavelet $\psi^2_{((s_1,s_2),t)}$ is then given by the coefficients

$$\begin{aligned}
A_1 &= \frac{\sqrt{21}}{4(s_1+s_2)}\,, \quad A_2 = -\frac{\sqrt{21}}{4t}\,, \\
B_i^1 &= -\frac{1}{14}A_1 + \frac{\lambda^{s_2} + \lambda^{-s_2}}{\lambda^{-s_1-s_2} - \lambda^{s_1+s_2}}(\lambda^{i-s_1} + \lambda^{s_1-i}), \quad i = 1, \ldots, s_1\,, \\
B_i^2 &= -\frac{1}{14}A_1 + \frac{\lambda^{s_1} + \lambda^{-s_1}}{\lambda^{-s_1-s_2} - \lambda^{s_1+s_2}}(\lambda^{i-s_2} + \lambda^{s_2-i}), \quad i = 1, \ldots, s_2\,,
\end{aligned} \tag{7.5}$$

$$C_j = -\frac{1}{14}A_2 - \frac{\lambda^j + \lambda^{t-j}}{1-\lambda^t}, \quad j = 1, \dots, t-1\,,$$
$$U = -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{(\lambda^{s_1} + \lambda^{-s_1})(\lambda^{s_2} + \lambda^{-s_2})}{\lambda^{-s_1-s_2} - \lambda^{s_1+s_2}} - \frac{1+\lambda^t}{1-\lambda^t}\,.$$

3) A third alternative is $K_1 = 1/(\lambda^{s_1}+\lambda^{-s_1})$, $K_2 = -1/(\lambda^{s_2}+\lambda^{-s_2})$, $K_3 = -1/(1-\lambda^t)$, $K_4 = +\lambda^t/(1-\lambda^t)$. For the prewavelet $\psi^3_{((s_1,s_2),t)}$ we first set

$$K(\ell,m) = \frac{\lambda^{-\ell+m} - \lambda^{\ell-m}}{(\lambda^\ell + \lambda^{-\ell})(\lambda^m + \lambda^{-m})}\,,$$

and obtain the coefficients

$$\begin{aligned}
A_1 &= -\frac{5\sqrt{21}}{8(s_1+s_2)}K(s_1,s_2)\,, \quad A_2 = -\frac{7\sqrt{21}}{8t}K(s_1,s_2)\,,\\
B_i^1 &= -\frac{1}{14}A_1 + \frac{\lambda^{i-s_1} + \lambda^{s_1-i}}{\lambda^{s_1} + \lambda^{-s_1}}, \quad i = 1, \dots, s_1\,,\\
B_i^2 &= -\frac{1}{14}A_1 - \frac{\lambda^{i-s_2} + \lambda^{s_2-i}}{\lambda^{s_2} + \lambda^{-s_2}}, \quad i = 1, \dots, s_2\,,\\
C_j &= -\frac{1}{14}A_2 - \frac{\lambda^j - \lambda^{t-j}}{1-\lambda^t}, \quad j = 1, \dots, t-1\,,\\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2\,.
\end{aligned} \tag{7.6}$$

The linear independence is again established by the regularity of a 3×3 matrix, this time for the respective coefficients A_1, $B^1_{s_1}$, and $B^2_{s_2}$.

7.3 A simple $((s_1, s_2), (t_1, t_2))$ edge

Once again, by Lemma 7.1, any choice of coefficients, with arbitrary values for the constants K_1, K_2, K_3, and K_4,

$$\begin{aligned}
B_i^1 &= -\frac{1}{14}A_1 + K_1(\lambda^{i-s_1} + \lambda^{s_1-i}), \quad i = 1, \dots, s_1\,,\\
B_i^2 &= -\frac{1}{14}A_1 + K_2(\lambda^{i-s_2} + \lambda^{s_2-i}), \quad i = 1, \dots, s_2\,,\\
C_j^1 &= -\frac{1}{14}A_2 + K_3(\lambda^{j-t_1} + \lambda^{t_1-j}), \quad j = 1, \dots, t_1\,,\\
C_j^2 &= -\frac{1}{14}A_2 + K_4(\lambda^{j-t_2} + \lambda^{t_2-j}), \quad j = 1, \dots, t_2\,,
\end{aligned}$$

satisfies all orthogonality equations except (a_1), (a_2), $(f_1^1 = g_1^1)$, and $(f_1^2 = g_1^2)$.

1) One choice is $K_1 = (\lambda^{s_2} + \lambda^{-s_2})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_2 = (\lambda^{s_1} + \lambda^{-s_1})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_3 = (\lambda^{t_2} + \lambda^{-t_2})/(\lambda^{-t_1-t_2} - \lambda^{t_1+t_2})$, $K_4 = (\lambda^{t_1} + \lambda^{-t_1})/(\lambda^{-t_1-t_2} - \lambda^{t_1+t_2})$, such that a prewavelet $\psi^1_{((s_1,s_2),(t_1,t_2))}$ is given by the coefficients

$$\begin{aligned}
A_1 &= -\frac{3\sqrt{21}}{2(s_1+s_2)}\,, \quad A_2 = -\frac{3\sqrt{21}}{2(t_1+t_2)}\,, \\
B^1_i &= -\frac{1}{14}A_1 + \frac{\lambda^{s_2}+\lambda^{-s_2}}{\lambda^{-s_1-s_2}-\lambda^{s_1+s_2}}(\lambda^{i-s_1}+\lambda^{s_1-i}), \quad i = 1,\dots,s_1\,, \\
B^2_i &= -\frac{1}{14}A_1 + \frac{\lambda^{s_1}+\lambda^{-s_1}}{\lambda^{-s_1-s_2}-\lambda^{s_1+s_2}}(\lambda^{i-s_2}+\lambda^{s_2-i}), \quad i = 1,\dots,s_2\,, \\
C^1_j &= -\frac{1}{14}A_2 + \frac{\lambda^{t_2}+\lambda^{-t_2}}{\lambda^{-t_1-t_2}-\lambda^{t_1+t_2}}(\lambda^{j-t_1}+\lambda^{t_1-j}), \quad j = 1,\dots,t_1\,, \\
C^2_j &= -\frac{1}{14}A_2 + \frac{\lambda^{t_1}+\lambda^{-t_1}}{\lambda^{-t_1-t_2}-\lambda^{t_1+t_2}}(\lambda^{j-t_2}+\lambda^{t_2-j}), \quad j = 1,\dots,t_2\,, \\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{(\lambda^{s_1}+\lambda^{-s_1})(\lambda^{s_2}+\lambda^{-s_2})}{\lambda^{-s_1-s_2}-\lambda^{s_1+s_2}} + \frac{(\lambda^{t_1}+\lambda^{-t_1})(\lambda^{t_2}+\lambda^{-t_2})}{\lambda^{-t_1-t_2}-\lambda^{t_1+t_2}}\,.
\end{aligned} \tag{7.7}$$

2) Another alternative is $K_1 = (\lambda^{s_2} + \lambda^{-s_2})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_2 = (\lambda^{s_1} + \lambda^{-s_1})/(\lambda^{-s_1-s_2} - \lambda^{s_1+s_2})$, $K_3 = -(\lambda^{t_2} + \lambda^{-t_2})/(\lambda^{-t_1-t_2} - \lambda^{t_1+t_2})$, $K_4 = -(\lambda^{t_1} + \lambda^{-t_1})/(\lambda^{-t_1-t_2} - \lambda^{t_1+t_2})$, with the prewavelet $\psi^2_{((s_1,s_2),(t_1,t_2))}$ having the coefficients

$$\begin{aligned}
A_1 &= \frac{\sqrt{21}}{4(s_1+s_2)}\,, \quad A_2 = -\frac{\sqrt{21}}{4(t_1+t_2)}\,, \\
B^1_i &= -\frac{1}{14}A_1 + \frac{\lambda^{s_2}+\lambda^{-s_2}}{\lambda^{-s_1-s_2}-\lambda^{s_1+s_2}}(\lambda^{i-s_1}+\lambda^{s_1-i}), \quad i = 1,\dots,s_1\,, \\
B^2_i &= -\frac{1}{14}A_1 + \frac{\lambda^{s_1}+\lambda^{-s_1}}{\lambda^{-s_1-s_2}-\lambda^{s_1+s_2}}(\lambda^{i-s_2}+\lambda^{s_2-i}), \quad i = 1,\dots,s_2\,, \\
C^1_j &= -\frac{1}{14}A_2 - \frac{\lambda^{t_2}+\lambda^{-t_2}}{\lambda^{-t_1-t_2}-\lambda^{t_1+t_2}}(\lambda^{j-t_1}+\lambda^{t_1-j}), \quad j = 1,\dots,t_1\,, \\
C^2_j &= -\frac{1}{14}A_2 - \frac{\lambda^{t_1}+\lambda^{-t_1}}{\lambda^{-t_1-t_2}-\lambda^{t_1+t_2}}(\lambda^{j-t_2}+\lambda^{t_2-j}), \quad j = 1,\dots,t_2\,, \\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{(\lambda^{s_1}+\lambda^{-s_1})(\lambda^{s_2}+\lambda^{-s_2})}{\lambda^{-s_1-s_2}-\lambda^{s_1+s_2}} - \frac{(\lambda^{t_1}+\lambda^{-t_1})(\lambda^{t_2}+\lambda^{-t_2})}{\lambda^{-t_1-t_2}-\lambda^{t_1+t_2}}\,.
\end{aligned} \tag{7.8}$$

3) For a third prewavelet $\psi^3_{((s_1,s_2),(t_1,t_2))}$, $K_1 = 1/(\lambda^{s_1} + \lambda^{-s_1})$, $K_2 = -1/(\lambda^{s_2} + \lambda^{-s_2})$, $K_3 = -1/(\lambda^{t_1} + \lambda^{-t_1})$, $K_4 = 1/(\lambda^{t_2} + \lambda^{-t_2})$, and we again use the term $K(\ell, m)$ introduced in the previous subsection to obtain the coefficients

$$\begin{aligned}
A_1 &= -\frac{\sqrt{21}}{8(s_1+s_2)}(5K(s_1,s_2) - 7K(t_1,t_2))\,, \\
A_2 &= -\frac{\sqrt{21}}{8(t_1+t_2)}(7K(s_1,s_2) - 5K(t_1,t_2))\,,
\end{aligned}$$

$$
\begin{aligned}
B_i^1 &= -\frac{1}{14}A_1 + \frac{\lambda^{i-s_1}+\lambda^{s_1-i}}{\lambda^{s_1}+\lambda^{-s_1}}, \quad i=1,\dots,s_1\,,\\
B_i^2 &= -\frac{1}{14}A_1 - \frac{\lambda^{i-s_2}+\lambda^{s_2-i}}{\lambda^{s_2}+\lambda^{-s_2}}, \quad i=1,\dots,s_2\,,\\
C_j^1 &= -\frac{1}{14}A_2 - \frac{\lambda^{j-t_1}+\lambda^{t_1-j}}{\lambda^{t_1}+\lambda^{-t_1}}, \quad j=1,\dots,t_1\,,\\
C_j^2 &= -\frac{1}{14}A_2 + \frac{\lambda^{j-t_2}+\lambda^{t_2-j}}{\lambda^{t_2}+\lambda^{-t_2}}, \quad j=1,\dots,t_2\,,\\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2\,.
\end{aligned} \tag{7.9}
$$

The matrix for A_1, $B^1_{s_1}$, and $B^2_{s_2}$, similar to the previous subsection, yields linear independence.

7.4 A simple $((s,0),(t,0))$ edge

This case can be handled using the previous one with parameters $s_1 = s$, $s_2 = 0$, $t_1 = t$, $t_2 = 0$, yielding the following two prewavelets.

1) With factors $K_1 = 2/(\lambda^{-s} - \lambda^s)$, $K_2 = 2/(\lambda^{-t} - \lambda^t)$, a prewavelet $\psi^1_{((s,0),(t,0))}$ is given by the coefficients

$$
\begin{aligned}
A_1 &= -\frac{3\sqrt{21}}{2s}\,, \quad A_2 = -\frac{3\sqrt{21}}{2t}\,,\\
B_i &= -\frac{1}{14}A_1 + \frac{2}{\lambda^{-s}-\lambda^s}(\lambda^{i-s}+\lambda^{s-i}), \quad i=1,\dots,s\,,\\
C_j &= -\frac{1}{14}A_2 + \frac{2}{\lambda^{-t}-\lambda^t}(\lambda^{j-t}+\lambda^{t-j}), \quad j=1,\dots,t\,,\\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{2(\lambda^s+\lambda^{-s})}{\lambda^{-s}-\lambda^s} + \frac{2(\lambda^t+\lambda^{-t})}{\lambda^{-t}-\lambda^t}\,.
\end{aligned} \tag{7.10}
$$

2) Using $K_1 = 2/(\lambda^{-s} - \lambda^s)$, $K_2 = -2/(\lambda^{-t} - \lambda^t)$, another prewavelet $\psi^2_{((s,0),(t,0))}$ has the coefficients

$$
\begin{aligned}
A_1 &= \frac{\sqrt{21}}{4s}\,, \quad A_2 = -\frac{\sqrt{21}}{4t}\,,\\
B_i &= -\frac{1}{14}A_1 + \frac{2}{\lambda^{-s}-\lambda^s}(\lambda^{i-s}+\lambda^{s-i}), \quad i=1,\dots,s\,,\\
C_j &= -\frac{1}{14}A_2 - \frac{2}{\lambda^{-t}-\lambda^t}(\lambda^{j-t}+\lambda^{t-j}), \quad j=1,\dots,t\,,\\
U &= -\frac{1}{14}A_1 - \frac{1}{14}A_2 + \frac{2(\lambda^s+\lambda^{-s})}{\lambda^{-s}-\lambda^s} - \frac{2(\lambda^t+\lambda^{-t})}{\lambda^{-t}-\lambda^t}\,.
\end{aligned} \tag{7.11}
$$

As $Q(u)$ only has dimension 2 in this case, these two coefficient sets already form a basis, with linear independence following from the regularity of a 2×2 matrix formed by A_1 and A_2.

8. Prewavelet Bases

We will now make some comments on the results obtained in the previous sections and ask the question ‘do the prewavelets derived in Section 7 form a basis of the wavelet space W^0?’.

Kotyczka and Oswald [8] investigated prewavelets on an infinite type-1 triangulation, also called three directional mesh. Following our terminology (with obvious extensions), this triangulation is simple. In fact, all edges are interior and exactly six edges emanate from each vertex. Consequently, Lemma 6.3 is applicable to all edges, and three independent solutions are given by (7.1), (7.2), and (7.3), with $s = t = 6$ in this special case. Indeed, the three solutions are the ones described by Kotyczka and Oswald [8], p. 240. Namely, after multiplying by a factor of $72/\sqrt{21}$, we find $\psi^1_{(6,6)}$ given by $U = 34$, $A_1 = A_2 = -18$, $B_1 = B_5 = C_1 = C_5 = -2$, $B_2 = B_4 = C_2 = C_4 = 2$, and $B_3 = C_3 = 1$. Similarly, with a factor of $144/\sqrt{21}$, $\psi^2_{(6,6)}$ is given by $U = 0$, $A_1 = -A_2 = 6$, $B_1 = B_5 = -C_1 = -C_5 = -7$, $B_2 = B_4 = -C_2 = -C_4 = 1$, and $B_3 = -C_3 = -1$. Finally, with a factor of 24, $\psi^3_{(6,6)}$ is given by $U = A_1 = A_2 = B_3 = C_3 = 0$, $B_1 = -B_5 = -C_1 = C_5 = 5$, and $B_2 = -B_4 = -C_2 = C_4 = -1$. In [8] it is shown that the elements $\psi^1_{(6,6)}$ there form a Riesz basis of the infinite-dimensional wavelet space, while $\psi^2_{(6,6)}$ and $\psi^3_{(6,6)}$ do not.

Now let us consider whether the prewavelets derived in Section 7 form a basis of W^0 in general. Following equation (6.7), for ψ_u on an (s,t) edge, the sufficient condition (4.2) reduces to

$$U > \sum_{i=1}^{s-1} |B_i| + \sum_{j=1}^{t-1} |C_j| \,. \tag{8.1}$$

Since $U = 0$ for $\psi^3_{(s,t)}$ and $U = 0$ also for $\psi^2_{(s,t)}$ when $s = t$, this sufficient condition does not hold. Therefore we turn our attention to $\psi^1_{(s,t)}$.

Lemma 8.1. *For an (s,t) edge, the coefficients of the prewavelet $\psi^1_{(s,t)}$ satisfy the inequality (8.1) for $3 \le s,t \le 26$.*

Proof: Recalling that $-1 < \lambda < 0$ and since $A_1 < 0$ in (7.1),

$$\sum_{i=1}^{s-1} |B_i| \le -\frac{(s-1)}{14} A_1 + \frac{2|\lambda|(1-|\lambda|^{s-1})}{(1-\lambda^s)(1-|\lambda|)}.$$

So

$$\begin{aligned} D(s,t) &:= U - \sum_{i=1}^{s-1} |B_i| - \sum_{i=1}^{t-1} |C_i| \\ &\ge \frac{(s-2)}{14} A_1 + \frac{(t-2)}{14} A_2 + \frac{1+\lambda^s}{1-\lambda^s} + \frac{1+\lambda^t}{1-\lambda^t} \\ &\quad + \frac{2\lambda}{1+\lambda} \left(\frac{1-|\lambda|^{s-1}}{1-\lambda^s} + \frac{1-|\lambda|^{t-1}}{1-\lambda^t} \right). \end{aligned}$$

Now substituting the values of A_1 and A_2 and noting that $(1+\lambda^s)/(1-\lambda^s) \geq (1+\lambda^3)/(1-\lambda^3)$ and $(1-|\lambda|^s)/(1-\lambda^s) \leq 1$ for $s \geq 3$,

$$D(s,t) \geq E(s,t) := -\left(2 - \frac{2}{s} - \frac{2}{t}\right)\frac{3}{28}\sqrt{21} + 2\frac{1+\lambda^3}{1-\lambda^3} + \frac{4\lambda}{1+\lambda}.$$

Then a simple calculation shows that $E(26,26) > 0$ while $E(27,27) < 0$. Therefore since $\partial E/\partial s \leq 0$ and $\partial E/\partial t \leq 0$ for $s,t \geq 3$, this proves the result. ■

Following (6.10), the condition (4.2) for an $((s,0),(t,0))$ (boundary) edge is

$$U > \sum_{i=1}^{s} |B_i| + \sum_{j=1}^{t} |C_j|. \tag{8.2}$$

Lemma 8.2. *For an $((s,0),(t,0))$ (boundary) edge, the coefficients of the prewavelet $\psi^1_{((s,0),(t,0))}$ satisfy the inequality (8.2) for all $s,t \geq 1$.*

Proof: From (7.10) we have

$$\begin{aligned}\sum_{i=1}^{s} |B_i| &\leq -\frac{s}{14}A_1 + \frac{2}{1-\lambda^{2s}}\sum_{i=1}^{s}(|\lambda|^i + |\lambda|^{2s-i}) \\ &\leq -\frac{s}{14}A_1 - \frac{2\lambda}{1+\lambda}\frac{1-|\lambda|^s}{1-\lambda^s}\frac{1+|\lambda|^{s-1}}{1+\lambda^s}.\end{aligned}$$

So noting that $(1-|\lambda|^s)/(1-\lambda^s) \leq 1$ and since $(1+|\lambda|^{s-1})/(1+\lambda^s) \leq 2/(1+\lambda)$ for $s \geq 1$, we find

$$\begin{aligned}U - \sum_{i=1}^{s}|B_i| - \sum_{i=1}^{t}|C_i| &\geq \frac{3}{28}\sqrt{21}\left(-2 + \frac{1}{s} + \frac{1}{t}\right) + 2\left(\frac{1+\lambda^{2s}}{1-\lambda^{2s}} + \frac{1+\lambda^{2t}}{1-\lambda^{2t}}\right) + \frac{8\lambda}{(1+\lambda)^2} \\ &\geq -\frac{3}{14}\sqrt{21} + 4 + \frac{8\lambda}{(1+\lambda)^2} > 0.\end{aligned}$$

■

Using Lemmas 8.1 and 8.2 and some numerical computations we are thus able to formulate the following theorem. Let us say that the degree of a vertex v in the triangulation $\mathcal{T}^0$ is the number of its neighbours, in other words $|V_v^0|$.

Theorem 8.3. *Suppose $\mathcal{T}^0$ is any triangulation such that the degree of every vertex is at most 21. Given its uniform refinement $\mathcal{T}^1$, and the decomposition $S^1 = S^0 \oplus W^0$, a basis of locally supported prewavelets for W^0 is given by choosing for each edge e, the appropriate prewavelet $\psi^1_{(s,t)}$, $\psi^1_{((s_1,s_2),t)}$, $\psi^1_{((s_1,s_2),(t_1,t_2))}$, or $\psi^1_{((s,0),(t,0))}$.*

Proof: It is sufficient to show that inequality (4.2) holds for every prewavelet ψ^1. Due to (6.8), for an edge of type (s,t), Lemma 8.1 shows that the condition (4.2) (in this case inequality (8.1)) holds for $s,t \leq 26$. In fact direct numerical evaluation shows that inequality (8.1) holds for all $s, t \leq 291$. For an edge of type $((s_1, s_2), t)$, condition (4.2) becomes

$$U > \sum_{i=1}^{s_1} |B_i^1| + \sum_{i=1}^{s_2} |B_i^2| + \sum_{j=1}^{t-1} |C_j| \,. \tag{8.3}$$

Numerical computations show that this inequality is satisfied by $\psi^1_{((s_1,s_2),t)}$ for $s_1 + s_2 + 1 \leq 53$ and $t \leq 53$. From (6.9), for an edge of type $((s_1, s_2), (t_1, t_2))$, condition (4.2) becomes

$$U > \sum_{i=1}^{s_1} |B_i^1| + \sum_{i=1}^{s_2} |B_i^2| + \sum_{j=1}^{t_1} |C_j^1| + \sum_{j=1}^{t_2} |C_j^2| \tag{8.4}$$

and, by numerical computation, is satisfied for $s_1 + s_2 + 1 \leq 21$, and $t_1 + t_2 + 1 \leq 21$. Finally Lemma 8.2 shows that for all s, t, the element $\psi^1_{((s,0),(t,0))}$ satisfies condition (4.2). ■

For comparison, we note that [6] contains a simple counterexample to show that the elements ψ^2 do not always form a basis of W^0.

9. Conclusions and Numerical Examples

We have constructed a set of prewavelets $\Psi^0 = \{\psi_u^1\}_{u \in V^1 \setminus V^0}$ of small support and shown that they form a basis of W^0 in most practical situations, by showing that the matrix Q of Lemma 4.1 is diagonally dominant. Due to uniform refinement, as mentioned in Section 3, the same analysis can clearly be applied to construct prewavelet bases Ψ^j in W^j, $j = 0, 1, 2, \ldots$.

In a forthcoming paper [7] a more sophisticated analysis of the matrix Q will be used to establish that Ψ^j is always a basis of W^j, i.e., independent of the degrees of the vertices and connectedness of edges. Moreover it can be shown that

$$\Phi^0 \cup \bigcup_{j=0}^{\infty} \Psi^j,$$

where $\Phi^0 = \{\phi_v^0\}_{v \in V^0}$, is a Riesz basis with respect to the weighted L_2 norm induced by the inner product (3.1).

Further properties of the prewavelets are addressed in [6] and [7]. For now let us simply conclude the paper by presenting the results of some numerical tests. A model of Norwegian terrain containing 128×128 points on a regular rectangular grid is shown top left in Figure 10. Treating the grid as a (refined) type-1 triangulation we decomposed the data over five levels, shown in Figure 10, using an iterative solver for the inverse wavelet transform. The coefficients of the seven different prewavelets occurring on such a triangulation are given in [6], scaled to be integers. For an arbitrary initial triangulation $\mathcal{T}^0$ the prewavelet coefficients need only be computed once and stored in a table and so the *construction* of the matrices used in the inverse wavelet transforms has no adverse effect on computational efficiency.

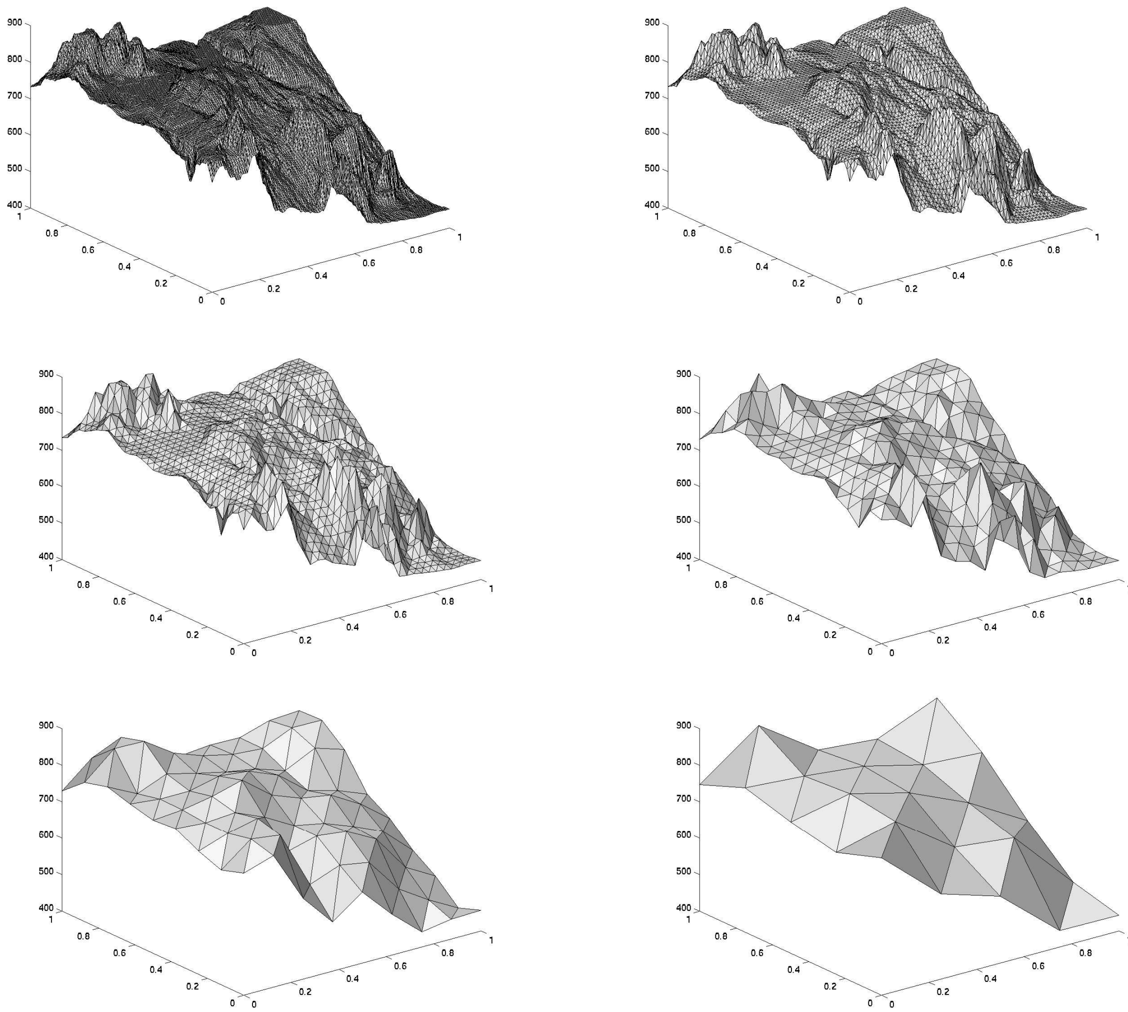

Fig. 10. Decomposition of terrain.

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