

On the positivity of B-spline Wronskians

Michael S. Floater*

August 5, 2024

Abstract

A proof that Wronskians of non-zero B-splines are positive is given, using only recursive formulas for B-splines and their derivatives. This could be used to generalize the de Boor-DeVore geometric proof of the Schoenberg-Whitney conditions and total positivity of B-splines to Hermite interpolation. For Wronskians of maximal order with respect to a given degree, positivity follows from a simple formula.

Keywords: B-splines, Wronskians, collocation, Schoenberg-Whitney conditions, total positivity.

1 Introduction

In order to generalize the Schoenberg-Whitney conditions [17] and the total positivity property of B-splines to Hermite interpolation it is necessary to show that Wronskians of B-splines are positive whenever the B-splines themselves are positive. In this paper we prove this latter property using only recursions for B-splines and their derivatives.

The fact that a Hermite B-spline collocation matrix is non-singular under the Schoenberg-Whitney conditions was shown by Karlin and Ziegler [12, Theorem 3] as a special case of a more general result for Chebyshev splines. A more direct proof using Rolle's theorem was derived by de Boor [7]. Schumaker [18] proved it by bounding the number of zeros of a spline, and a version of this latter proof appears again in [19, Theorem 4.67].

*Department of Mathematics, University of Oslo, Moltke Moes vei 35, 0851 Oslo, Norway, email: michaelf@math.uio.no

A proof of the total positivity of collocation matrices of B-spline values (all minors are non-negative) was derived by Karlin [11, Ch. 10, Theorem 4.1] based on determinant identities. de Boor in [7] also specified which minors are positive and generalized to the Hermite case [7, Theorem 2].

Later, de Boor and DeVore [8] had the idea of deriving these results using Böhm's knot insertion algorithm [1], which seems a simpler approach. Due to the linearity of determinants with respect to their columns, one can express the determinant of a given collocation matrix of B-spline values as a non-negative linear combination of the determinants of collocation matrices of refined B-splines. Once enough knots have been inserted between each pair of interpolation points, any resulting collocation matrix is diagonal and trivially has a non-negative determinant, which is also positive if the diagonal elements are positive; see [8, Theorem 2].

A similar approach was used to obtain the generalization to Hermite interpolation [8, Theorem 3]. The same knot-insertion strategy can be applied in order to isolate the distinct interpolation points. It then remains to show that the submatrix associated with each distinct interpolation point has a positive determinant if its B-splines are positive at that point. These determinants are Wronskians of non-zero B-splines, and it was proposed in [8] to establish their positivity by further, repeated knot insertion. This, however, requires dealing with one-sided derivatives, and motivates looking for an alternative proof, that only involves 'real' derivatives, i.e., that does not involve values or derivatives of B-splines that only make sense as right or left limits at a knot. Indeed, there is such a proof, that of Theorem 1 below, which only uses the two recursion formulas for derivatives (3) and (4). For B-spline Wronskians of maximal order, positivity follows from a simple formula (Lemma 1) of which we give some examples.

The positivity of B-spline Wronskians of first order (2×2) also leads to a proof of the hodograph property of rational spline curves in B-spline form [9] and the log-concavity property of B-splines [10]. Wronskians also play a central role in the theory of Chebyshev systems [11, 13, 14, 15, 20].

It is common practice to carry out Hermite interpolation by splines of degree d on an interval $[a, b]$ by letting the first $d+1$ knots equal a and the last $d+1$ equal b and take the limits of the B-splines and their derivatives from the right at a and from the left at b . For this reason, we also derive the positivity of one-sided Wronskians at a knot of multiplicity $d+1$ in Theorems 2 and 3.

2 Wronskian of maximal order

Let $\mathbf{t} = (t_i)_i$ be a knot vector, i.e., a real sequence such that $t_i \leq t_{i+1}$ for all i . Let $d \geq 0$ and suppose that $t_i < t_{i+d+1}$. Then we define the B-spline $B_{i,d}$ by

$$B_{i,d}(x) := (t_{i+d+1} - t_i)[t_i, t_{i+1}, \dots, t_{i+d+1}](\cdot - x)_+^d, \quad x \in \mathbb{R}, \quad (1)$$

where $[t_i, \dots, t_{i+d+1}]$ is a divided difference and $u_+ := \max\{u, 0\}$. We recall that $B_{i,d}$ is a piecewise polynomial of degree d which is positive in (t_i, t_{i+d+1}) , zero outside $[t_i, t_{i+d+1}]$, and has continuity of order C^{d-m} at a knot of $\mathbf{t}$ of multiplicity m .

Suppose that for some $m \geq 0$,

$$t_{\mu-m} < t_{\mu-m+1} = \dots = t_{\mu} = x < t_{\mu+1}. \quad (2)$$

Thus, if $m = 0$ then $t_{\mu} < x < t_{\mu+1}$ and x is between knots, while if $m \geq 1$ then x is at a knot of multiplicity m .

Next let $d \geq m$. Then the non-zero B-splines of degree d at x are $B_{\mu-d,d}, \dots, B_{\mu-m,d}$. Let W_d denote their Wronskian at x ,

$$W_d := \begin{vmatrix} B_{\mu-d,d}(x) & \dots & B_{\mu-m,d}(x) \\ B'_{\mu-d,d}(x) & \dots & B'_{\mu-m,d}(x) \\ \vdots & & \vdots \\ B_{\mu-d,d}^{(d-m)}(x) & \dots & B_{\mu-m,d}^{(d-m)}(x) \end{vmatrix},$$

and for $0 \leq r \leq d$, let

$$d^{[r]} := d^r (d-1)^{(r-1)} \dots (d-r+1)^1.$$

Here and elsewhere an empty product is defined to equal 1 (as in the case $r = 0$).

Lemma 1 For $d \geq m$,

$$W_d = d^{[d-m]} \Big/ \prod_{k=m+1}^d \prod_{j=\mu-k+1}^{\mu-m} (t_{j+k} - t_j).$$

Proof. The proof is by induction on $d \geq m$. If $d = m$ then the formula holds because

$$W_m = B_{\mu-m,m}(x) = 1.$$

Otherwise, $d > m$. Then since $B_{\mu-d,d}, \dots, B_{\mu-m,d}$ sum to 1 at x ,

$$B_{\mu-d,d}(x) = 1 - \sum_{j=\mu-d+1}^{\mu-m} B_{j,d}(x),$$

$$B_{\mu-d,d}^{(k)}(x) = 0 - \sum_{j=\mu-d+1}^{\mu-m} B_{j,d}^{(k)}(x), \quad k = 1, \dots, d-m.$$

Substituting these into the first column of W_d gives

$$W_d = \begin{vmatrix} 1 - \sum_{j=\mu-d+1}^{\mu-m} B_{j,d}(x) & B_{\mu-d+1,d}(x) & \cdots & B_{\mu-m,d}(x) \\ 0 - \sum_{j=\mu-d+1}^{\mu-m} B'_{j,d}(x) & B'_{\mu-d+1,d}(x) & \cdots & B'_{\mu-m,d}(x) \\ \vdots & \vdots & & \vdots \\ 0 - \sum_{j=\mu-d+1}^{\mu-m} B_{j,d}^{(d-m)}(x) & B_{\mu-d+1,d}^{(d-m)}(x) & \cdots & B_{\mu-m,d}^{(d-m)}(x) \end{vmatrix}.$$

Then adding all columns except the first to the first gives

$$W_d = \begin{vmatrix} B'_{\mu-d+1,d}(x) & \cdots & B'_{\mu-m,d}(x) \\ \vdots & & \vdots \\ B_{\mu-d+1,d}^{(d-m)}(x) & \cdots & B_{\mu-m,d}^{(d-m)}(x) \end{vmatrix}.$$

Next, applying to every element of this determinant the recursion [5, 3]

$$B_{j,d}^{(k)}(x) = d \left(\frac{B_{j,d-1}^{(k-1)}(x)}{t_{j+d} - t_j} - \frac{B_{j+1,d-1}^{(k-1)}(x)}{t_{j+d+1} - t_{j+1}} \right) \quad (3)$$

and defining the column vectors

$$\mathbf{w}_j := \frac{d}{t_{j+d} - t_j} [B_{j,d-1}(x), B'_{j,d-1}(x), \dots, B_{j,d-1}^{(d-m-1)}(x)]^T,$$

we find

$$W_d = |\mathbf{w}_{\mu-d+1} - \mathbf{w}_{\mu-d+2}, \dots, \mathbf{w}_{\mu-m-1} - \mathbf{w}_{\mu-m}, \mathbf{w}_{\mu-m}|.$$

Then, adding the last column to the second to last column, then adding the second to last column to the third to last column, and so on, we deduce that

$$W_d = |\mathbf{w}_{\mu-d+1}, \dots, \mathbf{w}_{\mu-m}| = \frac{d^{d-m}}{\prod_{j=\mu-d+1}^{\mu-m} (t_{j+d} - t_j)} W_{d-1},$$

and the result follows from the induction hypothesis. $\square$

3 Examples

Let us consider some examples of the formula of Lemma 1. The first two non-trivial cases are

$$W_{m+1} = \frac{m+1}{t_{\mu+1} - t_{\mu-m}},$$

and

$$W_{m+2} = \frac{(m+2)^2(m+1)}{(t_{\mu+1} - t_{\mu-m-1})(t_{\mu+2} - t_{\mu-m})(t_{\mu+1} - t_{\mu-m})}.$$

For special cases of the knot vector, we obtain simpler formulas for general degree. Suppose that $\mathbf{t}$ is uniform, with $t_i = i$. Then the B-splines are cardinal and there are two cases. If $m = 0$ then $\mu < x < \mu + 1$ and then

$$W_d = \frac{d^{[d]}}{d^{[d]}} = 1, \quad d \geq 0.$$

If $m = 1$ then $x = \mu$ and then

$$W_d = \frac{d^{[d-1]}}{d^{[d-1]}} = 1, \quad d \geq 1.$$

Suppose $m = 0$ and

$$t_{\mu-d} = \cdots = t_{\mu} = 0, \quad \text{and} \quad t_{\mu+1} = \cdots = t_{\mu+d+1} = 1.$$

Then $0 < x < 1$ and $B_{\mu-d,d}(x), \dots, B_{\mu,d}(x)$ are the Bernstein polynomials of degree d at x ,

$$B_{\mu-d+j,d}(x) = \binom{d}{j} x^j (1-x)^{d-j}, \quad j = 0, 1, \dots, d.$$

In this case,

$$W_d = d^{[d]}, \quad d \geq 0.$$

Finally, consider the case $m = 0$ of Lemma 1 for any knot vector such that $t_{\mu} < t_{\mu+1}$. It is well known that the B-splines $B_{\mu-d,d}, \dots, B_{\mu,d}$ are linearly independent in $(t_{\mu}, t_{\mu+1})$. This was already established by Curry and Schoenberg [4] and also follows, for example, from Marsden's identity [16]. The case $m = 0$ of Lemma 1 gives an alternative proof of this since it implies that the Wronskian W_d as a function of $x \in (t_{\mu}, t_{\mu+1})$ is non-zero. We note that W_d is also constant for $x \in (t_{\mu}, t_{\mu+1})$ which is expected since $B_{\mu-d,d}, \dots, B_{\mu,d}$ are polynomials of degree $\leq d$ in $(t_{\mu}, t_{\mu+1})$.

4 Wronskian of general order

Suppose again that $\mathbf{t}$, m , and x are as in (2). Then choose some $d \geq m$ and r , $0 \leq r \leq d - m$, and an increasing sequence of indices $\mathbf{j} = (j_0, j_1, \dots, j_r)$ such that

$$\mu - d \leq j_0 < j_1 < \dots < j_r \leq \mu - m,$$

and consider the Wronskian of order r ,

$$W_{\mathbf{j},d} := \begin{vmatrix} B_{j_0,d}(x) & \cdots & B_{j_r,d}(x) \\ B'_{j_0,d}(x) & \cdots & B'_{j_r,d}(x) \\ \vdots & & \vdots \\ B_{j_0,d}^{(r)}(x) & \cdots & B_{j_r,d}^{(r)}(x) \end{vmatrix}.$$

Theorem 1 $W_{\mathbf{j},d} > 0$.

Proof. For fixed $r \geq 0$, the proof is by induction on $d \geq m + r$. If $d = m + r$ then $r = d - m$ and $\mathbf{j} = (\mu - d, \dots, \mu - m)$ and so $W_{\mathbf{j},d} = W_d > 0$ by Lemma 1. Otherwise, $d > m + r$. Then recall the recursion [6, 2]

$$B_{j,d}^{(k)}(x) = \frac{d}{d-k} \left(\alpha_j B_{j,d-1}^{(k)}(x) + \beta_j B_{j+1,d-1}^{(k)}(x) \right), \quad (4)$$

where

$$\alpha_j := \frac{x - t_j}{t_{j+d} - t_j}, \quad \beta_j := \frac{t_{j+d+1} - x}{t_{j+d+1} - t_{j+1}}.$$

Since $r < d - m \leq d$ we can apply this to all the elements of $W_{\mathbf{j},d}$. Then, using the linearity of $W_{\mathbf{j},d}$ with respect to its rows and columns,

$$W_{\mathbf{j},d} = \prod_{k=1}^r \frac{d}{d-k} \sum_{\epsilon \in \{0,1\}^{r+1}} \gamma_{\epsilon} W_{\mathbf{j}+\epsilon,d-1},$$

where $\epsilon := (\epsilon_0, \epsilon_1, \dots, \epsilon_r)$, and

$$\gamma_{\epsilon} := \prod_{k=0}^r ((1 - \epsilon_k) \alpha_{j_k} + \epsilon_k \beta_{j_k}) > 0.$$

Let $\epsilon \in \{0,1\}^{r+1}$ and set $\mathbf{j}' := \mathbf{j} + \epsilon$. Then $\mathbf{j}'$ is non-decreasing. If any two elements of $\mathbf{j}'$ are equal then the corresponding columns of $W_{\mathbf{j}',d-1}$ are equal.

If $j'_0 = \mu - d$ then the first column of $W_{\mathbf{j}', d-1}$ is zero and if $j'_r = \mu - m + 1$ then the last column of $W_{\mathbf{j}', d-1}$ is zero. Thus in all these three cases, $W_{\mathbf{j}', d-1} = 0$. Otherwise,

$$\mu - d + 1 \leq j'_0 < j'_1 < \cdots < j'_r \leq \mu - m, \quad (5)$$

and then, by the induction hypothesis, $W_{\mathbf{j}', d-1} > 0$. Thus $W_{\mathbf{j}, d} > 0$ if there is at least one $\epsilon \in \{0, 1\}^{r+1}$ with the property that $\mathbf{j}' := \mathbf{j} + \epsilon$ satisfies (5). To see that there is one, observe that since $d > m + r$, there must be some index $p \in \{\mu - d, \dots, \mu - m\}$ that does not belong to $\mathbf{j}$. Then $\epsilon \in \{0, 1\}^{r+1}$ defined by

$$\epsilon_k := \begin{cases} 1 & \text{if } j_k < p, \\ 0 & \text{if } j_k > p, \end{cases}$$

has this property. □

5 Knots of multiplicity $d + 1$

Suppose that

$$t_{\mu-d} = \cdots = t_\mu = x < t_{\mu+1},$$

and for some r , $0 \leq r \leq d$, consider the Wronskian at x with limiting values from the right,

$$W_+ := \begin{vmatrix} B_{\mu-d,d}(x+) & \cdots & B_{\mu-d+r,d}(x+) \\ B'_{\mu-d,d}(x+) & \cdots & B'_{\mu-d+r,d}(x+) \\ \vdots & & \vdots \\ B_{\mu-d,d}^{(r)}(x+) & \cdots & B_{\mu-d+r,d}^{(r)}(x+) \end{vmatrix}.$$

Its positivity follows from

Theorem 2

$$W_+ = \frac{d^{[r]}}{\prod_{i=1}^r (t_{\mu+i} - x)^{r-i+1}}.$$

Proof. One could use the recursion (3) to show that W_+ is lower triangular and to compute its diagonal elements by induction on degree. Alternatively, we can simply take a limiting case of the formula of Lemma 1. Suppose initially that

$$t_{\mu-d+r} < t_{\mu-d+r+1} = \cdots = t_\mu = x < t_{\mu+1}.$$

Then we can let $m := d - r$ in Lemma 1, so that

$$W_d = d^{[r]} \Big/ \prod_{k=d-r+1}^d \prod_{j=\mu-k+1}^{\mu-d+r} (t_{j+k} - t_j).$$

Then letting $t_{\mu-d}, \dots, t_{\mu-d+r} \rightarrow x$, the denominator converges to

$$\prod_{k=d-r+1}^d \prod_{j=\mu-k+1}^{\mu-d+r} (t_{j+k} - x) = \prod_{k=1}^r ((t_{\mu+1} - x) \cdots (t_{\mu+k} - x)),$$

and the result follows. $\square$

Conversely, suppose that

$$t_\mu < x = t_{\mu+1} = \cdots = t_{\mu+d},$$

and for some r , $0 \leq r \leq d$, let

$$W_- := \begin{vmatrix} B_{\mu-r,d}(x-) & \cdots & B_{\mu,d}(x-) \\ B'_{\mu-r,d}(x-) & \cdots & B'_{\mu,d}(x-) \\ \vdots & & \vdots \\ B_{\mu-r,d}^{(r)}(x-) & \cdots & B_{\mu,d}^{(r)}(x-) \end{vmatrix}.$$

Theorem 3

$$W_- = \frac{d^{[r]}}{\prod_{i=1}^r (x - t_{\mu-i+1})^{r-i+1}}.$$

Proof. Similar to W_+ , one could show by (3) that W_- is ‘lower triangular’, although this time the diagonal of the triangle goes from the lower left element to the upper right. However, it is again easier to take an appropriate limit of the formula of Lemma 1. Suppose initially that

$$t_\mu < x = t_{\mu+1} = \cdots = t_{\mu+d-r} < t_{\mu+d-r+1}.$$

Then we can let $m := d - r$ in Lemma 1 and replace μ by $\mu + d - r$ and we find

$$W_d = d^{[r]} \Big/ \prod_{k=d-r+1}^d \prod_{j=\mu+d-r-k+1}^{\mu+d-r} (t_{j+k} - t_j).$$

Then letting $t_{\mu+d-r+1}, \dots, t_{\mu+d} \rightarrow x$, the denominator converges to

$$\prod_{k=d-r+1}^d \prod_{j=\mu+d-r-k+1}^{\mu+d-r} (x - t_j) = \prod_{k=1}^r ((x - t_{\mu-k+1}) \cdots (x - t_\mu)),$$

and the result follows. $\square$

References

- [1] W. Böhm, *Inserting new knots into B-spline curves*, Comp.-Aided Design **12** (1980), 199–201.
- [2] K. R. Butterfield, *The computation of all the derivatives of a B-spline basis*, J. Inst. Maths Applies **17** (1976), 15–25.
- [3] M. G. Cox, *The numerical evaluation of B-splines*, J. Inst. Maths Applies **10** (1972), 134–149.
- [4] H. B. Curry and I. J. Schoenberg, *On Pólya frequency functions IV: the fundamental spline functions and their limits*, J. Analyse Math. **17** (1966), 71–107.
- [5] C. de Boor, *On calculating with B-splines*, J. Approx. Theory **6** (1972), 50–62.
- [6] ———, *Splines as linear combinations of B-splines. A survey*, Approximation theory, II (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, eds.), Academic Press, 1976, pp. 1–47.
- [7] ———, *Total positivity of the spline collocation matrix*, Indiana Univ. J. Math. **25** (1976), 541–551.
- [8] C. de Boor and R. DeVore, *A geometric proof of total positivity for spline interpolation*, Math. Comp. **45** (1985), 497–504.
- [9] M. S. Floater, *Evaluation and properties of the derivative of a NURBS curve*, Mathematical Methods in CAGD (T. Lyche and L. L. Schumaker, eds.), Academic Press, Boston, 1992, pp. 261–274.
- [10] ———, *Log-concavity of B-splines*, J. Approx. Theory **300** (2024), 106042, pp. 1–5.
- [11] S. Karlin, *Total Positivity, Vol. I*, Stanford University Press, Stanford, California, 1968.
- [12] S. Karlin and Z. Ziegler, *Tchebysheffian spline functions*, SIAM J. Numer. Anal. **3** (1966), 514–543.

- [13] Y. J. Lee and C. A. Micchelli, *On collocation matrices for interpolation and approximation*, J. Approx. Theory **174** (2013), 148–181.
- [14] T. Lyche, C. Manni, and H. Speleers, *Tchebycheffian B-splines revisited: an introductory exposition*, Methods for Geometric Modeling and Numerical Simulation, INdAM Series Vol. 35 (C. Gianelli and H. Speleers, eds.), Springer, 2019, pp. 179–216.
- [15] E. Mainar, J. M. Peña, and B. Rubio, *Accurate computations with Wronskian matrices*, Calcolo **58** (2021), Article 1, pp. 1–15.
- [16] M. J. Marsden, *An identity for spline functions with applications to variation-diminishing spline approximation*, J. Approx. Theory **3** (1970), 7–49.
- [17] I. J. Schoenberg and A. Whitney, *On Pólya frequency functions III: the positivity of translation determinants with an application to the interpolation problem by spline curves*, Trans. Amer. Math. Soc. **74** (1953), 246–259.
- [18] L. L. Schumaker, *Zeros of spline functions and applications*, J. Approx. Theory **18** (1976), 152–168.
- [19] ———, *Spline functions: basic theory, 3rd edition*, Cambridge University Press, Cambridge, 2007.
- [20] H. Speleers, *Algorithm 1020: Computation of multi-degree Tchebycheffian B-splines*, ACM Trans. Math. Softw. **48** (2022), Article 12, 30 pages.