Maximum principles for infinite dimensional time changed processes

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Abstract

We study an infinite dimensional controlled stochastic evolution equation. This equation is driven by a time-changed Lévy process with value in a separable Hilbert space, and a time dipendent unbounded linear operator. In this framework we deal with two information flows which we are going to consider as partial to each other. We provide a sufficient and a necessary maximum principle for both filtrations, and study the corresponding backward stochastic differential equation.

1 Introduction

Our goal is to study the optimization problem of maximizing the performance functional

$$J(u) = \mathbb{E}\left[\int_0^T F(t, X(t), u(t))dt + G(X(T))\right]$$

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where the dynamics for the forward process are given by

$$X_t^u = x + \int_0^t (A(s)X_s^u + \beta(s, X_s^u, \lambda_s, u_s))\, ds + \int_0^t\int_{\mathbb{H}} \sigma(s, X_s^u, \lambda_s, \xi)\mu(ds, d\xi),$$

where X is a process with values in a separable Hilbert space $\mathbb{H}$ driven by a time changed Lévy process μ. Here we work with a time change

$$\Lambda(t,\Delta)dt\nu(d\xi) := \mathbb{1}_{\{(t,0)\}}(t)\lambda_t^B dt + \int_\Delta \mathbb{1}_\Delta(t,\xi)\nu(d\xi)\lambda_t^H dt, \quad t \in [0,T],\ \Delta \in \mathcal{B}_{\mathbb{H}}$$

This problem is studied for example in [AH10, AH11] in the case of a continuous driver μ, and, for example, in [DNS14] in the finite dimensional case. Our approach, similarly to what has been done in [DNS14], is based on the analysis of the noise and the information flow. We notice that there are two filtrations of interest in such a problem: the first one being $\mathbb{F}$, namely the smallest right continuous filtration to which μ is adapted, and the second one being $\mathbb{G}$, i.e. the filtration generated by $\mathbb{F}$ and the entire history of the time changed process. We will regard $\mathbb{G}$ as an initial enlargement of $\mathbb{F}$.

In this paper we also deal with the existence and uniqueness of a particular form of BSDE in infinite dimension related to the maximum principle approach. Namely we will study an equation of the form

$$Y(t) = \zeta - \int_0^t A(s)Y(s)ds - \int_0^t F(s, Y(s), \phi(s,\cdot)\Lambda^{1/2}(s,\cdot))ds + \int_0^t\int_{\mathbb{H}} \phi(s,\xi)\mu(ds,d\xi) + N(ds)$$

The study of this equation is strongly based on a representation result due to [Ouv75] and follows the lines of [AH09], where the author once again considered a continuous martingale as a driver. We will provide existence result for such equation, and show the crucial role played by the backward for the optimization results.

Our goal will be to provide both a necessary and a sufficient maximum principle, allowing us to fully characterize optimal controls for the optimization problem.

The present work is structured as follows: in Section 2 we introduce the framework material for the infinite-dimensional time-changed Lèvy measure, in Section 3 we study a class of Backward stochastic differential equations related to the maximum principle approach, in Section 4 we provide a sufficient and a necessary maximum principle with respect to two different information flows which we are going to regard as partial to each other. Lastly, in Section 5 we provide an example of an application of our maximum principle to a lifted Volterra process.

2 Framework

We want to consider a controlled type-process with dynamics given by

$$X_t^u = x + \int_0^t \left(A(s)X_s^u + \beta(s, X_s^u, \lambda_s, u_s)\right) ds + \int_0^t\int_{\mathbb{H}} \sigma(s, X_s^u, \lambda_s, \xi)\mu(ds, d\xi), \tag{2.1}$$

where $\beta : [0,T] \times \mathbb{H} \times U \times \longrightarrow \mathbb{H}$, $\sigma : [0,T] \times \mathbb{H} \times \mathbb{R} \longrightarrow \mathbb{H}$, and μ is a Hilbert valued random measure as introduced in the following subsection. Our goal is going to be to maximize a performance functional of the form:

$$J(u) = \mathbb{E}\left[\int_0^T F(t, X(t), u(t))dt + G(X(T))\right] \tag{2.2}$$

where the control u is a stochastic process in the set of admissible, $\mathbb{F}$ predictable controls $\mathcal{A}^{\mathbb{F}}$. Similarly to what we did in the finite case in [DNG20b], we are going to present a sufficient and a necessary maximum principle with respect to both $\mathbb{F}$, namely the smallest right continuous filtration generated by μ, and with respect to $\mathbb{G}$, namely the filtration including information about the future values of the time change process.

Inspired by the work done by [Par79, Par07], and similarly to what has been done in [AH10, AH09, AH11], we are also going to assume that we are in a rigged Hilbert space $(V, \mathbb{H}, V')$, i.e. we take V to be a separable

Hilbert space embedded continuously and densly in $\mathbb{H}$. This framework was introduced by [Lio69] and we are going to need it in order to use a variational approach to some BSDEs that are introduced in the following part.

Lastly we take our admissible controls $\mathcal{A}^{\mathbb{F}}$ to take value in U a nonempty convex subset of $\mathcal{O}$, a separable Hilbert space. Analogously we define the set of admissible controls with respect to the enlarged information flow $\mathbb{G}$.

Whenever we consider inner products in V or in $\mathcal{O}$ we will explicitly write $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_{\mathcal{O}}$ respectivly. If not stated the inner products take values in $\mathbb{H}$.

2.1 Square integrable martingales in Hilbert spaces

Definition 2.1. Two elements M and N in $\mathcal{M}^2_{[0,T]}(\mathbb{H})$, the class of square integrable martingales, are said to be very strongly orthogonal (VSO) if

$$\mathbb{E}\left[M(\tau) \otimes N(\tau)\right] = \mathbb{E}\left[M(0) \otimes N(0)\right]$$

for all $[0,T]$ valued stopping times τ.

We also define the following spaces:

Definition 2.2. Given $\mathbb{H}$ a separable Hilbert space, we define

- $L^2([0,T];\mathbb{H})_{\mathcal{F}} := \left\{ f : [0,T] \times \Omega \longrightarrow K, \mathbb{F} \text{ predictable and } \mathbb{E}\left[\int_0^T |f(t)|_K^2 dt\right] < \infty \right\}$
- $\mathcal{S}^2(\mathbb{H}) := \left\{ f : [0,T] \times \Omega \longrightarrow \mathbb{H} \text{ cadlag, adapted and } \mathbb{E}\left[\sup_{t\in[0,T]} |f(t)|_{\mathbb{H}}^2\right] < \infty \right\}$
- $\mathcal{R}$ the space of functions $\varphi \in L^2(\mathbb{H})$ such that

$$\|\varphi\|_{\mathcal{R}} := |\varphi(0)|_{\mathbb{H}} + \int_{\mathbb{H}_0} \|\varphi(\xi)\|^2_{L^2(\mathbb{H})} \nu(d\xi) < \infty$$

- $\mathcal{I}^{\mathbb{F}}$ the subspace of $L^2(\mathbb{H} \times \Omega, \mathcal{B}_{\mathbb{H}} \times \mathcal{F}, \Lambda \times \mathbb{P})$ of the random fields admitting a $\mathbb{F}$ predictable modification, with the norm

$$\|(\phi_1,\phi_2)\|_{\mathcal{I}} = \mathbb{E}\left[\int_0^T |\phi_1(t)|^2_{\mathbb{H}} dt\right] + \mathbb{E}\left[\int_0^T \left|\int_{\mathbb{H}} \phi_2(t,\xi)\tilde{\mathcal{Q}}^{1/2}_{\mu}(t,\xi)\nu(d\xi)\right|^2_{L^2(\mathbb{H})} dt\right]$$

Analogously we define $\mathcal{I}^{\mathbb{G}}$ the subspace of $L^2(\mathbb{H}\times\Omega, \mathcal{B}_{\mathbb{H}}\times\mathcal{F}, \Lambda\times\mathbb{P})$ of the random fields admitting a $\mathbb{G}$ predictable modification, with the above norm.

- $\mathfrak{B}^2_{\mathbb{F}}(\mathbb{H}) = L^2([0,T];\mathbb{H})_{\mathcal{F}} \times \mathcal{I}^{\mathbb{F}}$ and similarly for $\mathbb{G}$.
- $\mathfrak{R}$ is the subspace of $\mathcal{R}$ such that $\int_{\mathbb{H}} \phi(\cdot,\xi)\nu(d\xi)$ belongs to $L^2(\mathbb{H})$.

Theorem 2.3. *(*[Ouv75]*) Let M in $\mathcal{M}^2_{[0,T]}(\mathbb{H})$ and $\mathcal{H}_1 := \left\{\int X dM \text{ s.t. } X \in \mathcal{I}\right\} \subset \mathcal{M}^2_{[0,T]}(H)$. Let $\mathcal{H}_2$ be the orthogonal component of $\mathcal{H}_1$ in $\mathcal{M}^2_{[0,T]}(H)$. Then every element of $\mathcal{H}_2$ is VSO to every element of $\mathcal{H}_1$. In particular, every element $L \in \mathcal{M}^2_{[0,T]}(H)$ can be written uniquely as*

$$L = \int X dM + N$$

with M VSO to N, $X \in \mathcal{I}$ and $N \in \mathcal{H}_2$.

2.2 Conditional cylindrical Lévy process

Given a Hilbert space $\mathbb{H}$, we consider the space

$$\mathbb{X} := [0,T]\times\mathbb{R} := ([0,T]\times\{0\}) \cup ([0,T]\times\mathbb{H}_0),$$

where $\mathbb{H}_0 = \mathbb{H}\backslash\{0\}$. The Borel σ-algebra on $\mathbb{H}$ is denoted $\mathcal{B}_{\mathbb{H}}$. For all $a = (a_1,...,a_n)$ with $a_i \in \mathbb{H}^*$ for all $i = 1,...,n$, we define

$$\pi_a(x) = \langle x, a\rangle = (\langle x,a_1\rangle, ..., \langle x,a_n\rangle) \in \mathbb{R}^n$$

Remark 2.4. Being $\mathbb{H}$ a Hilbert space, and thus a separable Banach space, the cylindrical σ algebra and the Borel σ algebra coincide.

Let $\mathcal{L}$ be the space of all $\mathbb{H}$-valued two dimensional stochastic processes $\lambda = (\lambda^B, \lambda^H)$ such that, for each component $k = B, H$ we have that

1. $\pi_a(\lambda_t^k) \geq 0$ $P - a.s.$ for all $t \in [0,T]$, $a \in (\mathbb{H}^*)^n$
2. λ^k is stochastically continuous for all $\varepsilon > 0$ and almost all $t \in [0,T]$,
3. $\mathbb{E}\left[\int_0^T \lambda_t^k dt\right] < \infty$.

These processes represent the stochastic time change rate. Let ν be a finite measure on the Borel sets of $\mathbb{H}_0$ such that $\int_{\mathbb{H}_0} \xi^2 \nu(d\xi) < \infty$.

With the above, define the random measure Λ on $\mathcal{B}_{\mathbb{H}}$ by

$$\Lambda(t,\Delta)dt\nu(d\xi) := \mathbb{1}_{\{(t,0)\}}(t)\lambda_t^B dt + \int_\Delta \mathbb{1}_\Delta(t,\xi)\nu(d\xi)\lambda_t^H dt, \quad t \in [0,T],\ \Delta \in \mathcal{B}_{\mathbb{H}}$$

Denote the σ-algebra generated by the values of Λ by $\mathcal{F}^\Lambda$ and denote Λ^B, Λ^H the restrictions of Λ to $[0,T] \times \{0\}$ and $[0,T] \times \mathbb{H}_0$, respectively.

For later use we also introduce the filtration

$$\mathbb{F}^\Lambda = \{\mathcal{F}_t^\Lambda, t \in [0,T]\}$$

generated bu the values $\Lambda([0,t])$, $t \in [0,T]$. Also define $\mathcal{F}^\Lambda := \mathcal{F}_T^\Lambda$.

We now introduce the random measures B and H as a generalization of what is done in [DNS14] Definition 2.1:

Definition 2.5. The conditional cylindrical Gaussian measure (given $\mathcal{F}^\Lambda$) B is a signed random measure on the Borel sets of $[0,T] \times \{0\}$ satisfying

A1. $P\left(B(\Delta) \leq x | \mathcal{F}^\Lambda\right) = P\left(B(\Delta) \leq x | \Lambda^B(\Delta)\right) = \Phi\left(\frac{x}{\sqrt{\Lambda^B(\Delta)}}\right)$, $x \in \mathbb{R}, \Delta \subseteq [0,T] \times \{0\}$.

A2. $B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$ whenever $\Delta_1, \Delta_2 \subseteq [0,T] \times \{0\}$ such that $\Delta_1 \cap \Delta_2 = \emptyset$.

The conditional Poisson measure (given $\mathcal{F}^\Lambda$) H is a random measure on the Borel sets of $[0,T] \times \mathbb{H}_0$ satisfying

A3. $P\left(H(\Delta)=k|\mathcal{F}^{\Lambda}\right)=P\left(H(\Delta)=k|\Lambda^{H}(\Delta)\right)=\frac{\Lambda^{H}(\Delta)^{k}}{k!}e^{-\Lambda^{H}(\Delta)}$, $k\in\mathbb{N},\Delta\subseteq[0,T]\times\mathbb{H}_0$.

A4. $H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent given $\mathcal{F}^{\Lambda}$ whenever $\Delta_1,\Delta_2\subseteq[0,T]\times\mathbb{H}_0$ such that $\Delta_1\cap\Delta_2=\emptyset$.

Also we assume

A5. B and H are conditionally independent given $\mathcal{F}^{\Lambda}$.

And we define $\widetilde{H}(\Delta):=H(\Delta)-\Lambda^{H}(\Delta)$, $\Delta\subset\mathbb{X}$.

Definition 2.6. We define the signed random measure μ on the Borel subsets of $\mathbb{H}$ by

$$\mu(\Delta):=B\left(\Delta\cap[0,T]\times\{0\}\right)+\widetilde{H}\left(\Delta\cap[0,T]\times\mathbb{H}_0\right),\ \Delta\subseteq\mathbb{X} \tag{2.3}$$

The random measures B and H are related to a time-changed cylindrical Wiener process and a time changed pure jump cylindrical Lévy process. To illustrate, define the processes

$$B_t:=B([0,t]\times\{0\}) \qquad \Lambda_t^B:=\int_0^t\lambda_s^B ds$$
$$\eta_t:=\int_0^t\int_{\mathbb{H}_0}\xi\widetilde{H}(ds,d\xi) \qquad \Lambda_t^H:=\int_0^t\lambda_s^H ds,$$

for $t\in[0,T]$ and compute the conditional characteristic functions of B and η. First, from (A1) and (A3) we have that

$$\begin{aligned}\mathbb{E}[e^{i\langle c,B_t\rangle}|\mathcal{F}^{\Lambda}]=\mathbb{E}[e^{icB_t}|\Lambda_t^B]&=\exp\left\{\int_0^t\frac{1}{2}c^2\lambda_s^B ds\right\}\\&=\exp\left\{\frac{1}{2}\langle c^2,\Lambda_t^B\rangle\right\},\ c\in\mathbb{H},\end{aligned}$$

which corresponds to a time changed Wiener process motion with time change process Λ^B. If $\lambda^b\equiv 1$ we get back remark 4.22 of [PZ07]

Correspondingly we have that

$$\mathbb{E}[e^{i\langle c,\eta_t\rangle}|\mathcal{F}^\Lambda] = \exp\left\{\left\langle \int_{\mathbb{H}_0}[e^{ic\xi}-1-ic\xi]\nu(d\xi), \Lambda_t^H\right\rangle\right\},\ c\in\mathbb{H}.$$

once again if $\lambda^H \equiv 1$ we obtain Proposition 4.18 of [PZ07].

Moreover, if we consider $B_t^a = \pi_a(B_t)$ and $\eta_t^a = \pi_a(\eta_t)$, we have that

$$\mathbb{E}[e^{icB_t^a}|\mathcal{F}^\Lambda] = \exp\left\{\frac{1}{2}c^2\pi_a(\Lambda_t^B)\right\},\ c\in\mathbb{R},$$

and

$$\mathbb{E}[e^{ic\eta_t^a}|\mathcal{F}^\Lambda] = \exp\left\{\left(\int_{\mathbb{R}_0^n}[e^{ic\xi}-1-ic\xi]\nu_a(d\xi)\right)\pi_a(\Lambda_t^H)\right\},\ c\in\mathbb{R},$$

with $\nu_a = \nu\circ\pi_a$, thus getting back to the results in [AR10].

Definition 2.7. We call the process μ as in (2.3) a conditional cylindrical Lévy process (given $\mathbb{F}^\Lambda$).

Remark 2.8. Notice that, if we take $a = 1 \in \mathbb{R}$, we get back the time-change results from [**?**].

Since we are dealing with a time change process there are two information flows which we are going to consider:

- $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ namely the smallest right continuous filtration to which μ is adapted
- $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]} = \{\mathcal{F}_t \vee \mathcal{F}^\Lambda\}_{t\in[0,T]}$

For our martingale μ, we are going to denote with $\langle\mu\rangle$ the predictable quadratic variation of μ and with $\tilde{\mathcal{Q}}_\mu$ the predictable process taking values in $L^1(H)$ which is associated with the Doléans measure of $\mu\otimes\mu$. We denote with

$$\langle\langle \mu_t(\Delta)\rangle\rangle_t = \int_0^t \int_\Delta \tilde{\mathcal{Q}}_\mu(s,\xi)\nu(d\xi)\langle\mu\rangle(s,\Delta)ds$$

Notice also that $\langle\langle \mu\rangle\rangle_t = \int_0^t \mathcal{Q}(s,z)ds\nu(dz)$ for an opportune $\mathcal{Q}(s,z)$ to be specified below.

Remark 2.9. For the study of the above processes we refer to [Mé82]. In our framework we have that both $\mathcal{Q}$ and $\tilde{\mathcal{Q}}_\mu$ can be expressed in relation with the random measure μ. In particular, if we denote with $d_{\mu\otimes\mu}$ the Doléans function of $\mu\otimes\mu$, and with α_μ its trace. This means that we have

$$\frac{d\tilde{\mathcal{Q}}_\mu}{d\alpha_\mu} = \nabla d_{\mu\otimes\mu}$$

in the Radon-Nikodym derivative sense.

Moreover, since

$$\begin{aligned}\int_0^t \int_\Delta \langle\langle\mu\rangle\rangle(ds,d\xi) &= = \int_0^t \mathbb{1}_{\{0\}}(s,0)\lambda_s^B ds + \int_0^t \int_{\mathbb{H}_0} \mathbb{1}_{\mathbb{H}_0}(s,\xi)\lambda_s^H \nu(d\xi)ds \\ &= \int_0^t \int_{\mathbb{H}} \Lambda(s,\xi)\nu(d\xi)ds\end{aligned}$$

we have that $\langle\langle\mu\rangle\rangle$ can be expressed in terms of an integral with respect to $\nu(d\xi)$ and ds.

Proposition 2.10. *Let $\mu \in \mathcal{M}^2_{[0,T]}(H)$. Suppose that there exists a predictable process $\mathcal{Q}(\cdot)$ such that, for each (t,ω), $\Lambda(t,\xi,\omega)$ is symmetric, positive definite nuclear operator on $\mathbb{H}$ and $\langle\langle\mu\rangle\rangle = \Lambda$. If for some positive definite nuclear operator $\mathcal{Q}$ on H, $\Lambda(t,\xi) \leq \mathcal{Q}$, for all $t \in [0,T], \xi \in \mathbb{R}$, then, given $\phi \in \mathcal{I}^{\mathbb{F}}$, we have that*

$$\mathbb{E}\left[\int_0^T \| \int_{\mathbb{H}} \phi \circ \tilde{\mathcal{Q}}_\mu^{1/2} \|_2^2 \langle\mu\rangle(dt,\Delta)\right] \leq \mathbb{E}\left[\int_0^T \| \int_{\mathbb{H}} \phi\|_\Lambda^2 dt\right]$$

where $\| \int_\Delta \phi \|_\Lambda := \| \int_\Delta \phi(s,\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)\|_{L^2(\mathbb{H})}$ *for each* $\Delta \in \mathcal{B}_{\mathbb{H}}$.

Proof. Notice that, by definition of $q(t,z) = Trace(\Lambda(s,z))$, and recalling that

$$\langle \mu \rangle(s) = \int_{\mathbb{H}} q(s,\xi)\nu(d\xi),$$

and that

$$\tilde{\mathcal{Q}}^{1/2}(s,\xi) = \frac{\Lambda(s,\xi)}{\int_{\mathbb{H}} q(s,\xi)\nu(d\xi)}$$

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \| \int_{\mathbb{H}} \phi(s,z)\tilde{\mathcal{Q}}^{1/2}(s,z)\nu(dz)\| \langle \mu \rangle ds\right] \\
&= \mathbb{E}\left[\int_0^T \| \int_{\mathbb{H}} \phi(s,\xi)\left(\frac{\Lambda(s,\xi)\nu(d\xi)}{\int_{\mathbb{H}} q(s,\xi)\nu(d\xi)}\right)^{1/2} \|_2^2 \int_{\mathbb{H}} q(s,\xi)\nu(d\xi) ds\right] \\
&= \mathbb{E}\left[\int_0^T \| \int_{\mathbb{H}} \phi(s,\xi)\Lambda(s,\xi)\nu(d\xi)\|_2^2 ds\right]
\end{aligned}$$

□

3 Backward-Hamiltonian

In order to study our optimization problem we have to consider a backward stochastic differential equation of the following form

$$\begin{aligned}
dY(t) &= -F(t, Y(t), \phi(t,\cdot)\mathcal{Q}^{1/2}(t))dt + \int_{\mathbb{H}_0} \phi(t,z)\mu(dt,dz) - dN(t) \\
Y(T) &= \zeta, \qquad (3.1)
\end{aligned}$$

where $t \in [0,T]$. Here ζ is the terminal value, and b satisfies the following hypothesis:

Hypothesis 3.1.

- $b : [0,T] \times \Omega \times \mathbb{H} \times (\mathcal{R} \times L^2(\mathbb{H})) \longrightarrow \mathbb{H}$ *is measurable*
- $\mathbb{E}\left[\int_0^T |F(t,0,0)|_{\mathbb{H}}^2 dt\right] < \infty$

- *It exists a constant $k > 0$ such that for all y_1, y_2 in $\mathbb{H}$ and for all φ_1, φ_2 in $\mathcal{R}$:*

  $$|F(t, y_1, \varphi_1) - F(t, y_2, \varphi_2)|^2_{\mathbb{H}}$$
  $$\leq k \left(|y_1 - y_2|^2_{\mathbb{H}} + |\varphi_1(0) - \varphi_2(0)|^2_{L^2(\mathbb{H})} \lambda^B + \int_{R_0} |\varphi_1(0) - \varphi_2(0)|^2_{L^2(\mathbb{H})} \nu(dz) \lambda^H \right)$$

  uniformly in t, ω.

- $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{H})$

In this chapter we are going to consider this BSDE under the information flow $\mathbb{G}$.

A solution of this BSDE is a triple (Y, ϕ, N) of predictable processes that satisfy the integral form of this BSDE for each t, in addition to some integrability conditions. Here N is a square integrable martingale required to be very strongly orthogonal to μ.

The process $\mathcal{Q}(s, \xi)$ is called the local characteristic operator, or local covariation operator of the martingale $\mu(s)$.

We notice that moreover we have that $\tilde{\mathcal{Q}}_\mu(t, \xi) = \frac{\Lambda(t,\xi)}{\int_{\mathbb{H}} q(t,\xi)\nu(d\xi)}$ and $\langle \mu \rangle(dt) = \int_{\mathbb{H}} q(t, \xi)\nu(d\xi)dt$, where $q(t, \xi) = Trace(\Lambda(t, \xi))$. In particular this means that if $g \in \mathcal{I}$ then

$$\mathbb{E}\left[\left| \int_0^T \int_{\mathbb{H}} g(s, \xi)\mu(ds, d\xi) \right|^2_{\mathbb{H}} \right] = \mathbb{E}\left[\int_0^T \left| \int_{\mathbb{H}} g(s, \xi) \frac{\Lambda^{1/2}(s, \xi)\nu(dz)}{(\int_{\mathbb{H}} q(s, \xi)\nu(d\xi))^{1/2}} \right|^2_{L^2(\mathbb{H})} \int_{\mathbb{H}} q(s, \xi)\nu(d\xi) ds \right]$$
$$= \mathbb{E}\left[\int_0^T \left| \int_{\mathbb{H}} g(s, \xi)\Lambda^{1/2}(s, \xi)\nu(d\xi) \right|^2_{L^2(\mathbb{H})} ds \right]$$

Theorem 3.2. *Assume that the previous Hypothesis hold. then there exists a unique solution (Y, ϕ, N) in $\mathfrak{B}^2_{\mathbb{G}}(\mathbb{H}) \times \mathcal{M}^2[0, T](\mathbb{H})$. Moreover $Y(t) \in \mathcal{S}^2(\mathbb{H})$.*

Proof. We will use a fixed point theorem for the contraction principle, similarly to what we already did in [DNG20b] for the finite dimensional case, and similarly to what is done in [AH09] for the continuous martingale driver case.

Notice now that given $(x,\varphi) \in \mathcal{I}$ the follwing local martingale belongs to $\mathcal{M}^2_{[0,T]}(\mathbb{H})$

$$\kappa(t) = \mathbb{E}\left[\zeta + \int_0^T F(s, y(s), \varphi(s,\cdot))ds|\mathcal{F}_t\right], 0 \le t \le T$$

If we now define the map Ψ on $\mathcal{I}$ by $\Psi(y,\varphi) = (Y,\phi)$ where

$$Y(t) = \mathbb{E}\left[\zeta + \int_0^T F(s, y(s), \varphi(s,\cdot))ds|\mathcal{F}_t\right]$$

and ϕ is given by using the representation of the martingale κ in Theorem 2.3 as:

$$\kappa(t) = Y(0) + \int_0^t \int_{\mathbb{H}} \phi(s,\xi)\mu(ds,d\xi) + N(t),\ t \in [0,T] \tag{3.2}$$

Such that N is an $\mathbb{H}$-valued càdlàg local martingale VSO to μ. Moreover we notice that thanks to our assumptions

$$\mathbb{E}\left[|\kappa(t)|^2\right] \le 2\mathbb{E}\left[|\zeta|^2\right] + 2T\mathbb{E}\left[\int_0^T |F(s, y(s), \varphi(s,\cdot)\Lambda^{1/2}(s,\cdot))|ds\right] < \infty \tag{3.3}$$

We thus have that

$$
\begin{aligned}
\|Y\|_{\mathcal{S}^2(\mathbb{H})} &\leq \mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{E}\left[\zeta + \int_t^T F(s,y(s),\phi(s,\cdot)\Lambda^{1/2}(s,\cdot))ds\middle|\mathcal{F}_t\right]\right|^2\right] \\
&\leq 2\mathbb{E}\left[\sup_{t\in[0,T]}\mathbb{E}\left[|\zeta||\mathcal{F}_t\right]^2\right] \\
&+ 2\mathbb{E}\left[\sup_{t\in[0,T]}\mathbb{E}\left[\int_0^T F(s,y(s),\phi(s,\cdot)\Lambda^{1/2}(s,\cdot))\middle|\mathcal{F}_t\right]^2\right] \\
&\leq 8\left[\mathbb{E}[|\zeta|^2] + T\mathbb{E}\left[\int_0^T |F(s,y(s),\phi(s,\cdot)\Lambda^{1/2}(s,\cdot))|^2 ds\right]\right] < \infty
\end{aligned}
$$

due to Doob's inequality. In particular one has that

$$
\mathbb{E}\left[\int_0^T |Y(t)|^2 dt\right] \leq T\|Y\|^2_{\mathcal{S}^2(\mathbb{H})} < \infty
$$

We now observe that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |\int_{\mathbb{H}} \phi(t,z)\Lambda^{1/2}(t,z)\nu(dz)|^2 dt\right] + \mathbb{E}\left[|N(T)|\right] \\
&= \mathbb{E}\left[\iint_{\mathbb{X}} |\phi(t,z)\tilde{\mathcal{Q}}_\mu^{1/2}(t,z)\nu(dz)|\langle\mu\rangle(dt)\right] + \mathbb{E}\left[|N(T)|\right] \\
&= \mathbb{E}\left[\iint_{\mathbb{X}} \phi(t,z)\mu(dt,dz) + N(T)|^2\right] \\
&= \mathbb{E}\left[|\kappa(T)-\kappa(0)|^2\right] \\
&\leq 4\mathbb{E}\left[|\kappa(T)|^2\right] \\
&< \infty
\end{aligned}
$$

by using Jensen's inequality and (3.3). We have thus that $Y \in L^2_{\mathcal{G}}(0,T;\mathbb{H})$, $\phi \in \mathcal{I}^{\mathbb{G}}$ and $N \in \mathcal{M}^2_{[0,T]}(\mathbb{H})$, so in particular we have that Ψ goes from $\mathfrak{B}^2_{\mathbb{G}}(\mathbb{H})$ into itself. Moreover $(Y,\phi,N) \in \mathfrak{B}^2_{\mathbb{G}}(\mathbb{H}) \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$ is a solution of the BSDE (3.1) if and only if (Y,ϕ) is a fixed point of Ψ.

We now want to show that this map is actually a contraction, we thus take two elements (y_i, φ_i), $i = 1, 2$ with corresponding image (Y_i, ϕ_i, N_i), $i = 1, 2$ in $\mathcal{I} \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$. by using the mapping Ψ as done earlier. We denote by Δy, $\Delta\varphi$, etc. the difference processes $y_1 - y_2$, $\varphi_1 - \varphi_2$, etc.

Let now γ be a real number, we have that

$$\begin{aligned}
&\mathbb{E}\left[e^{\gamma t}|\Delta Y(t)|^2\right] + \gamma\mathbb{E}\left[\int_t^T e^{\gamma s}|\Delta Y(s)|^2 ds\right] \\
&\quad + \mathbb{E}\left[\int_t^T e^{\gamma s}|\int_{\mathbb{H}} \Delta\phi(s,\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2 ds\right] + \mathbb{E}\left[\int_t^T e^{\gamma s}\langle\Delta N\rangle(ds)\right] \\
&\quad = 2\mathbb{E}\left[\int_t^T e^{\gamma s}\langle\Delta Y(s), F(s, y_1(s), \varphi_1(s,\cdot)\Lambda^{1/2}(s,\cdot)) - F(s, y_2(s), \varphi_2(s,\cdot)\Lambda^{1/2}(s,\cdot))\rangle ds\right]
\end{aligned}$$

thus by Lipshitzianity of F and by choosing $\gamma = 2k_1 + 1$ we have that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\gamma s}|\Delta Y(s)|^2 ds + \int_0^T e^{\gamma s}|\int_{\mathbb{H}} \Delta\phi(s,\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2 ds + \int_0^T e^{\gamma s}\langle\Delta N\rangle(ds)\right] \\
&\quad \leq \frac{1}{2}\left(\mathbb{E}\left[\int_0^T e^{\gamma s}|\Delta y(s)|^2 ds + \int_0^T |\int_{\mathbb{H}} \Delta\varphi(s,\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2 ds\right]\right)
\end{aligned}$$

In particular

$$\|(\Delta Y, \Delta\phi)\|^2_{\mathcal{I}} \leq \frac{1}{2}\|(\Delta y, \Delta\varphi)\|^2_{\mathcal{I}},$$

and thus we have that Ψ is a contraction on $\mathfrak{B}^2_{\mathbb{G}}(\mathbb{H})$ equipped with the norm

$$\|(Y,\phi)\|_{\mathcal{I}} := \mathbb{E}\left[\int_0^T e^{\gamma s}|Y(s)|^2 ds + \int_0^T e^{\gamma s}|\int_{\mathbb{H}} \phi(s,\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2 ds\right]^{1/2}$$

hence it has a unique fixed point.

□

If we now consider the more general equation

$$\begin{aligned}
&dY(t) = -A(t)Y(t) - F(t, Y(t), \phi(t,\cdot)\Lambda^{1/2}(t,\cdot))dt + \int_{\mathbb{R}} \phi(t,\xi)\mu(dt, d\xi) + N(dt) \\
&Y(T) = \zeta
\end{aligned} \tag{3.4}$$

where A is a predictable unbounded linear operator on $\mathbb{H}$ we can add some more hypothesis to the ones we already stated in order to get a similar existence result for the solution of such BSDE. In particulare we assume that $A(t,\omega)$ is a linear operator on $\mathbb{H}$, $\mathbb{F}$ measurable, belongs to $L(V,V^*)$ uniformly in (t,ω) and satisfies

Hypothesis 3.3. *Assume that:*

(H0)

- $2\langle A(t,\omega)y,y\rangle_V + \alpha|y|_V^2 \leq c|y|^2$ *a.e.t* $\in [0,T]$, *a.s.*$\forall$ $y \in V$ *for some* $\alpha, c > 0$
- $A(t,\omega)$ *is uniformly continuous*

Theorem 3.4. *Under Hypothesis 3.1-*(*H*0)*, there exists a unique solution* (Y,ϕ,N) *of* (3.4) *in* $\mathfrak{B}^2_{\mathbb{G}}(\mathbb{H}) \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$

Proof. The proof of this Theorem requires some additional results, which can be found in the appendix.

As for the uniqueness part one can prove it by following the lines of Lemma 7.1.

Concerning the proof of the existence of a solution we exploit the work done in Lemma 7.2 and we define recursively

$$\begin{aligned} Y_n(t) = \zeta + \int_t^T (A(s)Y_n(s) + F(s, Y_{n-1}(s), \phi_n(s,\cdot)\Lambda^{1/2}(s,\cdot)))ds \\ - \int_t^T \phi_n(s,\xi)\mu(ds,d\xi) - \int_t^T N_n(ds), \quad t \in [0,T] \end{aligned} \tag{3.5}$$

for $n \geq 1$ and where $Y_0 = 0$. The solutions of the previous equation (Y_n,ϕ_n,N_n) are in $L^2([0,T];V) \times \mathcal{I}^{\mathbb{G}} \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$ for each $n \geq 1$. If we now apply Ito's formula, together with Hypothesis 3.1, we find that

$$\mathbb{E}\left[|Y_{n+1}(t)-Y_n(t)|^2_{\mathbb{H}}\right]+\frac{1}{2}\mathbb{E}\left[\int_t^T|\int_{\mathbb{H}}(\phi_{n+1}(s,\xi)-\phi_n(s,\xi))\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2ds\right]$$
$$+\mathbb{E}\left[\int_t^T\langle N_{n+1}-N_n\rangle(s)\right]+\alpha\mathbb{E}\left[\int_t^T|Y_{n+1}(s)-Y_n(s)|^2_V ds\right]$$
$$\leq(c+2k+1)\mathbb{E}\left[\int_t^T\left(|Y_{n+1}(s)-Y_n(s)|^2_{\mathbb{H}}+|Y_n(s)-Y_{n-1}(s)|^2_{\mathbb{H}}\right)ds\right]$$

if we now denote by

$$P_n(t):=\mathbb{E}\left[\int_t^T|Y_n(s)-Y_{n-1}(s)|^2_{\mathbb{H}}ds\right]$$

we can rewrite the previous inequality as

$$-\frac{d}{dt}P_{n+1}(t)-(c+2k+1)P_{n+1}(t)\leq(c+2k+1)P_n(t)$$

which, in particular, means that

$$-\frac{d}{dt}(P_{n+1}(t)e^{(c+2k+1)t})\leq(c+2k+1)e^{(c+2k+1)t}P_n(t).$$

If we now integrate over $t\in[0,T]$ and iterate the inequality we have that

$$P_{n+1}(t)\leq\left[(c+2k+1)e^{(c+2k+1)T}\right]^n\frac{(T-t)^n}{n!}P_1(0)$$

and thus, being $\{P_n\}$ convergent, we have that $\{Y_n\}$, $\{\phi_n\}$ and $\{N_n\}$ are Cauchy sequences, and thus convergent. If we define Y, ϕ and N the limits of those three sequences, we can notice that N and μ are orthogonal. Lastly thanks to our Hypothesis 3.1, taking the limit for $n\to\infty$ in (3.5) we have that the following equality holds a.s.

$$
\begin{aligned}
Y(t) = \zeta + \int_t^T (A(s)Y(s) + F(s, Y(s), \phi(s,\cdot)\Lambda^{1/2}(s,\cdot)))ds \\
- \int_t^T\int_{\mathbb{H}} \phi(s,\xi)\mu(ds,d\xi) - \int_t^T N(ds), \quad t \in [0,T]
\end{aligned}
$$

and thus (Y, ϕ, N) is a solution of (3.4). □

4 Sufficient maximum principles

Our goal in this section is to find a sufficient maximum principle for the optimization problem

$$
J(\hat{u}) = \sup_{u\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[\int_0^T F(t, X(t)u(t))dt + G(X(T))\right] \tag{4.1}
$$

Throughout this section we are going to assume the following conditions in addition to ***(H0)***

Hypothesis 4.1. *We assume that:*

(H1 Suff) $\beta : [0,T]\times\mathbb{H}\times[0,\infty)^2 U \longrightarrow \mathbb{H}$ *is such that it exists a constant* $K > 0$ *such that for all* $x_1, x_2 \in \mathbb{H}$

$$
|\beta(t, x_1, \lambda, u) - \beta(t, x_2, \lambda, u)|^2 \leq K_2|x_1 - x_2|^2
$$

for all $u \in U$*,* $t \in [0,T]$*, and* $\sigma : [0,T] \times \mathbb{H} \times \mathbb{R} \longrightarrow L_{\mathcal{Q}}(\mathbb{H})$ *is Lipschitz in* x *uniformly for* $t \in [0,T]$*,* $\xi \in \mathbb{H}$*.*

(H2 Suff) $F : [0,T] \times \mathbb{H} \times U \longrightarrow \mathbb{R}$ *and* β *are continuously Frechet differentiable with respect to both* x *and* u *with bounded derivatives.* $G : \mathbb{H} \longrightarrow \mathbb{H}$ *is convex and is continuously Frechet differentiable with bounded derivative*

(H3 Suff) $\bar{\sigma} : [0,T] \times \mathbb{H}[0,\infty)^2 \times \mathbb{H} \longrightarrow L^2(\mathbb{H})$ *defined by* $\bar{\sigma}(t, x, \lambda, \xi) = \sigma(t, x, \lambda, \xi)\Lambda^{1/2}(t, \xi)$ *is Frechet differentiable with respect to* x *with bounded* *derivative.*

We define the Hamiltonian as

$$\mathcal{H} : [0,T] \times \mathbb{H} \times \mathbb{H}[0,\infty)^2 \times U \times L^2(\mathbb{H}) \longrightarrow \mathbb{R}$$

$$\begin{aligned}\mathcal{H}(t,x,\lambda,\nu,y,\varphi) &:= -F(x,t,u) + \langle \beta(t,x,\lambda,u), y\rangle \\ &\quad + \langle \int_{\mathbb{H}} \sigma(t,x,\lambda,\xi)\Lambda^{1/2}(t,\xi)\nu(d\xi), \int_{\mathbb{H}} \varphi(\xi)\Lambda^{1/2}(t,\xi)\nu(d\xi)\rangle_2\end{aligned} \tag{4.2}$$

and the associated backward

$$\begin{aligned}dY(t) &= -[A^*(t)Y^u(t) + \nabla_x \mathcal{H}(X^u,\lambda,u(t),Y^u(t),\phi^u(t,\cdot)\Lambda^{1/2}(t,\cdot))]dt \qquad (4.3)\\ &\quad + \int_{\mathbb{H}} \phi^u(t,\xi)\mu(dt,d\xi) + dN^u(t) \\ Y^u(T) &= -\nabla G(X^u(T))\end{aligned}$$

where we denoted with ∇G the gradient of G which is defined by using the directional derivative $DG(x)(h)$ at a point $x \in H$ in the direction $h \in \mathbb{H}$ as $\langle \nabla G(x), h\rangle_{\mathbb{H}} = DG(x)(h)$. The operator A^* is the adjoint operator of A.

Remark 4.2. Notice that the associated backward equation (4.3) admits a unique solution in this case.

Since we are interested in considering the optimization problem with respect to the information flow $\mathbb{F}$ we now define

$$\mathcal{H}^{\mathbb{F}}(t,x,\nu,y,\varphi) = \mathbb{E}\left[\mathcal{H}(t,x,\nu,y,\varphi)|\mathcal{F}_t\right] \tag{4.4}$$

and we notice that, thanks to our hypothesis 4.1, we can rewrite $\mathcal{H}^{\mathbb{F}}$ as

$$\begin{aligned}\mathcal{H}^{\mathbb{F}} &= -F(x,t,u) + \langle \beta(t,x,u), \mathbb{E}\left[y|\mathcal{F}_t\right]\rangle \\ &\quad + \langle \int_{\mathbb{H}} \sigma(t,x,\xi)\Lambda^{1/2}(t,\xi)\nu(d\xi), \int_{\mathbb{H}} \mathbb{E}\left[\varphi(\xi)\Lambda^{1/2}(t,\xi)|\mathcal{F}_t\right]\nu(d\xi)\rangle_2\end{aligned}$$

Theorem 4.3. *Let $u \in \mathcal{A}^{\mathbb{F}}$ and assume that there exists a unique solution to X^u to* (2.1) *and (Y^u, ϕ^u, N^u) to* (4.3)*. Suppose that the following assumptions hold:*

- *$G(x)$ is a convex function.*
- *$\mathcal{H}^{\mathbb{F}}(t,\cdot,\cdot Y^u, \phi^u(t,\cdot)\Lambda^{1/2}(t,\cdot))$ is concave for all $t \in [0,T]$ a.s.*
- *$\mathcal{H}^{\mathbb{F}}(t, \hat{X}(t), \hat{u}(t), Y^{\hat{u}}, \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot)) = \max_{u\in U} \mathcal{H}^{\mathbb{F}}(t, \hat{X}(t), u(t), Y^{\hat{u}}, \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot))$ for a.e. $t \in [0,T]$ a.s.*

Then $(\hat{X}, \hat{u})$ is an optimal pair for the control problem (4.1) *with forward dynamics given by* (2.1)*.*

Proof. Define

$$\begin{aligned}I_1 &= \mathbb{E}\left[\int_0^T F(t,\hat{X}(t),\hat{u}(t)) - F(t,X(t),u(t))dt\right] \\ I_2 &= \mathbb{E}\left[G(\hat{X}(T)) - G(X^u(T))\right]\end{aligned}$$

We notice that

$$J(\hat{u}) - J(u) = I_1 + I_2$$

Moreover, thanks to the convexity of G we have that

$$\begin{aligned}I_2 &\leq \mathbb{E}\left[\langle \nabla G(\hat{X}(T)), \hat{X}(T) - X(T)\right] \\ &= -\mathbb{E}\left[\langle Y^{\hat{u}}(T), \hat{X}(T) - X(T)\rangle\right]\end{aligned}$$

Define now

$$I_3 = \mathbb{E}\left[\int_0^T \langle \hat{F}(t) - F(t), Y^{\hat{u}}(t)\rangle dt\right]$$
$$I_4 = \mathbb{E}\left[\int_0^T \langle \int_{\mathbb{H}} (\hat{\sigma}(t,\xi) - \sigma(t,\xi))\Lambda^{1/2}(t,\xi)\nu(d\xi), \int_{\mathbb{H}} \mathbb{E}\left[\hat{\phi}(t,\xi)\Lambda^{1/2}(t,\xi)|\mathcal{F}_t\right]\nu(d\xi)\rangle_2 dt\right]$$
$$I_5 = \mathbb{E}\left[\int_0^T \langle \nabla_x \hat{\mathcal{H}}(t), \hat{X}(t) - X(t)\rangle dt\right]$$

where we wrote $\hat{F}(t)$ for $F(t, \hat{X}, \hat{u}(t))$, $\hat{\sigma}(t,\xi)$ for $\hat{\sigma}(t, \hat{X}(t), \xi)$, etc. when no confusion arises.

We have that

$$I_2 \leq -I_3 - I_4 + I_5$$

In order to prove this we recall the definition of $\mathcal{Q}$ in Proposition 2.10 and apply the Ito formula to compute $d\langle Y^{\hat{u}}(t), \hat{X}(t) - X(t)\rangle$ as follows

$$\begin{aligned}
& d\langle Y^{\hat{u}}(t), \hat{X}(t) - X(t)\rangle \\
& \leq \langle \mathbb{E}[Y^{\hat{u}}(t)|\mathcal{F}_t], A(t)(\hat{X}(t) - X(t))\rangle dt + \langle \hat{\beta}(t) - \beta(t), \mathbb{E}[Y^{\hat{u}}(t)|\mathcal{F}_t]\rangle dt \\
& + \langle \mathbb{E}[Y^{\hat{u}}(t)|\mathcal{F}_t], \int_{\mathbb{H}} (\hat{\sigma}(t,\xi) - \sigma(t,\xi))\mu(dt, d\xi)\rangle \\
& - \langle A^* \mathbb{E}[Y^{\hat{u}}(t)|\mathcal{F}_t], \hat{X}(t) - X(t)\rangle dt - \langle \nabla_x \hat{\mathcal{H}}^{\mathbb{F}}(t), \hat{X}(t) - X(t)\rangle dt \\
& + \langle \hat{X}(t) - X(t), \int_{\mathbb{H}} \mathbb{E}[\hat{\phi}(t,\xi)|\mathcal{F}_t]\mu(dt, d\xi)\rangle + \langle \hat{X}(t) - X(t), dN(t)\rangle \\
& + \langle \int_{\mathbb{H}} (\hat{\sigma}(t,\xi) - \sigma(t,\xi))\Lambda^{1/2}(t,\xi)\nu(d\xi), \int_{\mathbb{H}} \mathbb{E}[\hat{\phi}(t,\xi)\Lambda^{1/2}(t,\xi)|\mathcal{F}_t]\nu(d\xi)\rangle_2 dt
\end{aligned}$$

Where we have used the orthogonality between μ and N and the definition of Λ.

If we now define

$$I_6 = \mathbb{E}\left[\int_0^T \hat{\mathcal{H}}^{\mathbb{F}}(t) - \mathcal{H}^{\mathbb{F}}(t) dt\right]$$

we have that

$$I_1 = -I_6 + I_3 + I_4$$

As we only need to use the definition of $\bar{\sigma}$ and taking expectations.

We thus have obtained that

$$J(\hat{u}) - J(u) \leq I_5 - I_6 \tag{4.5}$$

Then by using the concavity of $\mathcal{H}$ we find that

$$\int_0^T (\hat{\mathcal{H}}^{\mathbb{F}}(t) - \mathcal{H}^{\mathbb{F}}(t))dt \geq \int_0^T \langle \nabla_x \hat{\mathcal{H}}^{\mathbb{F}}(t), \hat{X}(t) - X(t) \rangle dt + \int_0^T \langle \nabla_u \hat{\mathcal{H}}^{\mathbb{F}}(t), \hat{u}(t) - u(t) \rangle dt$$

By using the conditional maximum principle (iii), we have that

$$\int_0^T \langle \nabla_u \hat{\mathcal{H}}^{\mathbb{F}}(t), \hat{u}(t) - u(t) \rangle dt \geq 0$$

and thus we get that $I_6 \geq I_5$ which in turn gives us that

$$J(\hat{u}) - J(u) \leq 0$$

which completes the proof. □

Also, when considering the enlarged filtration $\mathbb{G}$, and considering the optimization problem

$$J(\hat{u}) = \sup_{u \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\left[\int_0^T F(t, X(t)u(t))dt + G(X(T))\right] \tag{4.6}$$

we have that

Theorem 4.4. *Let $u \in \mathcal{A}^{\mathbb{G}}$ and assume that there exists a unique solution to X^u to* (2.1) *and (Y^u, ϕ^u, N^u) to* (4.3)*. Suppose that the following assumptions hold:*

- *$G(x)$ is a convex function.*

- *$\mathcal{H}(t, \cdot, \cdot Y^u, \phi^u(t,\cdot)\Lambda^{1/2}(t,\cdot))$ is concave for all $t \in [0,T]$ a.s.*

- *$\mathcal{H}(t, \hat{X}(t), \hat{u}(t), Y^{\hat{u}}, \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot)) = \max_{u\in U} \mathcal{H}(t, \hat{X}(t), u(t), Y^{\hat{u}}, \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot))$ for a.e. $t \in [0,T]$ a.s.*

Then $(\hat{X}, \hat{u})$ is an optimal pair for the control problem (4.6) *with dynamics for the forward process X given by* (2.1)*.*

Proof. The proof is similar to the one of Theorem 4.3 □

5 Necessary maximum principles

Throughout this section we are going to assume the following hypothesis on our coefficients in addition to ***(H0)***:

Hypothesis 5.1. ***(H1 Nec)*** *β, σ, F and G are continuously Frechet differentiable with respect to x. β is continuously Frechet differentiable with respect to u and the derivatives β_x, β_u, σ_x, F_x are uniformly bounded. Lastly $|G_x|_{\mathbb{H}} \leq C(1 + |x|_{\mathbb{H}})$ for some constant $C > 0$.*

(H2 Nec) *F_x satisfies Lipschitz condition with respect to u uniformly in t and x.*

Theorem 5.2. *Suppose (H0), (H1 Nec), and (H2 Nec) hold true. Assume that $(\hat{X}, \hat{u})$ is an optimal pair for the control problem* (2.1)-(4.1)*. Then there exists a unique triple $(\hat{Y}, \hat{\phi}, \hat{N})$ solving the corresponding BSDE* (4.3) *such that the following variational inequality holds:*

$$\langle \nabla_u \mathcal{H}^{\mathbb{F}}(t, \hat{X}(t), \hat{u}(t), \hat{Y}(t), \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot)), \hat{u}(t) - u\rangle_{\mathcal{O}} \geq 0$$

for all $u \in U$ a.e. $t \in [0,T]$ a.s.

Proof. We already know that the BSDE (4.3) admits a unique solution, hence we only have to prove the previous inequality. From (8.4) and (8.5) we get that

$$\mathbb{E}\left[\int_0^T \left[\langle \mathbb{E}\left[\hat{Y}(s)|\mathcal{F}_t\right], \delta_\varepsilon \beta(s) - \varepsilon \beta_u(s, \hat{X}(s), \hat{u}(s))u(s)\rangle - \delta_\varepsilon \mathcal{H}^{\mathbb{F}}(s)\right] ds\right] \geq o(\varepsilon)$$

Moreover, similarly to what we did in Lemma 8.3, by using the continuity and boundedness of β_u in *(H1 Nec)* and the dominated convergence Theorem, we have that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T \langle \mathbb{E}\left[\hat{Y}(s)|\mathcal{F}_t\right], \delta_\varepsilon \beta(s) - \varepsilon \beta_u(s, \hat{X}(s), \hat{u}(s))u(s)\rangle ds\right] \\
&\quad = \mathbb{E}\left[\int_0^T \left\langle \mathbb{E}\left[\hat{Y}(s)|\mathcal{F}_t\right], \int_0^1 [\beta_u(s, \hat{X}(s), \hat{u}(s) + \theta(u_\varepsilon(s) - \hat{u}(s))) - \beta_u(s, \hat{X}(s), \hat{u}(s))]u(s) d\theta \right\rangle ds\right] \\
&\quad \longrightarrow 0
\end{aligned}$$

as $\varepsilon \to 0^+$. In particular we have that

$$\mathbb{E}\left[\int_0^T \langle \mathbb{E}\left[\hat{Y}(s)|\mathcal{F}_t\right], \delta_\varepsilon \beta(s) - \varepsilon \beta_u(s, \hat{X}(s), \hat{u}(s))u(s)\rangle ds\right] = o(\varepsilon)$$

And thus

$$-\mathbb{E}\left[\int_0^T \delta_\varepsilon \mathcal{H}^{\mathbb{F}}(s) ds\right] \geq o(\varepsilon) \tag{5.1}$$

Therefore if we divide (5.1) by ε and let $\varepsilon \to 0^+$ we have

$$\mathbb{E}\left[\int_0^T \langle \nabla_u \mathcal{H}^{\mathbb{F}}(t, \hat{X}(t), \hat{u}(t), \hat{Y}(t), \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot)), u(t)\rangle_{\mathcal{O}} dt\right] \leq 0 \tag{5.2}$$

And as a result, by arguing like in [Ben92] and [AH11] we get the thesis. □

Analogously to what we did for the sufficient maximum principle we can also state the following Theorem which gives us a necessary maximum principle with respect to the information flow $\mathbb{G}$.

Theorem 5.3. *Suppose (H0), (H1 Nec), (H2 Nec) hold true. Assume that $(\hat{X}, \hat{u})$ is an optimal pair for the control problem* (2.1)-(4.6). *Then there* *exists a unique triple* $(\hat{Y}, \hat{\phi}, \hat{N})$ *solving the corresponding BSDE* (4.3) *such that the following variational inequality holds:*

$$\langle \nabla_u \mathcal{H}(t, \hat{X}(t), \hat{u}(t), \hat{Y}(t), \hat{\phi}(t,\cdot)\Lambda^{1/2}(t,\cdot)), \hat{u}(t) - u\rangle_{\mathcal{O}} \geq 0$$

for all $u \in U$ *a.e.* $t \in [0,T]$ *a.s.*

Proof. Analogous to the case with information flow $\mathbb{F}$. □

6 Example: a lifted Volterra process

We want to consider a process subject to consumption of the following form

$$X(t) = x_0 + \int_0^t K(t-s)(\beta(s, X_s) - c(s))ds + \int_0^t\int_{\mathbb{H}} K(t-s)\sigma(s, X_s, \xi)\mu(dsd\xi)$$

where $x_0 \in \mathbb{R}$ $\beta : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ the kernel $K(t)$ is a right shift operator. Our goal is to maximize a performance functional of the form

$$J(c) = \mathbb{E}\left[\frac{1}{2}X_T^2\right] \tag{6.1}$$

For more details on the Markovian lifts of Volterra processes we refer to [CDN19, CT20, CT19], and to our work done in [DNG20a]. We also refer to the Appendix for a quick summary on the Lift.

We thus know that we can rewrite the previous problem in an infinite dimensional setting in order to recover the Markov properties. In this case we have that, we have the following shift representation for the Kernel

$$K(t) = \int_0^\infty \mathcal{S}_t^* K(x)\delta_0(dx) = \int_0^\infty K(x+t)\delta_0(dx)$$

where $(\mathcal{S}_t^*)$ is the shift semigroup and the kernel K is the (weak) derivative of an absolutely continuous function such that $\int_0^\infty K^2(x)v(x) < \infty$ for some positive weight function v. In particular we are lifting our problem to the

Hilbert space

$$Y = \left\{ y \in AC(\mathbb{R}_+, \mathbb{R}) \text{ s.t. } \int_0^\infty |y'(x)|^2 v(x) dx < \infty \right\}$$

for some strictly positive weight function $v > 0$. We endow Y with the scalar product $\langle y, \zeta \rangle_v = y(0)v(0) + \int y'(x)\zeta(x)v(x)dx$. Moreover we consider

▷ $Z = \{y \in Y |\ y(0) = 0 \text{ and } y' \in Y\}$

▷ $Z^* = \{\nu |\ \nu = \zeta' \text{ for some } \zeta \in Y^*\}$

and we notice that $(\mathcal{S}_t^*)_t$ acts on Y^* and Z^* as $\mathcal{S}_t^* \lambda = \lambda(t + \cdot)$. In this case we have that

$$K(t) = \langle \mathcal{S}_t^* \nu, g \rangle_v = \langle \mathcal{S}_t^* K(x), 1 \rangle_v = \langle K(t + x), 1 \rangle_v$$

In this case our forward process can be rewritten on Y as

$$d\mathcal{Z}_t(x) = \mathcal{A}^* \mathcal{Z}_t(x) dt + K(x) \left[(\beta^1(t, \mathcal{Z}_t) - c(t)) dt + \int_{\mathbb{H}} \sigma^1(t, \mathcal{Z}_t, \xi) \mu(dt d\xi) \right] \tag{6.2}$$

where $\mathcal{A}^* = \frac{d}{dx}$ and $X(t) = Z_t(0)$. In particular, we can rewrite the optimization problem (6.2)-(6.1) as

$$d\mathcal{Z}_t(x) = \mathcal{A}^* \mathcal{Z}_t(x) dt + \left[K(x) \beta^1(t, \mathcal{Z}_t) - K(x) c(t) \right] dt + \int_{\mathbb{H}} K(x) \sigma^1(t, \mathcal{Z}_t, \xi) \mu(dt d\xi) \tag{6.3}$$

and our goal is going to be to maximize

$$\sup_{u \in \mathcal{A}^{\mathbb{F}}} \mathbb{E} \left[\frac{1}{2} (\mathcal{Z}_t^1)^2 \right] \tag{6.4}$$

where we have set $\beta^1(\cdot, \mathcal{Z}_t) = \beta(\cdot, \langle 1, \mathcal{Z}_t \rangle_v)$ and similarly for σ^1 and where $\mathcal{Z}_t^1 = \langle \mathcal{Z}_t, 1 \rangle_v$.

In order to solve this optimization problem we are going to first consider it under the enlarged information flow $\mathbb{G}$.

$$\sup_{u\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\left[\frac{1}{2}(\mathcal{Z}_t^1)^2\right] \tag{6.5}$$

in this case our Hamiltonian (4.2) becomes

$$\begin{aligned}\mathcal{H} &= \langle \big[\beta^1(t,\mathcal{Z}_t) - c(t)\big] K(x), Y_t(x)\rangle_v \\ &\quad + \langle \int_{\mathbb{H}} K(x)\sigma^1(t,\mathcal{Z}_t,\xi)\Lambda^{1/2}(t,\xi)\nu(d\xi), \int_{\mathbb{H}} \phi_t(\xi)(x)\Lambda^{1/2}(t,\xi)\nu(d\xi)\rangle_2\end{aligned}$$

with the backward equation (4.3) given by

$$\begin{aligned}dY_t(x) &= -A^*Y_t(x) + \nabla_z\mathcal{H}(\mathcal{Z}_t(x), c(t), Y_t(x), \phi_t(\cdot)(x)\Lambda^{1/2}(t,\cdot))dt \\ &\quad + \int_{\mathbb{H}} \phi_t(\xi)(x)\mu(dt,d\xi) + dN(t) \qquad (6.6)\\ Y_T(x) &= -\mathcal{Z}_T^1 \qquad (6.7)\end{aligned}$$

and by exploiting both the necessary and the sufficient condition for the maximum given by Theorem 4.4-5.3 we notice that the maximum is attained if and only if

$$\frac{\partial\mathcal{H}}{\partial c} = 0 \tag{6.8}$$

which means that

$$0 = K(x)\langle 1, Y_t(dx)\rangle = K(x)Y_t(0).$$

which gives us an implicit characterization of the optimal control c in terms of the backward equation.

If we now move to the information flow $\mathbb{F}$ we notice that the condition (6.8) simply transforms in

$$\frac{\partial\mathcal{H}^{\mathbb{F}}}{\partial c} = 0 \tag{6.9}$$

which translates into having that

$$0 = K(x)\langle 1, \mathbb{E}[Y_t(x)|\mathcal{F}_t]\rangle = K(x)\mathbb{E}[Y_t(0)|\mathcal{F}_t]$$

Notice that, in this case, we would have obtained the same condition if we had followed, for example, the work in [?] and considered the finite-dimensional optimization problem.

7 Appendix A: additional results for the existence of the Backward

Lemma 7.1. *Suppose that* $F \in L^2([0,T];\mathbb{H})$ *and Hypothesis 3.1. Then*

$$\begin{aligned} Y(t) = \zeta + \int_t^T (A(s)Y(s) + F(s))ds \\ - \int_t^T \int_{\mathbb{H}} \phi(s,\xi)\mu(ds,d\xi) - \int_t^T dN(s) \quad t \in [0,T] \end{aligned} \tag{7.1}$$

admits a unique solution $(Y,\phi,N) \in L^2([0,T];V) \times \mathcal{I}^{\mathbb{G}} \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$.

Proof. (Uniqueness)
let (Y_1,ϕ_1,N_1) and (Y_2,ϕ_2,N_2) two solutions of (7.1). The Ito formula, together with Hypothesis 3.1 proves that

$$\begin{aligned} &\mathbb{E}\left[|Y_1(t) - Y_2(t)|^2_{\mathbb{H}}\right] + \mathbb{E}\left[\int_t^T \left|\int_{\mathbb{H}} (\phi_1(s,\xi) - \phi_2(s,\xi))\Lambda^{1/2}(s,\xi)\nu(d\xi)\right|^2_{L^2(\mathbb{H})} ds\right] \\ &\quad + \mathbb{E}\left[\int_t^T \langle N_1 - N_2\rangle(ds)\right] + \alpha\mathbb{E}\left[\int_t^T |Y_1(t) - Y_2(t)|^2_V\, ds\right] \\ &\quad \le c\mathbb{E}\left[\int_t^T |Y_1(t) - Y_2(t)|^2_{\mathbb{H}}\, ds\right] \quad t \in [0,T]. \end{aligned}$$

Now thanks to Gronwall's inequality we get that $Y_1(t) = Y_2(t)$ a.s. for $t \in [0,T]$. This, together with the previous inequality gives the uniqueness of ϕ and N. □

Proof. (Existence)
Let $\{e_i\}_{i\ge 1}$ be an orthonormal basis of $\mathbb{H}$ such that $e_i \in V$ for all $i \ge 1$. Let

$\mathbb{H}_n := span(e_1, ..., e_n)$, $n \geq 1$, and consider the following system of equations in $\mathbb{H}^n \cong \mathbb{R}^n$:

$$
\begin{aligned}
Y_n^i(t) = \langle e_i, \zeta\rangle_{\mathbb{H}} + \int_t^T \langle e_i, A(s)\left(\sum_{j=1}^n Y_n^j(s)\cdot e_j\right)\rangle_{V\times V'} ds \\
+ \int_t^T \langle e_i, F(s)\rangle_{\mathbb{H}} ds - \int_t^T\int_{\mathbb{H}} \phi_n^i(s,\xi)\mu_n(ds,d\xi) - \int_t^T N_n^i(ds) \qquad (7.2)
\end{aligned}
$$

where $\mu_n(s,\xi) = \sum_{j=1}^n \mu_j(s,\xi)\cdot e_j$ and $\mu_j(s,\xi) = \langle \mu(s,\xi), e_j\rangle \in \mathcal{M}^2_{[0,T]}(\mathbb{H}_n)$. We thus have that ϕ_n is in $\mathcal{M}^2_{[0,T]}(\mathbb{H}_n)$. If we now define

$$
Y_n(t) := \sum_{i=1}^n Y_n^i(t)e_i \quad \phi_n(t,\xi) := \sum_{i=1}^n \phi_n^i(t,\xi)e_i \quad N_n(t) := \sum_{i=1}^n N_n^i(t)e_i
$$

for $t \in [0,T]$, then we can rewrite (7.2) as the following finite dimensional BSDE

$$
\begin{aligned}
Y_n(t) = \pi_n\zeta + \int_t^t (\Pi_n A(s)Y_n(s))ds + \int_t^T \pi_n F(s)ds \\
+ \int_t^T\int_{\mathbb{H}} \phi(s,\xi)\mu_n(ds,d\xi) - \int_t^T N(ds) \qquad (7.3)
\end{aligned}
$$

where $\Pi_n : V' \longrightarrow \mathbb{H}_n$ and $\pi_n : \mathbb{H} \longrightarrow \mathbb{H}_n$ are the orthogonal projection operators.

If we now fix n, equation (7.2) is actually a BSDE in $\mathbb{H}_n$ of the type of (3.1), and thus it has a unique solution $(Y_n, \phi_n, N_n) \in \mathfrak{B}^2_{\mathbb{G}}(\mathbb{H}_n) \times \mathcal{M}^2_{[0,T]}(\mathbb{H}_n)$.

The Ito formula, toghether with Grownall's inequality, leads now to

$$
\mathbb{E}\left[|Y_n(t)|^2_{\mathbb{H}}\right] \leq e^{(c+1)T}\left(\mathbb{E}\left[|\zeta|^2_{\mathbb{H}}\right] + \mathbb{E}\left[\int_0^T |F(s)|^2_{\mathbb{H}}\right]\right)
$$

consequently, one has that

$$\sup_{n\geq 1}\mathbb{E}\left[\int_0^T |Y_n(s)|^2_{\mathbb{H}}ds\right] < \infty, \qquad \sup_{n\geq 1}\mathbb{E}\left[\int_0^T |Y_n(s)|^2_V ds\right] < \infty,$$
$$\sup_{n\geq 1}\mathbb{E}\left[\int_0^T |\int_{\mathbb{H}} \phi(s,\xi)\pi_n\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2_{L^2(H)}ds\right] < \infty, \qquad \sup_{n\geq 1}\mathbb{E}\left[|N_n(T)|^2_{\mathbb{H}}\right] < \infty.$$

And thus it follows that, for some subsequence $\{n_k,\ k \geq 1\}$, $(Y_{n_k}, \phi_{n_k}\pi_n, N_{n_k})$ converges weakly in $L^2([0,T];V) \times \mathcal{I} \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$ for $k \longrightarrow \infty$ to (Y, ϕ, N). We are now left with proving that (Y, ϕ, N) is actually a solution to (7.1). To this end we take a function f so that $f(t) = \int_0^t \sigma(s)ds,\ t \in [0,T]$ and $\sigma \in L^2([0,T];\mathbb{R})$. For a fixed i we define $f_i := fe_i$ and by applying Ito's formula to (7.3) we have that

$$\int_0^T \langle Y_n(s), e_i\rangle_H \sigma(s)ds = \langle \zeta, f_i(T)\rangle + \int_0^T \langle A(s)Y_n(s), f_i(s)\rangle_{V\times V'}ds \tag{7.4}$$
$$+ \int_0^T \langle F(s), f_i(s)\rangle_H ds - \int_0^T \langle f_i(s), \int_{\mathbb{H}} \phi_n(s,\xi)\pi_n\mu(ds,d\xi)\rangle_H - \int_0^T \langle f_i(s), N_n(ds)\rangle_H$$

If we now take the continuous linear mappings

$$\Psi_1(\mu) := \int_0^T \langle f_i(s), \mu(ds,d\xi)\rangle_H,$$
$$\Psi_2(R) := \int_0^T \langle f_i(s), \int_{\mathbb{H}} R(s,\xi)\mu(ds,d\xi)\rangle_H,$$
$$\Psi_3(Y) := \int_0^T \langle A(s)Y(s), f_i(s)\rangle_{V\times V'}ds,$$

we have that, being Ψ_i linear and continuous, they are continuous also with respect to the weak topologies. We can thus replace n with n_k in (7.4) and pass to the weak limit in $L^2(\mathbb{R})$ for $k \longrightarrow \infty$ to conclude that

$$\int_0^T \langle Y(s), e_i\rangle_H \sigma(s)ds = \langle \zeta, f_i(T)\rangle + \int_0^T \langle A(s)Y(s), f_i(s)\rangle_{V\times V'} ds$$
$$+ \int_0^T \langle F(s), f_i(s)\rangle_H ds - \int_0^T \langle f_i(s), \int_{\mathbb{H}} \phi(s,\xi)\mu(ds, d\xi)\rangle_H - \int_0^T \langle f_i(s), N(ds)\rangle_H$$

And, being the previous equation true for every $i \geq 1$, it holds for $f_i(s) = vf(s)$ and $v \in V$. If we now choose for $t \in (0, T)$ the map

$$f_m(s) := \begin{cases} 1 & \text{if } s \geq t + \frac{1}{2m} \\ \frac{1}{2} - m(t-s) & \text{if } t - \frac{1}{2m} < s < t + \frac{1}{2m} \\ 0 & \text{if } s \leq t - \frac{1}{2m} \end{cases} \tag{7.5}$$

for any $m \geq 1$, then by applying the continuity of the mappings Ψ_i, $i = 1, 2, 3$ and letting $m \to \infty$ we finally get that, for almost all $t \in [0, T]$,

$$\langle Y(t), v\rangle_{\mathbb{H}} = \langle \zeta, v\rangle_{\mathbb{H}} + \int_t^T \langle A(s)Y(s), v\rangle_{V\times V'} ds + \int_t^T \langle F(s), v\rangle_{\mathbb{H}} ds$$
$$- \int_t^T \langle v, \int_{\mathbb{H}} \phi(s,\xi)\mu(ds, d\xi)\rangle_H - \int_t^T \langle v, N(ds)\rangle_H$$

And, being v separable, we have that, for $a.e.$ $t \in [0, T]$

$$Y(t) = \zeta + \int_t^T (A(s)Y(s) + F(s))ds - \int_t^T \int_{\mathbb{H}} \phi(s,\xi)\mu(ds, d\xi) - \int_t^T N(ds)$$

Lastly we prove that N is orthogonal to μ in the sense that, for any τ stopping time taking values in $[0, T]$

$$\mathbb{E}[\int_{\mathbb{H}} \mu(\tau, d\xi) \otimes N(\tau)] = 0$$

Here we sketch an idea of the proof, for more details we refer to Lemma 4.5 of [AH09].

Recall now that μ_n and N_n are VSO, so one can find a subsequence n_k such that

$$\mathbb{E}\left[\int_{\mathbb{H}} \mu_n(\tau, d\xi) \otimes N_{n_k}(\tau)\right] = 0$$

and now, since N_{n_k} converges weakly to N in $\mathcal{M}^2_{[0,T]}$ for $k \to \infty$ then $\langle g, N_{n_k}\rangle$ converges weakly to $\langle g, N\rangle$ in $\mathcal{M}^2_{[0,T]}$ for any $g \in \mathbb{H}$. Using now the optimal stopping theorem one gets that

$$\langle g, N_{n_k}(\tau \wedge \cdot)\rangle \to \langle g, N(\tau \wedge \cdot)\rangle$$

in $\mathcal{M}^2_{[0,T]}$. As a result we find that

$$\lim_{k\to\infty} \mathbb{E}\left[\int_{\mathbb{H}} \mu_n(T, d\xi) \otimes N_{n_k}(\tau)\right] = \mathbb{E}\left[\int_{\mathbb{H}} \mu_n(T, d\xi) \otimes N(\tau)\right] \tag{7.6}$$

Arguing as above one can let $n \to \infty$, use the convergence of μ_n and find that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{\mathbb{H}} \mu_n(T, d\xi) \otimes N(\tau)\right] = \mathbb{E}\left[\int_{\mathbb{H}} \mu(T, d\xi) \otimes N(\tau)\right] \tag{7.7}$$

and from (7.6) and (7.7), we get the desired result.

□

We can now consider a more general function F and we have that

Lemma 7.2. *Consider the following BSPDE*

$$Y(t) = \zeta + \int_t^T (A(s)Y(s) + F(s, \phi(s,\cdot)\Lambda^{1/2}(s,\cdot))ds \tag{7.8}$$

$$- \int_t^T\int_{\mathbb{H}} \phi(s,\xi)\mu(ds,d\xi) - \int_t^T N(ds), \quad t \in [0,T] \tag{7.9}$$

and assume that Hypothesis 3.1 hold. Then there exists a unique solution (Y, ϕ, N) *of* (7.8) *in* $L^2([0,T];V) \times \mathcal{I}^{\mathbb{G}} \times \mathcal{M}^2_{[0,T]}(\mathbb{H})$.

Proof. We shall only prove the existence as the uniqueness can be proven as in Lemma (7.1). We use an approximation technique and exploit Lemma

(7.1). We define $\phi_0 = 0$ and consider the BSPDE:

$$Y_n(t) = \zeta + \int_t^T (A(s)Y_n(s) + F(s, \phi_{n-1}(s,\cdot)\Lambda^{1/2}(s,\cdot)))ds$$
$$- \int_t^T\int_{\mathbb{H}} \phi_n(s,\xi)\mu(ds,d\xi) - \int_t^T N_n(ds), \quad t \in [0,T],$$

for $n \geq 1$. From the previous Lemma we have that this equation admits a unique solution (Y_n, ϕ_n, N_n) for any $n \geq 1$. By using Ito formula and our Hypothesis, we have that

$$\mathbb{E}\left[|Y_{n+1}(t) - Y_n(t)|^2_{\mathbb{H}}\right] + \mathbb{E}\left[\int_t^T |\int_{\mathbb{H}} (\phi_{n+1}(s,\xi) - \phi_n(s,\xi))\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2_{L^2(\mathbb{H})} ds\right]$$
$$+ \mathbb{E}\left[\int_t^T \langle N_{n+1} - N_n\rangle_H(ds)\right]$$
$$\leq (c+2k)\mathbb{E}\left[\int_t^T |Y_{n+1}(s) - Y_n(s)|^2_{\mathbb{H}} ds\right]$$
$$+ \frac{1}{2}\mathbb{E}\left[\int_t^T |\int_{\mathbb{H}} (\phi_{n+1}(s,\xi) - \phi_n(s,\xi))\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2_{L^2(\mathbb{H})} ds\right]$$
$$- \alpha\mathbb{E}\left[\int_t^T |Y_{n+1}(s) - Y_n(s)|^2_V ds\right]$$

if we now multiply both sides by $e^{(c+2k)t}$, integrate with respect to $t \in [0,T]$ and iterate in n we can find that

$$\int_0^T e^{(c+2k)t}\left(\mathbb{E}\left[\int_t^T |\int_{\mathbb{H}} (\phi_{n+1}(s,\xi) - \phi(s,\xi))\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2_{L^2(\mathbb{H})} ds\right]\right) dt \leq \left(\frac{1}{2}\right)^n K$$

where $K := \frac{1}{c+2k} e^{(c+2k)T}\mathbb{E}\left[\int_0^T |\int_{\mathbb{H}} \phi_1(s,\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)|^2_{L^2(\mathbb{H})} ds\right]$ and thus we can conclude that $\{Y_n\}$, $\{\phi_n\}$ $\{N_n\}$ are all Chauchy sequences and they converge to some limits Y, ϕ, N, which are the solution to (7.8). For more detail on this proof we refer to [AH09] Lemma 4.7. $\square$

8 Appendix B: estimates for the necessary maximum principle

For more details about these proofs we refer to [AH11]. Here these results are proved with respect to a general continuous, square integrable martingale M instead of our noise μ.

Let $\hat{u}$ be an optimal control and let $\hat{X}$ be the corresponding solution of (2.1). Let u be such that $\hat{u} + u \in \mathcal{A}$. For a given $\varepsilon \in [0, 1]$ consider the variational control

$$u_\varepsilon(t) = \hat{u}(t) + \varepsilon u(t) \tag{8.1}$$

We notice that by convexity of U u_ε is still in $\mathcal{A}$. Considering this control u_ε we can consider the X^{u_ε} as the solution of the forward equation (2.1) corresponding to u_ε and which we shall denote with X_ε. Let now p be the solution of the following linear equation

$$\begin{cases} dp(t) &= (A(t)p(t) + \beta_x(t, \hat{X}(t)\hat{u}(t)))dt + \beta_u(t, \hat{X}(t), \hat{u}(t))dt \\ &\quad + \int_{\mathbb{H}} \sigma_x(t, \hat{X}(t), \xi)p(t)\mu(dt, d\xi) \\ p(0) &= 0 \end{cases} \tag{8.2}$$

The following lemmas contain some estimates that will play an important role in proving the necessary maximum principle.

Lemma 8.1. *Suppose our Hypothesis (H0) and (H1 Nec) hold. Then*

$$\sup_{t \in [0,T]} \mathbb{E}\left[|p(t)|^2\right] < \infty \tag{8.3}$$

Proof. By applying Ito formula we have that

$$\begin{aligned}\mathbb{E}\left[|p(t)|^2\right] &= 2\mathbb{E}\left[\int_0^t \langle p(s), A(s)p(s)\rangle ds\right] \\ &+ 2E\left[\int_0^t \langle p(s), \beta_x(t,\hat{X},\hat{u}(s))p(s)\rangle ds\right] \\ &+ 2\mathbb{E}\left[\int_0^t \langle p(s), \beta_u(t,\hat{X},\hat{u}(s))u(s)ds\right] \\ &+ \mathbb{E}\left[\int_0^t \| \int_{\mathbb{H}} \sigma_x(s,\hat{X}(s),\xi)\Lambda^{1/2}(s,\xi)\nu(d\xi)\|_2^2 ds\right]\end{aligned}$$

and thus by applying Hypothesis *(H1 Nec)*, the Cauchy-Schwartz inequality and Proposition 2.10 we get that

$$\begin{aligned}&\mathbb{E}\left[|p(t)|^2\right] + \alpha\mathbb{E}\left[\int_0^t |p(t)|^2 ds\right] \\ &\qquad \le (c+C_1)\mathbb{E}\left[\int_0^t |p(s)|^2 ds\right] + C_2\mathbb{E}\left[\int_0^t |u(s)|^2 ds\right]\end{aligned}$$

where C_1 and C_2 are some positive constants. The Gronwall's inequality gives us the thesis. □

Lemma 8.2. *Assuming our Hypothesis (H0) and (H1 Nec), we have that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_\varepsilon(t) - \hat{X}(t)|^2\right] = O(\varepsilon^2)$$

Proof. Once again, by applying ito formula, taking expectation and using our hypothesis (H1 Nec) we have that

$$\begin{aligned}&\mathbb{E}\left[|X_\varepsilon(t) - \hat{X}(t)|^2\right] + \alpha\mathbb{E}\left[\int_0^t |X_\varepsilon(s) - \hat{X}(s)|^2\right] \\ &\quad \le (c+1)\mathbb{E}\left[\int_0^t |X_\varepsilon(s) - \hat{X}(s)|^2 ds\right] + \mathbb{E}\left[\int_0^t |\beta(s, X_\varepsilon(s), u_\varepsilon(s)) - \beta(s,\hat{X}(s), u_\varepsilon(s))|^2 ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t \|(\sigma(s, X_\varepsilon(s),\xi) - \sigma(s,\hat{X}(s),\xi))\tilde{\mathcal{Q}}_\mu^{1/2}(s,\xi)\|_2^2 \Lambda(s,d\xi)ds\right]\end{aligned}$$

but from *(H1 Nec)* we get that

$$\begin{aligned}
&\mathbb{E}\left[|\beta(s,X_\varepsilon(s),u_\varepsilon(s))-\beta(s,\hat{X}(s),\hat{u}(s))|^2 ds\right] \\
&\leq 2\mathbb{E}\left[\int_0^t |\beta(s,X_\varepsilon(s),u_\varepsilon(s))-\beta(s,\hat{X}(s),u_\varepsilon(s))|^2 ds\right] \\
&+ 2\mathbb{E}\left[\int_0^t |\beta(s,\hat{X}(s),u_\varepsilon(s))-\beta(s,\hat{X}(s),\hat{u}(s))|^2 ds\right] \\
&= 2E\left[\int_0^t |\tilde{\beta}_x(s,\varepsilon)(X_\varepsilon(s)-\hat{X}(s))|^2 ds\right] + 2\mathbb{E}\left[\int_0^t |\delta_\varepsilon\beta(s)|^2 ds\right] \\
&\leq 2C_4\mathbb{E}\left[\int_0^t |X_\varepsilon(s)-\hat{X}(s)|^2 ds\right] + 2C_3\varepsilon^2
\end{aligned}$$

where we have defined, for $y \in \mathbb{H}$,

$$\tilde{\beta}_x(s,\varepsilon)(y) = \int_0^1 \beta_x(s,\hat{X}(s)+\theta(X_\varepsilon(s)-\hat{X}(s)),u_\varepsilon(s))(y)d\theta$$

and

$$\delta_\varepsilon\beta(s) = \beta(s,\hat{X}(s),u_\varepsilon(s)) - \beta(s,\hat{X}(s),\hat{u}(s))$$

and used our hypothesis *(H1 Nec)* to prove that

$$\mathbb{E}\left[\int_0^T |\delta_\varepsilon\beta(s)|^2 ds\right] \leq \varepsilon^2 C_3$$

For more details on these inequalites we refer to [AH11] section 4. Similarly one can prove that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \| \int_{\mathbb{H}} (\beta(s,X_\varepsilon(s),\xi)-\beta(s,\hat{X}(s),\xi))\tilde{\mathcal{Q}}_\mu^{1/2}(s,\xi)\|_2^2 \langle\mu\rangle(ds)\right] \\
&\leq C_5\mathbb{E}\left[\int_0^t |X_\varepsilon(s)-\hat{X}(s)|^2 ds\right]
\end{aligned}$$

Once again our thesis is attained by using Gronwall's inequality. □

Lemma 8.3. *Define* $\eta_\varepsilon(t) = \frac{X_\varepsilon(t)-\hat{X}(t)}{\varepsilon} - p(t)$ *and assume (H0) and (H1 Nec).*

Then we have that

$$\lim_{\varepsilon\to 0^+}\sup_{t\in[0,T]}\mathbb{E}\left[|\eta_\varepsilon(t)|^2\right]=0$$

Proof. Using the definition of $p(t)$ in (8.2), and the definition of $X(t)$ in (2.1), we can deduce, as in the proof of Lemma 8.2 that

$$\mathbb{E}\left[|\eta_\varepsilon(t)|^2\right]+\alpha\mathbb{E}\left[\int_t|\eta_\varepsilon(s)|^2ds\right]$$
$$\leq (c+C_6)\int_0^t\mathbb{E}\left[|\eta_\varepsilon(s)|^2\right]ds+\rho(\varepsilon)$$

where

$$\begin{aligned}\rho(\varepsilon)=&\,2\mathbb{E}\left[\int_0^T|\tilde{\beta}_x(s,\varepsilon)-\beta_x(s,\hat{X}(s),\hat{u}(s))|^2ds\right]\\&+2\mathbb{E}\left[\iint_{\mathbb{X}}\|(\tilde{\sigma}_x(s,\varepsilon,\xi)-\sigma_x(s,\hat{X}(s),\xi))p(s)\tilde{\mathcal{Q}}_\mu^{1/2}(s,\xi)\|_2^2\Lambda(s,d\xi)ds\right]\\&+\mathbb{E}\left[\int_0^T\left|\frac{1}{\varepsilon}\delta_\varepsilon\beta(s)-\beta_u(s,\hat{X}(s),\hat{u}(s))u(s)\right|^2ds\right]\end{aligned}$$

where we defined for $y\in\mathbb{H}$

$$\tilde{\sigma}_x(s,\varepsilon,\xi)(y)=\int_0^1[\sigma_x(s,\hat{X}(s)+\theta(X_\varepsilon(s)-\hat{X}(s),\xi))(y)\tilde{\mathcal{Q}}_\mu^{1/2}(s,\xi)]d\theta$$

Now thanks to (H1 Nec) and the dominated convergence theorem we have that

$$\begin{aligned}&\mathbb{E}\left[|(\tilde{\beta}_x(s,\varepsilon)-\beta_x(s,\hat{X}(s),\hat{u}(s)))p(s)|^2ds\right]\\&=\mathbb{E}\left[\int_0^T\left|\int_0^1[\beta_x(s,\hat{X}(s)+\theta(X_\varepsilon(s)-\hat{X}(s)),u_\varepsilon(s))-\beta_x(s,\hat{X}(s),\hat{u}(s))]p(s)d\theta\right|^2ds\right]\\&\leq\int_0^T\int_0^1\mathbb{E}\left[\left|\beta_x(s,\hat{X}(s)+\theta(X_\varepsilon(s)-\hat{X}(s)),\hat{u}(s))-\beta_x(s,\hat{X}(s),\hat{u}(s))p(s)\right|^2\right]d\theta\ ds\\&\longrightarrow 0\text{ as }\varepsilon\to 0^+\end{aligned}$$

Similarly we have that

$$\mathbb{E}\left[\int_0^T \|[\tilde{\sigma}_x(s,\varepsilon) - \sigma_x(s,\hat{X}(s),\xi)]p(s)\tilde{\mathcal{Q}}_\mu^{1/2}(s,\xi)\|_2^2\Lambda(s,d\xi)ds\right] \longrightarrow 0$$

On the other hand, similarly to what we did for (8.4)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|\frac{1}{\varepsilon}\delta_\varepsilon\beta(s) - \beta_u(s,\hat{X}(s),\hat{u}(s))u(s)\right|^2 ds\right] \\
&\leq \int_0^T\int_0^1 \mathbb{E}\left[\left|[\beta_u(s,\hat{X}(s),\hat{u}(s)+\theta(u_\varepsilon(s)-\hat{u}(s))) - \beta_u(s,\hat{X}(s),\hat{u}(s))]u(s)\right|^2\right] d\theta\ ds \\
&\longrightarrow 0
\end{aligned}$$

by using (H1 Nec) and the dominated convergence theorem. Applying all these results we have thus that $\rho(\varepsilon)$ goes to zero as ε goes to zero, and thus, once again by Gronwall's inequality, we have the thesis. □

Theorem 8.4. *Suppose that (H0), (H1 Nec), (H2 Nec) hold true. Then for each $\varepsilon > 0$ we have that*

$$\begin{aligned}
J(u_\varepsilon) - J(\hat{u}) = &\ \varepsilon\mathbb{E}\left[G_x(\hat{X}(T))p(T)\right] \\
&+ \varepsilon\mathbb{E}\left[\int_0^T F_x(s,\hat{X}(s),\hat{u}(s))p(s)ds\right] \\
&+ \mathbb{E}\left[\int_0^T \left[F(s,\hat{X}(s),u_\varepsilon(s)) - F(s,\hat{X}(s),\hat{u}(s))\right] ds\right] + o(\varepsilon)
\end{aligned}$$

Proof. We write

$$J(u_\varepsilon) - J(\hat{u}) = I_1(\varepsilon) + I_2(\varepsilon)$$

where we define

$$\begin{aligned}
I_1(\varepsilon) &= \mathbb{E}\left[G(X_\varepsilon(T)) - G(\hat{X}(T))\right] \\
I_2(\varepsilon) &= \mathbb{E}\left[\int_0^T [F(s,X_\varepsilon(s),u_\varepsilon(s)) - F(s,\hat{X}(s),\hat{u}(s))]ds\right]
\end{aligned}$$

Now thanks to Lemma 8.2 and Lemma 8.3 and by dominated convergence we get that

$$\begin{aligned}
\frac{1}{\varepsilon}I_1(\varepsilon) &= \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^1 G_x(\hat{X}(t)+\theta(X_\varepsilon(T)-\hat{X}(T))(X_\varepsilon(T)-\hat{X}(T)))d\theta\right] \\
&= \mathbb{E}\left[\int_0^1 G_x(\hat{X}(T)+\theta(X_\varepsilon(T)-\hat{X}(T)))(p(T)+\eta_\varepsilon(T))d\theta\right] \\
&\longrightarrow \mathbb{E}\left[G_x(\hat{X}(t))p(T)\right]
\end{aligned}$$

as $\varepsilon \to 0^+$. In particular this means that

$$I_1(\varepsilon) = \varepsilon\mathbb{E}\left[G_x(\hat{X}(T))p(T)\right] + o(\varepsilon)$$

Similarly we get that one can write $I_2(\varepsilon)$ as

$$\begin{aligned}
I_2(\varepsilon) &= \varepsilon\mathbb{E}\left[\int_0^T F_x(s,\hat{X}(s),\hat{u}(s))p(s)ds\right] \\
&\quad + \mathbb{E}\left[\int_0^T [F(s,\hat{X}(s),u_\varepsilon(s)) - F(s,\hat{X}(s),\hat{u}(s))]ds\right] + o(\varepsilon)
\end{aligned}$$

once again we refer to [AH09] for more details concerning these calculations. The thesis now follows follows from our decomposition of $J(u_\varepsilon) - J(\hat{u})$. □

Lemma 8.5. *Suppose that (H0), (H1 Nec), (H2 Nec) hold, then*

$$\begin{aligned}
-\varepsilon\mathbb{E}\left[\langle\mathbb{E}[\hat{Y}(T)|\mathcal{F}_t],p(T)\rangle\right] + \varepsilon\mathbb{E}\left[\int_0^T F_x(s,\hat{X}(s),\hat{u}(s))p(s)ds\right] \\
+ \mathbb{E}\left[\int_0^T(-\delta_\varepsilon\mathcal{H}^{\mathbb{F}}(s) + \langle\hat{Y}(s),\delta_\varepsilon\beta(s)\rangle)ds\right] \geq o(\varepsilon)
\end{aligned} \tag{8.4}$$

where

$$\begin{aligned}
\delta_\varepsilon\mathcal{H}^{\mathbb{F}}(s) &= \mathcal{H}^{\mathbb{F}}(\hat{X}(s),u_\varepsilon(s),\hat{Y}(s),\hat{\phi}(s,\cdot)\mathcal{Q}^{1/2}(s)) \\
&\quad - \mathcal{H}^{\mathbb{F}}(\hat{X}(s),\hat{u}(s),\hat{Y}(s),\hat{\phi}(s,\cdot)\mathcal{Q}^{1/2}(s))
\end{aligned}$$

Proof. Since $\hat{u}$ is an optimal control, we have that $J(u_\varepsilon) - J(\hat{u}) \geq 0$. The rest follows from Theorem 8.4 and the definition of $\mathcal{H}$. □

Lemma 8.6.

$$\mathbb{E}\left[\langle \mathbb{E}\left[\hat{Y}(T)|\mathcal{F}_t\right], p(T)\rangle\right] = \mathbb{E}\left[\int_0^T F_x(s, \hat{X}(s), \hat{u}(s))p(s)ds\right] \tag{8.5}$$
$$+ \mathbb{E}\left[\int_0^T \langle \hat{Y}(s), \beta_u(s, \hat{X}(s), \hat{u}(s))u(s)\rangle ds\right]$$

Proof. We use Ito formula and the fact that

$$\langle \nabla_x \mathcal{H}^{\mathbb{F}}(\hat{X}(s), \hat{u}(s), \hat{Y}(s), \hat{\phi}(s,\cdot)\Lambda^{1/2}(s,\cdot)), p(t)\rangle$$
$$= -F_x(t, \hat{X}(t), \hat{u}(t))p(t) + \langle \beta_x(t, \hat{X}(t), \hat{u}(t))p(t), \mathbb{E}\left[\hat{Y}(t)|\mathcal{F}_t\right]\rangle$$
$$+ \langle \sigma(t, \hat{X}(t), \cdot)\Lambda^{1/2}(t,\cdot), \mathbb{E}\left[\hat{\phi}(t,\cdot)|\mathcal{F}_t\right]\Lambda^{1/2}(t,\cdot)\rangle_2$$

and that

$$\iint_{\mathbb{X}} \langle \sigma(s, \hat{X}(s), \xi)\mu(ds, d\xi), \mathbb{E}\left[\hat{\phi}(s,\xi)|\mathcal{F}_t\right]\mu(ds, d\xi)$$
$$= \iint_{\mathbb{X}} \langle \sigma(s, \hat{X}(s), \xi)\Lambda^{1/2}(s, d\xi), \mathbb{E}\left[\hat{\phi}(s,\xi)|\mathcal{F}_t\right]\Lambda^{1/2}(s, d\xi)\rangle_2 ds$$

□

9 Appendix C: notes on the Markovian lift

In this paper we focus on Markovian lifts as presented in [CT20]. We suppose to have a Volterra process of the form

$$V_t = f(t) + \int_0^t K(t-s)dX(s)$$

where $f(t)$ is a deterministic function, $K \in L^2_loc(\mathbb{R}_+;\mathbb{R})$ and X is a semimartingale depending on V. We suppose that the kernel K can be represented as $K(t) = \langle g, \mathcal{S}_t^* \nu \rangle$ with $(\mathcal{S}_t^*)_{t \geq 0}$ a strongly continuous semigroup acting on a banach space Y^*, $\nu \in Y^*$ (or in a slightly bigger space), $g \in Y$ and pairing $\langle \cdot, \cdot \rangle$.

Remark 9.1. In general we can consider a subspace $Z \subset Y$ with their relative duals $Y^* \subset Z^*$. In this case we should have that

- Z and Y are Banach spaces $Z \subset Y$ and Z embeds continuously into Y.
- The semigroup $\mathcal{S}^*$ with generator $\mathcal{A}^*$ acts in a strongly continuous way on Y^* and Z^* with respect to the respective norm topologies.
- The map $\mathcal{Z} \mapsto \mathcal{S}_t^* \mathcal{Z}$ is weak-* continuous on Y^* and on Z^* for every $t \geq 0$
- The pre adjoing operator of $\mathcal{A}^*$, generates a strongly continuous semigroup on Z with respect to the respective norm topology (but not necessarily on Y).

In this case we can have that $K(t) = \langle g, \mathcal{S}_t^* \nu \rangle$ with $\nu \in Z^*$ if $S_t^* \nu$ is in Y^*.

We are also assuming that $\mathcal{S}_t^* \nu \in Y^*$ for all $t > 0$ and that $\int_0^t \|\mathcal{S}_s^* \nu\|_{Y^*}^2 ds < \infty$ for all $t > 0$.

When the above hypothesis are satisfied, we have that, if we define

$$d\mathcal{Z}_t = \mathcal{A}^* \mathcal{Z}_t dt + \nu dX_t$$

then we are able to rewrite

$$V_t = \langle g, \mathcal{Z}_t \rangle$$

and $\mathcal{Z}_t$ is a generalized Feller process.

This method allows us to move from a finite to an infinite dimension setting and, in doing so, retrieve some Markov properties for the process V_t.

For more details on the lift of affine Volterra processes we refer to [CT20] in the 1-dimensional case and [CT19] for the d-dimensional one. For the case with a Lévy driver and non affine coefficients we refer to [CDN19].

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References

[AH09] A. Al-Hussein. Backward stochastic partial differential equations driven by infinite-dimensional martingales and applications. *Stochastics An International Journal of Probability and Stochastic Processes*, 81:601–626, 2009.

[AH10] A. Al-Hussein. Maximum principle for controlled stochastic evolution equations. *Int. Journal of Math Analysis*, 4:1447–1464, 2010.

[AH11] A. Al-Hussein. Necessary conditions for optimal control of stochastic evolution equations in Hilbert spaces. *Appl. Math. Optim*, 63:385–400, 2011.

[AR10] D. Applebaum and M. Riedle. Cylindrical Lévy processes in Banach spaces. *Proceedings of the London Mathematical Society*, 101:697–726, 2010.

[Ben92] A. Bensoussan. Stochastic control of partially observable systems. *Cambridge University Press*, 1992.

[CDN19] C. Cuchiero and G. Di Nunno. Notes - markovian lifts. 2019.

[CT19] C. Cuchiero and J. Teichman. Markovian lifts of positive semidefinite affine Volterra-type processes. *Decisions in Economics and Finance*, 42:407–448, 2019.

[CT20] C. Cuchiero and J. Teichman. Generalized Feller processes and Markovian lifts of stochastic Volterra processes: the affine case. *Journal of Evolution Equations*, pages 1–48, 2020.

[DNG20a] G. Di Nunno and M. Giordano. Lifting of Volterra processes: optimal control and HJB equations. 2020.

[DNG20b] G. Di Nunno and M. Giordano. Maximum principles for Volterra time changed processes. 2020.

[DNS14] G. Di Nunno and S. Sjursen. BSDEs Driven by Time-Changed Lévy Noises and Optimal Control. *Stochastic Processes and their Applications*, 124:1679–1709, 2014.

[FT02] M. Fuhrman and G. Tessitore. Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *The Annals of Probability*, 30:1397–1465, 2002.

[Lio69] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

[Mas08] F. Masiero. Stochastic optimal control problems and parabolic equations in Banach spaces. *SIAM Journal on Control and Optimization*, 47:251–300, 2008.

[Mé82] M. Métivier. *Semimartingales, a course on Stochastic Processes.* 1982.

[Ouv75] J.-Y. Ouvrard. Représentation de martingales vectorielles de carré intégrable à valeurs dans des espaces de Hilbert réels séparables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 33:195–208, 1975.

[Par79] E. Pardoux. Stochastic partial differential equations and filtering of diffusion processes. *Stochastics*, 3:127–176, 1979.

[Par07] E. Pardoux. Stochastic partial differential equations. *Lectures given in Fudan University, Shanghai*, 2007.

[PZ07] S. Peszat and K. Zabczyk. *Stochastic partial differential equations with Lévy noise*. 2007.