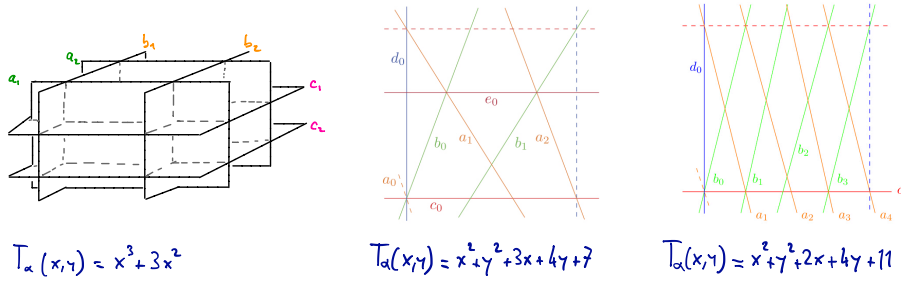


PROBLEM SET FOR ASGARD ‘24

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1. EXAMPLES

- (1) Compute the Tutte polynomials associated to the two periodic arrangements of pseudolines represented on the right-hand side of the following picture (a fundamental region is depicted).



- (2) Let $\mathcal{A}$ be the arrangement of 6 affine hyperplanes given by the planes containing the faces of the 3-dimensional cube with vertex set $\{(\pm 1, \pm 1, \pm 1)\}$ (depicted on the left-hand side in the previous figure).
- (a) Prove that the antipodal action of $\mathbb{Z}_2$ (i.e., the non-identity element acts by changing sign of all coordinates) acts transitively on $\mathcal{P}(\mathcal{A})$. If you'd like, verify the computation of the Tutte polynomial given above.
- (b) Draw the poset $\mathcal{P}(\mathcal{A})/\mathbb{Z}_2$. Prove that its reduced order complex¹ has 2-torsion, by convincing yourself that this order complex triangulates the real projective plane.
- (3) (From [3])

Consider the matrices

$$N_1 := \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix} \quad N_2 := \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}$$

and let $\mathcal{A}_1, \mathcal{A}_2$ be the toric arrangements defined by the columns of N_1, N_2 , respectively.

- (a) Prove that the column vectors of the matrices N_1 and N_2 define the same arithmetic matroid (i.e., same rank- and multiplicity functions).
- (b) Prove that the posets of layers $\mathcal{P}(\mathcal{A}_1)$ and $\mathcal{P}(\mathcal{A}_2)$ are not isomorphic. (Note: you may try to draw the posets using the sage library `arithmat`)

¹I.e., $\Delta((\mathcal{P}(\mathcal{A})/\mathbb{Z}_2) \setminus \hat{0})$

- by Roberto Pagaria and Giovanni Paolini [4]. Alternatively, see the hint in the footnote²)
- (c) As was computed as in [5], the integer (co)homology of the order complex of $\mathcal{P}(\mathcal{A}_1)$ is $(\mathbb{Z}, \mathbb{Z}_5, \mathbb{Z}^{48})$, thus it has 5-torsion. (Can you reproduce the computation? Use Sage and `arithmat`.)
 - (d) The group $\mathbb{Z}^3$ acts on $\mathcal{A}_1^\dagger$ and defines a $\mathbb{Z}^3$ -semimatroid α . Compute the number δ_α – or follow the computation done in [2, Example 9.6].
- (4) Consider a periodic line arrangement $\mathcal{A}$ and the associated group action $\alpha : \mathbb{Z}^2 \curvearrowright \mathcal{P}(\mathcal{A})$. In any case, by rank reasons, $\mathcal{P}(\mathcal{A})/\mathbb{Z}^2$ is homotopy Cohen-Macaulay. Find such an α with $\delta_\alpha > 1$.

2. TRANSLATIVE ACTIONS ON POSETS

- (1) Let Δ be a finite-dimensional simplicial complex and P its poset of cells (ordered by inclusion). Let G act on Δ by simplicial isomorphisms. The action of G on Δ induces an action of G on P . Prove that this action is translatable if and only if the action on Δ preserves a proper coloring³ of Δ .
- (2) Let P be a finite poset that admits a rank function. Consider any action of a group G on P by poset automorphisms.
Prove that this action induces an action on the order complex $\Delta(P)$, and that this latter action is always translatable.
- (3) Let P be a chain-finite, ranked poset with a translatable action of a group G . Prove that, for all $x \in P$, there is an isomorphism of posets $P_{\leq x} \simeq (P/G)_{\leq Gx}$.
(More difficult: prove that, for all $x \in P$, the poset $(P/G)_{\geq Gx}$ is isomorphic to $P_{\geq x}/\text{stab}(x)$.)
- (4) Use the previous exercise to prove the following. Let P be a chain-finite, bounded-below poset with a translatable action of a group G . If P is geometric, then so is P/G .

See: [2, Lemma 2.13]

See: [2, Lemma 2.14])

See: [1, Lemma 4.4.6])

REFERENCES

- [1] Christin Bibby and Emanuele Delucchi. Supersolvable posets and fiber-type abelian arrangements, 2024.
- [2] Alessio D’Alì and Emanuele Delucchi. Stanley-Reisner rings for symmetric simplicial complexes, G -semimatroids and abelian arrangements. *J. Comb. Algebra*, 5(3):185–236, 2021.
- [3] Roberto Pagaria. Two examples of toric arrangements. *J. Comb. Theory, Ser. A*, 167:389–402, 2019.
- [4] Roberto Pagaria and Giovanni Paolini. Arithmat: SageMath implementation of arithmetic matroids and toric arrangements. <https://github.com/giove91/arithmat>, 2019.
- [5] Roberto Pagaria and Giovanni Paolini. Representations of torsion-free arithmetic matroids. *Eur. J. Comb.*, 93:18, 2021. Id/No 103272.

²The property “we can number the hypersurfaces H_1, H_2, H_3, H_4 such that every connected component of $H_1 \cap H_2$ intersects every connected component of $H_3 \cap H_4$ ” holds for exactly one of the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$

³Recall: a “coloring” of a simplicial complex Δ of finite dimension d is an assignment of colors to the vertices of Δ such that adjacent vertices receive different colors.