18. Polynomial of several variables.

There are few differences among the theories of polynomials of two, three or four variables. We will content ourselves to write about polynomials of two variables.

We will not insist on the way to calculate the numerical value of such a polynomial through a generalization of Horner’s scheme, since that presents no particular difficulty.

19. Full polynomial.

A polynomial with undetermined coefficients will be said to be full if it contains with the term:

\[ a_{i,j} x^i y^j, \]

all the terms: \( a_{i',j'} x^{i'} y^{j'} \) with \( i' \leq i, j' \leq j \).

EXAMPLES: A complete polynomial in one variable of degree \( n \) is a full polynomial;
A complete polynomial of two variables, of degree \( n \) in \( x \) and \( p \) in \( y \), is a full polynomial;
A homogeneous polynomial of degree \( n \) is not a full polynomial.
20. Silhouette of a full polynomial.

Let us always consider polynomials of two variables. Let us draw in the first quadrant of the plane of $x, y$, a square grid with spacing 1 and let us mark the point $i, j$ if the term $a_{i,j}x^iy^j$ exists in the polynomial.

The set of marked points forms the silhouette of the full polynomial.

EXAMPLES:
a) Polynomial of degree 3 with respect to all its variables;
b) Polynomial of degree 2 in $x$ and 3 in $y$;
c) Polynomial:

$$\alpha x^3 + \beta xy^2 + ax^2 + bxy + cy^2 + dx + ey + f.$$  


A translation $x' = x + h, y' = y + k$, transforms a full polynomial into a full polynomial with the same silhouette. Indeed, in such a translation, the term:

$$a_{i,j}x^iy^j$$

gives uniquely terms in

$$x'^iy'^j \quad \text{with} \quad i' \leq i, \quad j' \leq j.$$  

Well, these terms exist in the polynomial.

22. Determining a polynomial from values.

The problem of determining a polynomial which is not full from some values can be indeterminate or impossible.
EXAMPLE: Let us determine $a$ and $b$, knowing the values of $P(x) = ax^2 + b$ for $x = 1$ and $x = -1$.

If the two proposed values are equal, the problem is indeterminate. If they are different, the problem is impossible.

The problem of determining a full polynomial from arbitrarily placed points can also be indeterminate or impossible. For example, let $a + bx + cy + dxy$ be determined by the values at the pairs

$$(0,0), \ (0,1), \ (0,2), \ (0,3).$$

The problem reduces to determining the polynomial $a+cy$ from the values for $y = 0,1,2,3$. This problem is, in general, impossible. If it is, by chance, possible, the proposed problem is indeterminate since $b$ and $d$ are arbitrary.

23. Grid associated with a full polynomial.

Let the full polynomial $\sum a_{i,j}x^iy^j$ be given.

We will denote by the grid associated with that full polynomial the set of points $(x_i, y_j)$.

The $x_i$ are fixed abscissae and the $y_j$ are fixed ordinates.

EXAMPLE: The silhouette of a full polynomial is a grid associated with that polynomial.

Given the polynomial of the silhouette of figure 40, one can take as the associated grid that of figure 41.

The numbering of the $x$ and $y$ is indicated next to the graph.
24. Determining a full polynomial from values at the vertices of an associated grid.

Given the abscissae \( x_0, x_1, \ldots \) and the ordinates \( y_0, y_1, \ldots \), consider the problem of determining the full polynomial:
\[
\sum a_{i,j} x^i y^j
\]
from the values it takes at the points \((x_i, y_j)\).

Let us perform a translation to have \( y_0 = 0 \).

We see that the conditions imposed on the points \((x_i, y_j)\) determine the terms \( x^i \). If we subtract these terms, we can make \( y \) a factor and we reduce it to the same problem with a smaller number of points. The property is therefore proved.

REMARK: For a polynomial of one variable, these hypotheses just reduce to saying that the polynomial with undetermined coefficients is complete of degree \( n \).

25. Form of Lagrange.

We will denote by this the form we obtain in making appear the polynomials taking the value 1 at one point, and the value 0 at any of the others. It is naturally written:
\[
\sum_{i,j} f_{i,j} L_{i,j}(x).
\]

In the general case, the \( L_{i,j}(x) \) are quite complicated to obtain. Here are some cases where they are simple.

26. Polynomials of degree given with respect to each variable.

The expression of Lagrange simplifies and reduces to the product:
\[
L_i(x)L_j(y).
\]

In the case of the figure:
\[
L_2(x)L_1(y).
\]
27. Polynomials of degree given with respect to all the variables.

If \( n \) is that degree, we can write the polynomial of Lagrange relative to \((i, j)\) in the form:

\[
C_i^x C_j^y C_{n-x-y}^{i+j}.
\]

Indeed, the first factor is of degree \( i \) in \( x \), the second of degree \( j \) in \( y \), the third of degree \( n - i - j \) with respect to the set \((x, y)\). The degree of the expression with respect to its set of variables does not exceed \( n \).

For \( x = i, y = j \), each of the terms has value 1.

Moreover, one or the other is zero for:

\[
\begin{align*}
x &= 0, 1, \ldots, i - 1; \\
y &= 0, 1, \ldots, j - 1; \\
x + y &= 0, 1, \ldots, i + j - 1.
\end{align*}
\]

Well, this represents all the points used other than the pair \((i, j)\).

28. Existing numerical data.

For polynomials of two variables of total degree 2, 3, 4, one will find the tabulation of the Lagrange forms in:

SALZER, Journal of Math. and Physics, 26 (1947), pp. 294–305:

\[
x, y \quad 0 (0, 1) 1 \quad \text{Exact values.}
\]