C*-algebras Associated to Algebraic Dynamics

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Outline

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2. Preliminaries II: Generalized Bunce-Deddens Algebras
3. Cuntz-Nica-Pimsner Algebras for Algebraic Dynamics
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Lattice ordered semigroups

Definition

A commutative, discrete semigroup \( P \) with unit \( e \) is called **lattice ordered**, if the following conditions are satisfied:

1. There exists a discrete group \( H \) together with an embedding of \( P \) into \( H \) such that \( P \cap P^{-1} = \{ e \} \) holds in \( H \).
2. Every two elements \( h_1 \) and \( h_2 \in H \) do have a unique least common upper bound with respect to the partial order defined by \( h_1 \leq h_2 \) iff \( h_1^{-1}h_2 \in P \).

\( h_1 \vee h_2 \) will denote this unique least upper bound.

Remark

A lattice ordered semigroup \( P \) is in particular an Ore semigroup, so we can actually take \( H = P^{-1}P \).
Definition

An action $\theta$ of a semigroup $\mathcal{P}$ on a discrete, abelian group $G$ by endomorphisms is called **exact**, if \( \bigcap_{t \in \mathcal{P}} \theta_t(G) = \{0_G\} \).

This is equivalent to \( \bigcup_{t \in \mathcal{P}} \ker \hat{\theta}_t \) being dense in $\hat{G}$.

Remark

Easy examples of exact semigroup actions are given by actions where one of the endomorphisms $\theta_t$ is exact in the sense that \( \bigcap_{n \in \mathbb{N}} \theta_{tn}(G) = \{0_G\} \), see for instance [CV12].

In fact, whenever $\mathcal{P}$ is commutative and finitely generated, this is all there is.
A notion of independence for commuting endomorphisms

**Definition**

Suppose $G$ is a discrete, abelian group, $\theta_1$ and $\theta_2$ are injective, commuting endomorphisms with finite cokernel. Then $\theta_1$ and $\theta_2$ are said to be *independent*, if the following equivalent conditions are satisfied:
A notion of independence for commuting endomorphisms

Definition

(i) $G/\theta_1(G) \xrightarrow{\theta_2} \theta_2(G)/\theta_2\theta_1(G)$ is an isomorphism.

(ii) $\theta_1(G) + \theta_2(G) = G$

(iii) $\theta_1(G) \cap \theta_2(G) = \theta_1\theta_2(G)$

(iv) $\hat{\theta}_2(\ker \hat{\theta}_1) = \ker \hat{\theta}_1$

(v) $\ker \hat{\theta}_1 \cap \ker \hat{\theta}_2$ is trivial.

(vi) $G/\theta_2(G) \oplus G/\theta_1(G) \longrightarrow G/\theta_1\theta_2(G)$ given by $([h], [g]) \mapsto \theta_1(h) - \theta_2(g)$ is an isomorphism.
Basic remarks on independence

Remark

i) The definition of independence and the equivalence of the conditions (i) to (v) are taken from [CV12].

ii) If \( \theta_1 \) and \( \theta_2 \) are independent, so are \( \theta_1^m \) and \( \theta_2^n \) for all \( m, n \in \mathbb{N} \).

iii) If \( \theta_1 \neq id \neq \theta_2 \) are independent, they are multiplicatively independent in the sense that \( \theta_1^m = \theta_2^n \) for \( m, n \in \mathbb{N}_0 \) implies \( n = m = 0 \). Otherwise, we would obtain a contradiction to ii).

Proposition

Independence is preserved under products of pairwise independent endomorphisms.
Definition
A triple \((G, \mathcal{P}, \theta)\) is called an **algebraic dynamics** if

1. \(G\) is a discrete, abelian group (with dual group \(K\)),
2. \(\mathcal{P}\) is a discrete, countable, l. o. semigroup with a unit \(e\) and
3. \(\theta\) is an exact action of \(\mathcal{P}\) on \(G\) by injective endomorphisms with finite cokernel such that relatively prime elements in \(\mathcal{P}\) act by independent endomorphisms.

Remark
For every algebraic dynamics, the mapping \(t \mapsto N_t := |\ker \hat{\theta}_t|\) gives a semigroup homomorphism \(\mathcal{P} \xrightarrow{N_*} (\mathbb{N}, \cdot)\).
Examples of algebraic dynamics

Example

(a) $G = \mathbb{Z}$, $\mathcal{P}$ generated by pairwise relatively prime elements $p_1, \ldots, p_k \in \mathbb{Z} \setminus \{-1, 0, 1\}$ acting by multiplication on $G$. Note that $\theta_{p_i}$ and $\theta_{p_j}$ are independent iff $p_i$ and $p_j$ are relatively prime in $\mathbb{Z}$.

(b) For $G = \mathbb{Z}^n$, $n \geq 1$, $\theta_1$ and $\theta_2$ given by commuting $n \times n$ matrices with integer entries and $|\det(\theta_i)| \geq 2$, the endomorphisms are independent if the determinants are relatively prime in $\mathbb{Z}$. On the other hand,

$$\theta_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{are independent.}$$
c) Restricting to $n = 2$ and $\theta_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, $|a_1| \lor |a_2| \geq 2$, we either have $a_1 \neq a_2$, forcing $\theta_2$ to be diagonal with entries relatively prime to $a_1$, $a_2$, respectively, or $\theta_2$ is of the form

$$\begin{pmatrix} k & b \ell \\ b \ell & k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b \ell & k \\ k & b \ell \end{pmatrix},$$

where $k, \ell \in \mathbb{Z}$, $|b| \geq 2$, $a \in b\mathbb{Z}$, $\overline{k} \in (\mathbb{Z}/a\mathbb{Z})^\times$ and $

\ell = 0 \implies |k| \geq 2.$
A group $\tilde{G}$ is called \textbf{residually finite}, if it has a separating family of finite index, normal subgroups and \textbf{sequentially residually finite}, if the above is true for a nested sequence $(L_n)_\mathbb{N}$.

\begin{proposition}[[Orf10, Proposition 5]]
If a group $\tilde{G}$ is sequentially residually finite with sequence $(L_n)_\mathbb{N}$, the induced projective system $(\tilde{G}/L_n)_\mathbb{N}$ defines a profinite completion $\tilde{G}_L$ of $\tilde{G}$ w. r. t. $(L_n)_\mathbb{N}$. $\tilde{G}_L$ is a Cantor space into which $\tilde{G}$ embeds densely.
\end{proposition}
Definition

If \( \tilde{G} \) is amenable and sequentially residually finite, 
\( C \left( \tilde{G}_L \right) \rtimes \tilde{G} \), where \( \tilde{G} \) acts by translation, is called a \textit{generalized Bunce-Deddens algebra}.

It is not hard to see that this action is free and minimal. With some more work, Stefanos Orfanos arrived at:

Proposition ([Orf10, Section 2 and 3])

\textit{Generalized Bunce-Deddens algebras are unital, separable, simple, nuclear, quasidiagonal, have rr 0, sr 1, comparability of projections and a unique trace.}
Generalized BD-algebras from algebraic dynamics

Proposition

For an algebraic dynamics $(G, \mathcal{P}, \theta)$, $C(G_\theta) \rtimes G$ is a generalized BD-algebra, where $G_\theta = \lim_{\leftarrow} G/\theta_t(G)$.
Example

(a) \( G = \mathbb{Z}, \ P \) generated by pairwise relatively prime elements \( p_1, \ldots, p_k \in \mathbb{Z} \setminus \{ -1, 0, 1 \} \) yields the (ordinary) BD-algebra of type \( \left( \prod_{i=1}^{k} p_i \right)^\infty \).

(b) If \( G = \mathbb{Z}^2, \ n \geq 1 \) and \( P \cong \mathbb{N}^k, \ k \geq 1 \) is given by diagonal matrices, where the corresponding diagonal entries are pairwise relatively prime, then the associated generalized BD-algebra will simply be the \( n \)-fold tensor product of the BD-algebras obtained by restricting to each component.
Example

c) For $G = \mathbb{Z}^n$, $n \geq 1$, $\theta_{e_1}, \ldots, \theta_{e_k}$, $k \geq 1$ given by pairwise independent, commuting $n \times n$ matrices with integer entries, the situation is more involved. Later: K-theoretic picture based on $\theta_{(1,\ldots,1)}$.

Disclaimer

From now on, $(G, \mathcal{P}, \theta)$ shall always be an algebraic dynamics.
Construction of CNP-algebras

Definition

\( \mathcal{O} [G, \mathcal{P}, \theta] \) is the universal \( C^* \)-algebra generated by

- a unitary representation \((u_g)_{g \in G}\) of \( G \) and
- an isometric representation \((s_t)_{t \in \mathcal{P}}\) of the semigroup \( \mathcal{P} \)

subject to the relations:

1. \( s_{t_1}^* s_{t_2} = s_{t_2} s_{t_1}^* \) for all relatively prime \( t_1, t_2 \in \mathcal{P} \).
2. \( s_t u_g = u_{\theta_t(g)} s_t \) for all \( t \in \mathcal{P}, g \in G \).
3. \( 1 = \sum_{[g] \in G/\theta_t(G)} e_{t,g} \) for all \( t \in \mathcal{P} \),

where \( e_{t,g} = u_g s_t s_t^* u_g^* \).
Elementary facts

Remark

i) For $t \in \mathcal{P}$ and $g_1, g_2 \in G$ such that $g_1 \equiv g_2 \mod \theta_t(G)$, $e_{t,g_2} = e_{t,g_1}$ by (2). Thus, the summation in (3) makes sense.

ii) Due to (2) and (3):

$$s_t^* u_g s_t = 1_{\theta_t(G)}(g) u_{\theta_t^{-1}}(g)$$

for all $t \in \mathcal{P}$, $g \in G$. 

iii) The linear span of
\[ \{ u_{g_1} s_{t_1} s_{t_2}^* u_{g_2}^* \mid g_1, g_2 \in G, \ t_1, t_2 \in \mathcal{P} \} \]

is dense in \( \mathcal{O}[G, \mathcal{P}, \theta] \).

iv) For all \( t_1, t_2 \in \mathcal{P} \) and \( g_1, g_2 \in G \):
\[
e_{t_1, g_1} = \sum_{[g_2] \in G/\theta_{t_2} (G)} e_{t_1 t_2, g_1 + \theta_{t_1} (g_2)}
\]

In particular, since \( \mathcal{P} \) is lattice ordered, all projections \( e_{t,g} \) commute.
Two canonical subalgebras

Definition

Denote by $\mathcal{F}$ and $\mathcal{D}$ the following $C^*$-subalgebras of $\mathcal{O}[G, P, \theta]$: 

$$
\mathcal{D} = C^* (\{ e_{t,g} \mid g \in G, \ t \in P \}) 
$$

$$
\mathcal{F} = C^* (\{ u_g, e_{t,g} \mid g \in G, \ t \in P \}) 
$$

$\mathcal{F}$ and $\mathcal{D}$ are referred to as the core and the diagonal of $\mathcal{O}[G, P, \theta]$. 
Remark

i) There is a natural gauge action $\gamma$ of $L = \hat{H}$, on $O [G, \mathcal{P}, \theta]$ given by $\gamma_\ell(u_g) = u_g, \quad \gamma_\ell(s_t) = \ell(t)s_t$
for $g \in G, \quad t \in \mathcal{P}$ and $\ell \in L$, where $H = \mathcal{P}^{-1}\mathcal{P}$ is the discrete, abelian group obtained as the enveloping group of $\mathcal{P}$. This gives $F = O [G, \mathcal{P}, \theta]^\gamma$.

ii) $F = \lim\longrightarrow F_t$, where $F_t = C^* (\{ u_g, \quad e_{t,g} \mid g \in G \})$ and the connecting maps are inclusions. Using iv) from the last remark and $C^* (\theta_t(G)) \cong C^*(G)$, we get that

$$F_t \cong C^*(G) \otimes M_{N_t}(\mathbb{C}).$$
Remark

iii) $\mathcal{F} \cong \mathcal{D} \rtimes G$, where the action is given by

$$g_1 \cdot e_{t,g_2} = u_{g_1}^* e_{t,g_2} u_{g_1} = e_{t,g_2 - g_1}.$$

iv) Similarly, $\mathcal{D} = \lim_{\rightarrow} \mathcal{D}_t$, where

$$\mathcal{D}_t = C^* \left( \{ e_{t,g} \mid g \in G \} \right),$$

with inclusions as connecting maps. As $e_{t,g_1} = e_{t,g_2}$ iff $g_1 \equiv g_2 \mod \theta_t(G)$ and $N_t < \infty$, $\mathcal{D}_t$ is finite dimensional and isomorphic to $C \left( G / \theta_t(G) \right)$ via $e_{t,g} \mapsto \mathbb{1}_{g + \theta_t(G)}$. 


Proposition

The spectrum of $\mathcal{D}$ can be identified with $G_\theta$ and $\mathcal{F}$ is isomorphic to $C(G_\theta) \rtimes G$, where $G_\theta = \lim_{\leftarrow} G/\theta_t(G)$ and the action of $G$ is induced by translation on $G_\theta$. In particular, $\mathcal{F}$ is a generalized Bunce-Deddens algebra.
The structure of CNP-algebras

**Proposition**

*There is a canonical, faithful conditional expectation*

\[ \mathcal{O} \left[ G, \mathcal{P}, \theta \right] \xrightarrow{E} \mathcal{D}. \]

In analogy to [CV12, Theorem 2.6]:

**Theorem**

\[ \mathcal{O} \left[ G, \mathcal{P}, \theta \right] \text{ is purely infinite and simple.} \]
Corollary

Let $(\xi_g)_{g \in G}$ denote the canonical ONB of $\ell^2(G)$ given by $\xi_g(h) = \delta_{g,h}$ for all $h$. For $t \in \mathcal{P}$ and $g \in G$ let $S_t, U_g \in \mathcal{L}(\ell^2(G))$ denote the operators defined by

$$S_t \xi_h = \xi_{\theta(t)h} \quad \text{and} \quad U_g \xi_h = \xi_{h+g} \quad \text{for all } h.$$ 

Then

$$\mathcal{O}[G, \mathcal{P}, \theta] \cong C^*\left(\{S_t, U_g \mid g \in G, \ t \in \mathcal{P}\}\right).$$
Back to the structure of CNP-algebras

Proposition

\( \mathcal{O} [G, \mathcal{P}, \theta] \) is isomorphic to the semigroup crossed product \( \mathcal{F} \rtimes_\gamma \mathcal{P} \), where \( \gamma_t(x) = s_t x s_t^* \) for all \( x \in \mathcal{F} \) and \( t \in \mathcal{P} \).

Therefore, \( \mathcal{O} [G, \mathcal{P}, \theta] \) is nuclear.

Corollary

\( \mathcal{O} [G, \mathcal{P}, \theta] \) is isomorphic to the crossed product \( \mathcal{D} \rtimes (G \rtimes \mathcal{P}) \), where the action is given by

\[
(g_1, t_1) \cdot e_{t_2, g_2} = e_{t_1 t_2, g_1 + \hat{\theta}_t}(g_2)
\]

for all \( g_1, g_2 \in G \) and \( t_1, t_2 \in \mathcal{P} \).
Notable aspects

Remark

i) The generalized BD-algebras $\mathcal{F}$ in question are classified by their Elliott invariant, cp. [Car11]. For finitely generated $\mathcal{P}$, $K_*(\mathcal{F})$ depends only on $K_*(G)$ and the induced map of a diagonal element in $\mathcal{P}$.

ii) The K-theory of $\mathcal{O}[G,\mathcal{P},\theta]$ is accessible via

$$\mathcal{O}[G,\mathcal{P},\theta] \cong \mathcal{F} \rtimes \gamma \mathcal{P}.$$
Open questions

(A) Is there an efficient procedure to compute the K-theory of $\mathcal{O} [G, \mathcal{P}, \theta]$ in this situation?

(B) Which UCT Kirchberg algebras occur?

(C) Which dynamical systems are prominent or interesting examples of this kind?

(D) What are the differences between this approach and the discretized one, based on graph $C^*$-algebras in combination with a partition of $K$ w. r. t. $(\mathcal{P}, \theta)$?

(E) To what extend can we recover information on the dynamics from these $C^*$-algebras?

(F) What might be a reasonable approach to relax the independence condition in order to allow for example for torsion in $H = \mathcal{P}^{-1}\mathcal{P}$?


