

# Scattering Amplitudes and Amplituhedra, Mathematical Aspects of Quantum Field Theory

Marcus Spradlin and Anastasia Volovich

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**Exercise 1.** (This exercise has essentially 16 parts but is mostly bookkeeping; you can try a few parts to familiarize yourself with the notation.) The three-particle amplitude (in Yang-Mills theory) is

$$A_3 = (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot (p_1 - p_2)) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot (p_2 - p_3)) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot (p_3 - p_1)).$$

Using the formulas

$$\epsilon_i^+ \cdot \epsilon_j^- = -\frac{\langle jr \rangle [ir]}{\langle ir \rangle [jr]}, \quad p_i \cdot \epsilon_j^+ = \frac{\langle ir \rangle [ji]}{\sqrt{2} \langle jr \rangle}, \quad p_i \cdot \epsilon_j^- = \frac{[ir] \langle ji \rangle}{\sqrt{2} [jr]}, \quad (1)$$

where  $r$  is a reference spinor that should drop out of all final answers (confirming this is part of the exercise), evaluate all  $2^3 = 8$  three-particle helicity amplitudes in the two cases (a) when  $\langle ij \rangle = 0$  for all  $i$  and  $j$ , and (b) when  $[ij] = 0$  for all  $i$  and  $j$ . In case (a) show that the only three non-zero helicity amplitudes are

$$A(1^+, 2^+, 3^-) = \frac{[12]^3}{[23][31]} \quad \text{and cyclic}, \quad (2)$$

while in case (b) the only three non-zero helicity amplitudes are

$$A(1^+, 2^-, 3^-) = \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 31 \rangle} \quad \text{and cyclic}. \quad (3)$$

**Exercise 2.** Let  $n = 4$ . Show that

$$\frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = \frac{[24]^4}{[12][23][34][41]}. \quad (4)$$

This demonstrates the equivalence of two different formulas for the helicity amplitude  $A(1^-, 2^+, 3^-, 4^+)$ . You'll need to use the momentum conservation equations:

$$\sum_{k=1}^4 \langle ik \rangle [kj] = 0 \quad \text{for all } i, j. \quad (5)$$

**Exercise 3.** Show that the Parke-Taylor formula for the MHV helicity amplitude

$$A(1^-, 2^+, \dots, j^-, \dots, n^+) = \frac{\langle 1j \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \quad (6)$$

with particles 1 and  $j$  having negative helicity, and the others positive helicity, satisfies the BCFW recursion relation. Reference: Section 3.1 of hep-th/0412308.

**Exercise 4.** (Discussed in lecture; fill in the details.) Compute the helicity amplitude  $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$  using BCFW recursion. Reference: Section 2.2 of hep-th/0412308.

**Exercise 5.** Consider the helicity amplitude  $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ , which is given by the formula

$$\frac{[4|P_{56}|1]^3}{[23][34][2|P_{34}|5]\langle 56\rangle\langle 61\rangle s_{234}} + \frac{[6|P_{12}|3]^3}{[61][12][2|P_{34}|5]\langle 34\rangle\langle 45\rangle s_{612}} \quad (7)$$

where we use the notation

$$[i|P_{jk}|l] = [ij]\langle jl\rangle + [ik]\langle kl\rangle \quad s_{ijk} = \langle ij\rangle[ij] + \langle ik\rangle[ik] + \langle jk\rangle[jk]. \quad (8)$$

Show that, despite the fact that each term has a factor of  $[2|P_{34}|5]$  in its denominator, the amplitude does not actually have a pole at  $[2|P_{34}|5] = 0$ ; i.e. the residues of the two terms add up to zero. We call this a *spurious pole*; it is exactly analogous to a spurious edge that arises from when computing the canonical form of a positive geometry by triangulation.

**Exercise 6\*.** Show that on the support of the  $n = 6$  momentum conservation equations

$$\sum_{k=1}^6 \langle ik\rangle[kj] = 0 \text{ for all } i, j, \quad (9)$$

the formula for  $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$  given in (8) equals

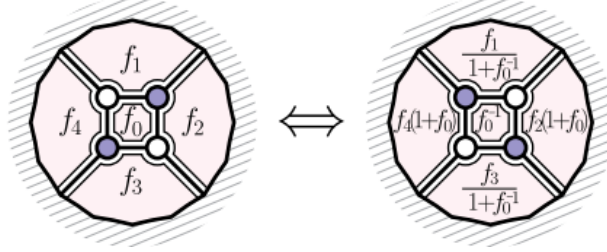
$$\frac{s_{123}^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|P_{23}|4][3|P_{45}|6]} + \frac{\langle 12\rangle^3[45]^3}{\langle 16\rangle[34][3|P_{45}|6][5|P_{16}|2]s_{612}} + \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[16][1|P_{23}|4][5|P_{61}|2]s_{234}}. \quad (10)$$

Note that this way of writing it makes it clear that the amplitude has no pole at  $[2|P_{34}|5] = 0$  (because there is no such factor anywhere in this expression), but it has three new spurious poles itself, all of which are absent from (8). *This problem is probably impossible without the help of a computer algebra system. It was the desire to find an “explanation” for the equality of the two expressions that led A. Hodges in arXiv:0905.1473 to the idea that they represent the volume of a polytope computed via two different triangulations.*

**Exercise 7.** Under a square move on a plabic graph, the five participating face variables are transformed from  $f_0, \dots, f_4$  to  $f'_0, \dots, f'_4$  according to the formula

$$f'_0 = \frac{1}{f_0}, f'_1 = \frac{f_1}{1 + 1/f_0}, f'_2 = f_2(1 + f_0), f'_3 = \frac{f_3}{1 + 1/f_0}, f'_4 = f_4(1 + f_0)$$

as shown in figure (4.68) of arXiv:1212.5605.



This could be embedded inside a larger plabic graph. Show that the associated canonical form is invariant up to sign:

$$\bigwedge_{i=0}^4 d \log f_i = - \bigwedge_{i=0}^4 d \log f'_i. \quad (11)$$

**Exercise 8.** For the following dilogarithm identities, compute the symbols of the left- and right-hand sides, which serves to partially verify them:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2} \mathrm{Li}_2(z^2), \quad (12)$$

$$\mathrm{Li}_2(1-z) + \mathrm{Li}_2(1-1/z) = -\frac{1}{2} (\log z)^2, \quad (13)$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log z \log(1-z), \quad (14)$$

$$\mathrm{Li}_2(-z) - \mathrm{Li}_2(1-z) + \frac{1}{2} \mathrm{Li}_2(1-z^2) = -\frac{\pi^2}{12} - \log z \log(1+z), \quad (15)$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2} (\log(-z))^2. \quad (16)$$