

Toric arrangements and equivariant matroid theory – Lecture Notes

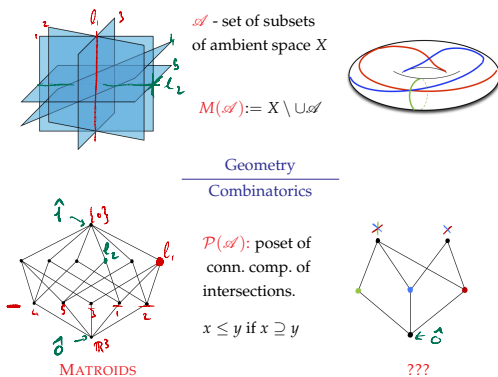
Emanuele Delucchi – ASGAR Math 2024

TORIC ARRANGEMENTS AND EQUIVARIANT MATROID THEORY

LECTURE 1: ARRANGEMENTS AND MATROIDS.

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MÖBIUS FUNCTIONS AND POSET POLYNOMIALS

Let $\mathcal{P}$ be a locally finite partially ordered set (poset).

The Möbius function of $\mathcal{P}$ is $\mu : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{Z}$, defined recursively by

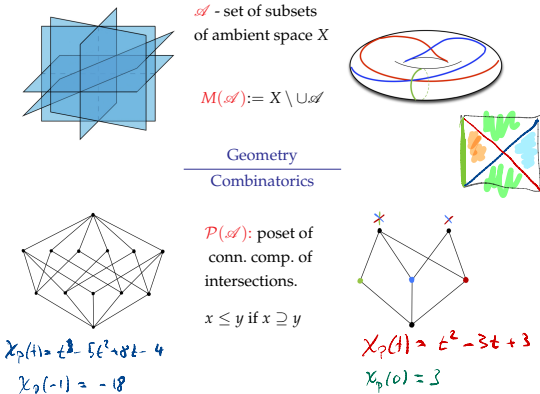
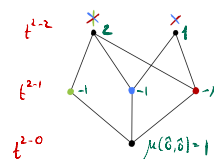
$$\begin{cases} \mu(x, y) = 0 & \text{if } x \not\leq y \\ \sum_{x \leq z \leq y} \mu(x, z) = \delta_{x, y} & \text{if } x \leq y \end{cases}$$

$\hookrightarrow \mu(x, x) = 1$

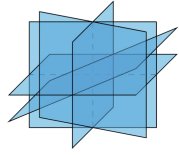
If $\mathcal{P}$ has a $\hat{0}$ and rank function ρ , its characteristic polynomial is

$$\chi_{\mathcal{P}}(t) := \sum_{x \in \mathcal{P}} \mu(\hat{0}, x) t^{\rho(\mathcal{P}) - \rho(x)}$$

$$\chi_{\mathcal{P}}(t) = t^2 - 3t + 3$$

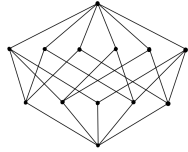


HYPERPLANES AND GEOMETRIC LATTICES



$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

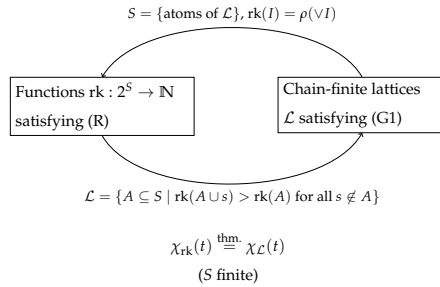
$$a_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, a_5 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$



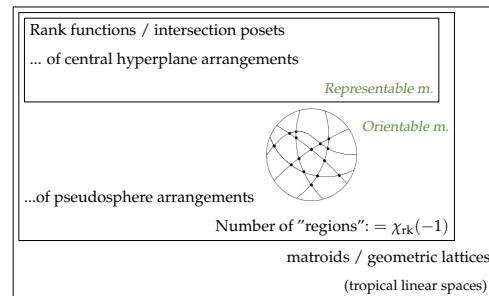
$$\text{rk}(I) := \dim \langle a_i \mid i \in I \rangle$$

$$= \text{codim} \bigcap_{i \in I} (a_i)^\perp =: \rho(I)$$

CRYPTOMORPHISMS (MATROIDS)



FINITE MATROIDS



Infinite example: set of all subspaces of V .

Let V be a vectorspace of (finite) dimension d over the field $\mathbb{K}$.

An **arrangement of hyperplanes** in V is any locally finite family $\mathcal{A} = \{H_i\}_{i \in S}$ where $H_i = \{v \in V \mid a_i(v) = b_i\}$ for some choice of tuples $(a_i)_{i \in S} \subseteq V^*$ and $(b_i)_{i \in S} \subseteq \mathbb{K}$.

The associated *rank function* is $\text{rk} : 2^S \rightarrow \mathbb{N}$, $\text{rk}(X) := \dim \langle a_i \mid i \in X \rangle$.

Call $\mathcal{A}$ **central** if $b_i = 0$ for all i (this implies $|S| < \infty$). In this case, $\mathcal{P}(\mathcal{A})$ satisfies the following.

Definition. A **geometric lattice** is a chain-finite lattice P such that

$$(G1) \ x \leq y \Leftrightarrow y \in x \vee a \text{ for some } a \geq \hat{0}.$$

Notice that a chain-finite meet-semiolattice is always bounded below. In any bounded-below poset P , the elements of $A(P) := \{a \in P \mid a \geq \hat{0}\}$ are called **atoms**.

Remark. Every geometric lattice admits a poset rank function ρ .

In turn, the rank function rk satisfies the following definition.

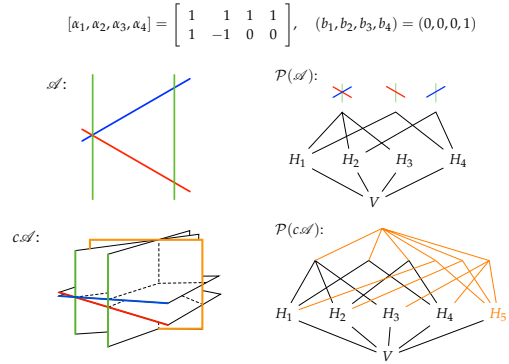
Definition. A **matroid rank function** on the ground set S is any monotone function $\text{rk} : 2^S \rightarrow \mathbb{N}$ s.t.

$$(R) : \begin{cases} 0 \leq \text{rk}(X) \leq |X| & \forall X \subseteq S \\ \text{rk}(X \cup Y) + \text{rk}(X \cap Y) \leq \text{rk}(X) + \text{rk}(Y) & \forall X, Y \subseteq S \\ \forall X \subseteq S \exists Y \subseteq X, |Y| < \infty, \text{rk}(X) = \text{rk}(Y). \end{cases}$$

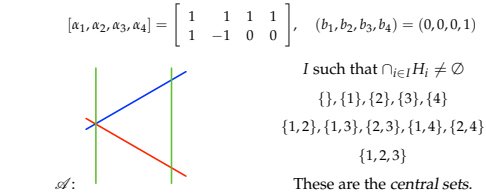
If $|S| < \infty$, define

$$\chi_{\text{rk}}(t) := \sum_{X \subseteq S} (-1)^{|X|} t^{\text{rk}(S) - \text{rk}(X)}.$$

AFFINE HYPERPLANE ARRANGEMENT ($\mathbb{K} = \mathbb{R}$)



AFFINE HYPERPLANE ARRANGEMENTS ($\mathbb{K} = \mathbb{R}$)



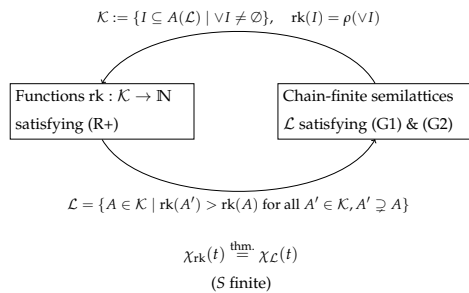
The family of central sets $\mathcal{K} \subseteq 2^S$ is an **abstract simplicial complex**.

(I.e.: if $I \in \mathcal{K}$ and $J \subseteq I$, then $J \in \mathcal{K}$)

For $I \in \mathcal{K}$ set $\text{rk}(I) := \dim(a_i \mid i \in I)$. This defines a **semimatroid**.

[Kawahara '04, Ardila '07]

CRYPTOMORPHISMS (SEMATROIDS)



We now consider an example of an affine hyperplane arrangement. Then, the poset of intersections has the following structure.

Definition. A **geometric semilattice** is any poset of the form $L \setminus L_{\geq a}$ where L is a geometric lattice and $a \in A(L)$ is an atom of L .

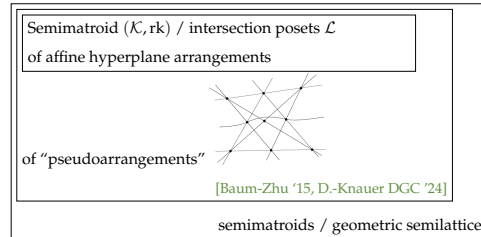
Equivalently, a geometric semilattice is any chain-finite meet-semilattice P that satisfies (G1) – thus admits a rank function ρ – and

(G2) for all $x, y \in P$ and for all $I \subseteq A(P)$ s.t. $y \in \vee I$ and $\rho(x) < \rho(y) = |I|$, there is $a \in I$ with $x \vee a \neq \emptyset$.

Definition. A **semimatroid rank function** is any monotone function $\text{rk} : \mathcal{K} \rightarrow \mathbb{N}$ from a finite-dimensional simplicial complex $\mathcal{K}$ to the natural numbers satisfying

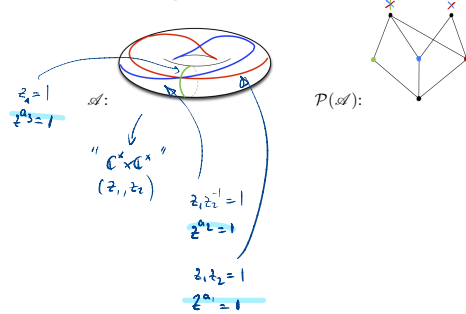
$$(R^s) : \begin{cases} 0 \leq \text{rk}(X) \leq |X| & \forall X \in \mathcal{K} \\ \text{rk}(X \cup Y) + \text{rk}(X \cap Y) \leq \text{rk}(X) + \text{rk}(Y) & \forall X, Y \subseteq S \text{ s.t. } X \cup Y \in \mathcal{K} \\ \text{for all } X, Y \in \mathcal{K}, \text{rk}(X) = \text{rk}(X \cap Y) \text{ implies } X \cup Y \in \mathcal{K} \\ \text{for all } X, Y \in \mathcal{K}, \text{rk}(X) < \text{rk}(Y) \text{ implies } X \cup a \in \mathcal{K} \text{ for some } a \in Y \setminus X \end{cases}$$

ABSTRACT THEORY



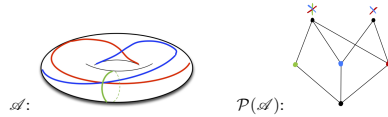
TORIC ARRANGEMENTS

$$[a_1, a_2, a_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$



TORIC ARRANGEMENTS

$$[a_1, a_2, a_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}; \text{ for } I \subseteq [n]: m(I) := \# \text{Tor}(\mathbb{Z}^d / \langle a_i \rangle_I).$$



The pair (rk, m) satisfies the axioms of an **arithmetic matroid**...

[d'Adderio-Moci '13, Brändén-Moci '14]

... but it does **not** determine $\mathcal{P}(\mathcal{A})$: **no cryptomorphism**.

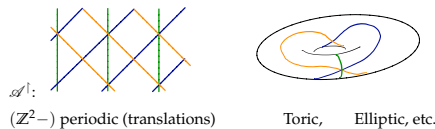
[Pagaria '17]

But $\mathcal{P}(\mathcal{A})$ determines ring $H^*(M(\mathcal{A}), \mathbb{Q})$

[Callegaro, D'Adderio, D., Migliorini, Pagaria]

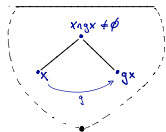
TRANSLATIVE ACTIONS

Understand abelian arrangements as "quotients" of linear, periodic ones.



$\mathcal{A}^\dagger$:
($\mathbb{Z}^2$ -) periodic (translations)

Toric, Elliptic, etc.



If x and gx have upper bound, then $x = gx$.

For all Abelian arrangements:

$$\mathcal{P}(\mathcal{A}) \simeq \mathcal{P}(\mathcal{A}^\dagger) / \mathbb{Z}^d$$

Let $a_1, \dots, a_n \in \Gamma \simeq \mathbb{Z}^d$ be an n -tuple of nonzero, full-rank elements.

Definition. The associated **toric arrangement** is the set $\mathcal{A} = \{H_1, \dots, H_n\}$ of subtori $H_i = \{z \in (\mathbb{C}^*)^d \mid z^{a_i} = 1\}$.

Fix $p, q \in \mathbb{N}$ and let $\mathbb{G} = \mathbb{R}^p \times (S^1)^q$.

Definition. The associated **abelian arrangement** in $\mathbb{G}^d$ is

$$\mathcal{A} = \{H_1, \dots, H_n\}, \quad H_i = \ker(\text{Hom}(\Gamma, \mathbb{G}) \rightarrow \mathbb{G}, \varphi \mapsto \varphi(a_i))$$

If $(p, q) = (2, 0)$ call $\mathcal{A}$ "linear";

if $(p, q) = (1, 1)$ the arrangement $\mathcal{A}$ is toric;

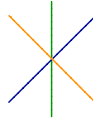
if $(p, q) = (0, 2)$ call $\mathcal{A}$ "elliptic".

EXAMPLE: COXETER ARRANGEMENTS

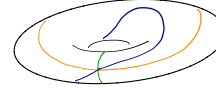
Let $\Phi : \{a_1, \dots, a_n\} \subseteq \mathbb{Z}^d$ roots of Coxeter system of type $ABCD$.

Let $\mathcal{A}_\Phi$ be the associated Abelian arrangement.

$\mathbb{G} = \mathbb{R}$



$\mathbb{G} = S^1$



Linear, toric, elliptic:

– Explicit description of $\mathcal{P}(\mathcal{A})$ via "enriched partitions" [Bibby '18]

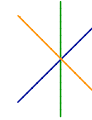
– $\mathcal{P}(\mathcal{A})$ is EL-shellable [D.-Girard-Paolini '19]

EXAMPLE: COXETER ARRANGEMENTS

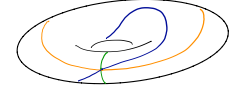
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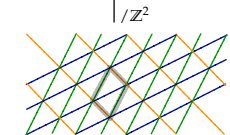
$\mathbb{G} = S^1$



Toric case: $\mathcal{A}_\Phi$ is "covered"

by the associated affine

Weyl reflection arrangement.



Given any abelian arrangement $\mathcal{A}$ let $\mathcal{A}^\dagger$ denote the lift of $\mathcal{A}$ to the universal cover of $\mathbb{G}^d$.

Note: in general, $\mathcal{A}^\dagger$ is an arrangement of subspaces (not necessarily of hyperplanes).

Yet, for every abelian $\mathcal{A}$ the poset $\mathcal{P}(\mathcal{A}^\dagger)$ is a geometric semilattice.

GROUP ACTIONS ON GEOMETRIC SEMILATTICES

From now:

$\mathcal{L}$: a chain-finite geometric semilattice;
 G : a group acting on $\mathcal{L}$ by poset automorphisms. $\left. \vphantom{\begin{matrix} \mathcal{L} \\ G \end{matrix}} \right\} \alpha : G \curvearrowright \mathcal{L}$

$$\mathcal{P}_\alpha = \mathcal{L}/G := \{Gx \mid x \in \mathcal{L}\}; \quad Gx \leq Gy \text{ iff } x \leq gy \text{ for some } g \in G.$$

This is a **poset** (eg., since $\mathcal{L}$ has a rank function)

Definition. A c.-f. poset P with $\hat{0}$ is **geometric** if it satisfies (G1), (G2).

Theorem. Let $G \curvearrowright P$ be a translatable action. If P is geometric, so is P/G .

Corollary. If the action α is translatable, then $\mathcal{P}_\alpha$ is a geometric poset.

So $\mathcal{P}(\mathcal{A})$ is geometric for every abelian arrangement $\mathcal{A}$.

GROUP ACTIONS ON GEOMETRIC SEMILATTICES

From now:

$\mathcal{L}$: a chain-finite geometric semilattice;
 G : a group acting on $\mathcal{L}$ by poset automorphisms. $\left. \vphantom{\begin{matrix} \mathcal{L} \\ G \end{matrix}} \right\} \alpha : G \curvearrowright \mathcal{L}$

$$\mathcal{P}_\alpha = \mathcal{L}/G := \{Gx \mid x \in \mathcal{L}\}; \quad Gx \leq Gy \text{ iff } x \leq gy \text{ for some } g \in G.$$

This is a **poset** with **rank function** ρ_α and set of **atoms** E_α . For $I \subseteq E_\alpha$ let

$$\text{rk}_\alpha(I) := \max \rho_\alpha(\vee I), \quad m_\alpha(I) := \# \vee I.$$

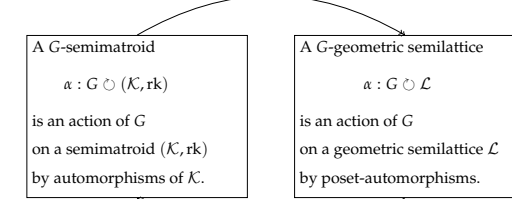
Theorem. $(E_\alpha, \text{rk}_\alpha)$ is a (semi)matroid if and only if α is translatable.

Definition.

$$T_\alpha(x, y) := \sum_{I \subseteq E_\alpha} m_\alpha(I) (x-1)^{\text{rk}_\alpha(E) - \text{rk}_\alpha(I)} (y-1)^{|I| - \text{rk}_\alpha(I)}$$

GROUP ACTIONS ON SEMIMATROIDS

Let G be a group



CRYPTOMORPHISM! ✓

If α is translatable: (1) $T_\alpha(x, y)$ has positive coefficients

$$(2) \chi_{\mathcal{P}_\alpha}(t) = (-1)^{\text{rk}(E_\alpha)} T_\alpha(1-t, 0) \quad [\text{D.-Riedel '18}]$$