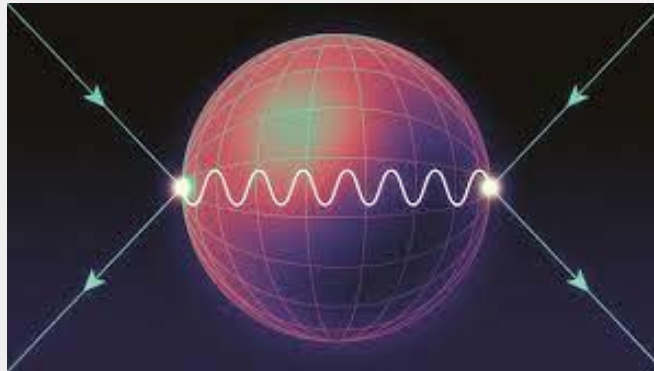
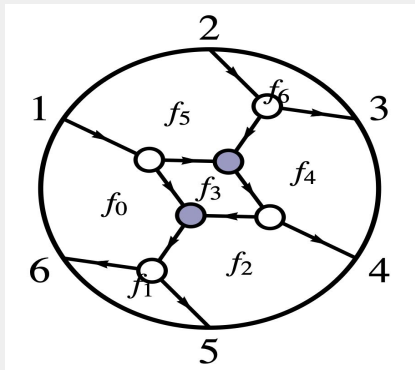
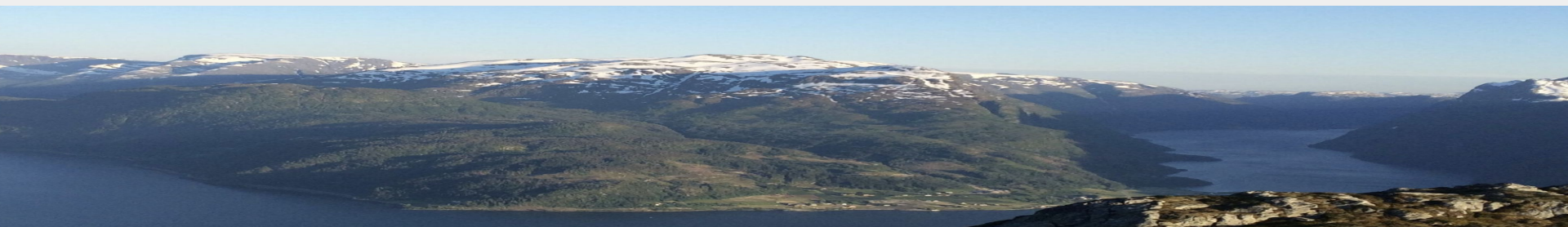


Mathematical Structures of Scattering Amplitudes



Marcus Spradlin and Anastasia Volovich
Nordfjordeid Summer School 2026



Plan of Lectures: Anastasia

- **Motivation for Scattering Amplitudes**
- **Feynman Rules**
- **Helicity Amplitudes**
- **BCFW recursion relations**
- **Examples: n -point MHV, 6-point NMHV**

Tree-level n-point MHV Amplitude

$$A(1^-, 2^+, \dots, (j-1)^+, j^-, (j+1)^+, \dots, (n-1)^+, n^+) = \frac{\langle 1 j \rangle^4}{\langle 4 5 \rangle \langle 5 6 \rangle \dots \langle n-1 n \rangle \langle n 1 \rangle \langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle}.$$

Tree-level 6-point NMHV Amplitude

$$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = \frac{[4|P_{32}|1\rangle^3}{[23][34][2|P_{34}|5\rangle\langle 56\rangle\langle 61\rangle s_{234}} + \frac{[6|P_{54}|3\rangle^3}{[61][12][2|P_{34}|5\rangle\langle 34\rangle\langle 45\rangle s_{612}}$$

References: Anastasia

De Pavlov Spradlin Volovich 2410.1175 p1-11

Witten hep-th/0312171 p3-10

Britto Cachazo Feng hep-th/0412308

Britto Cachazo Feng Witten hep-th/0501052

Plan of Lectures: Marcus

- Feynman Rules
- Helicity Amplitudes
- BCFW recursions
- Examples: n-point MHV, 6-point NMHV
- Momentum Twistors
- Plabic Graphs
- Amplituhedron
- Symbolology
- Cluster Algebras

References: Marcus

Arkani-Hamed, Bourjaily, Cachazo, Goncharov,
Postnikov, Trnka 1212.5605

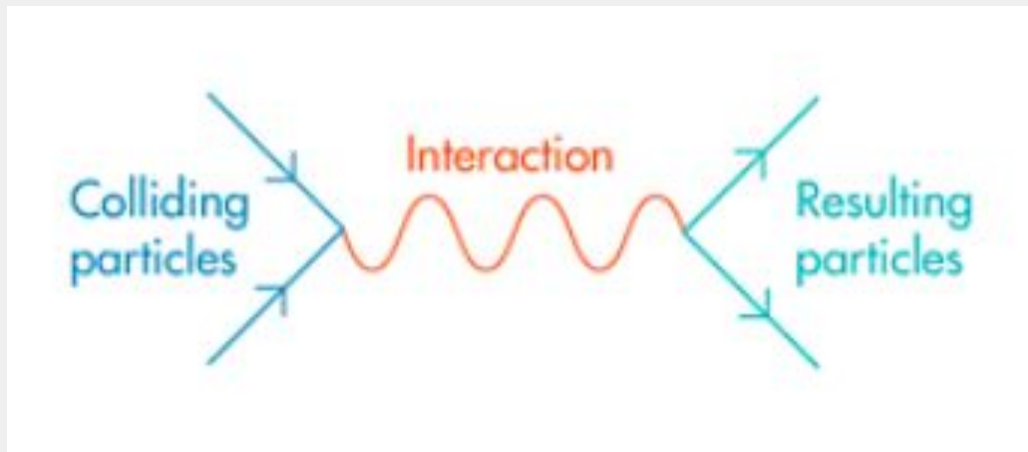
Arkani-Hamed, Trnka 1312.2007

Goncharov, Spradlin, Vergu, Volovich 1006.5703

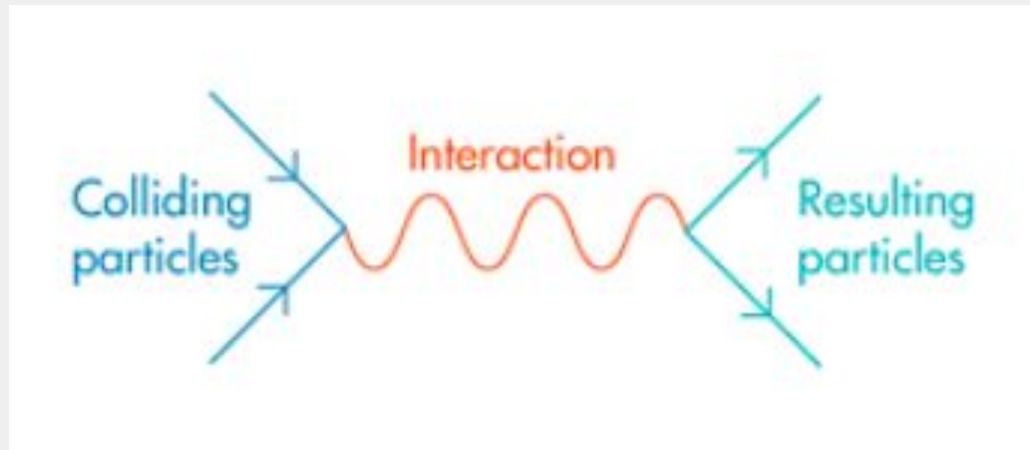
Golden, Goncharov, Spradlin, Vergu, Volovich
1305.1617

Scattering Amplitudes

encode elementary particle interactions and determine probabilities of various outcomes.



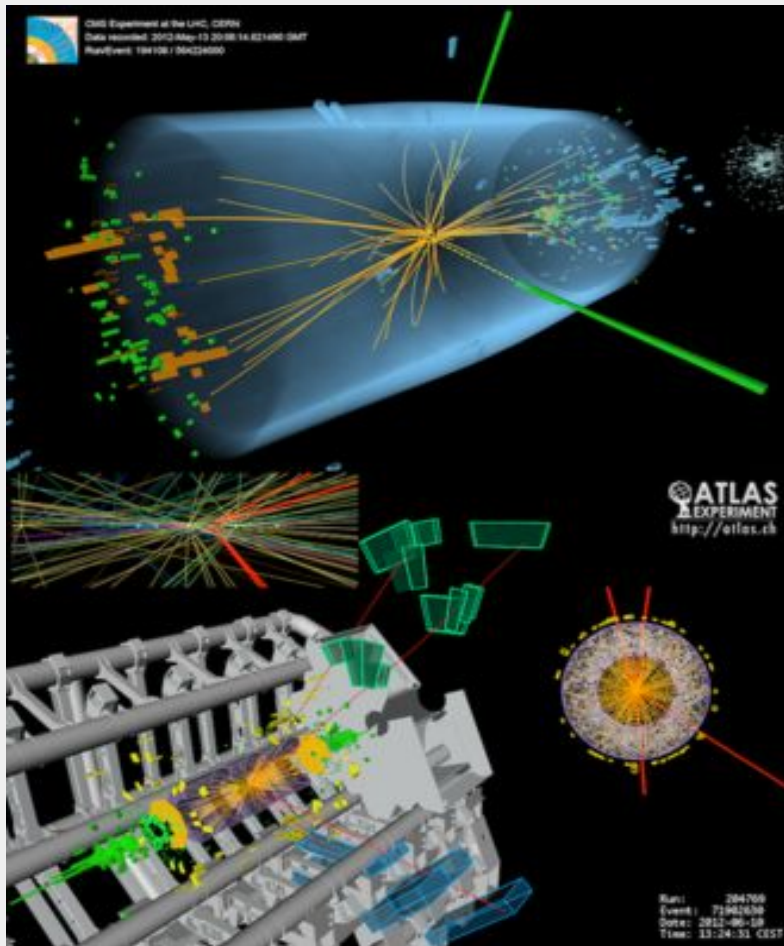
Scattering Amplitudes
encode elementary particle
interactions and determine
probabilities of various outcomes.



**Scattering Amplitudes are
the most basic observables in
fundamental physics.**

Particle Physics Experiments

accelerate particles to high energies, smash them together and see what comes out.

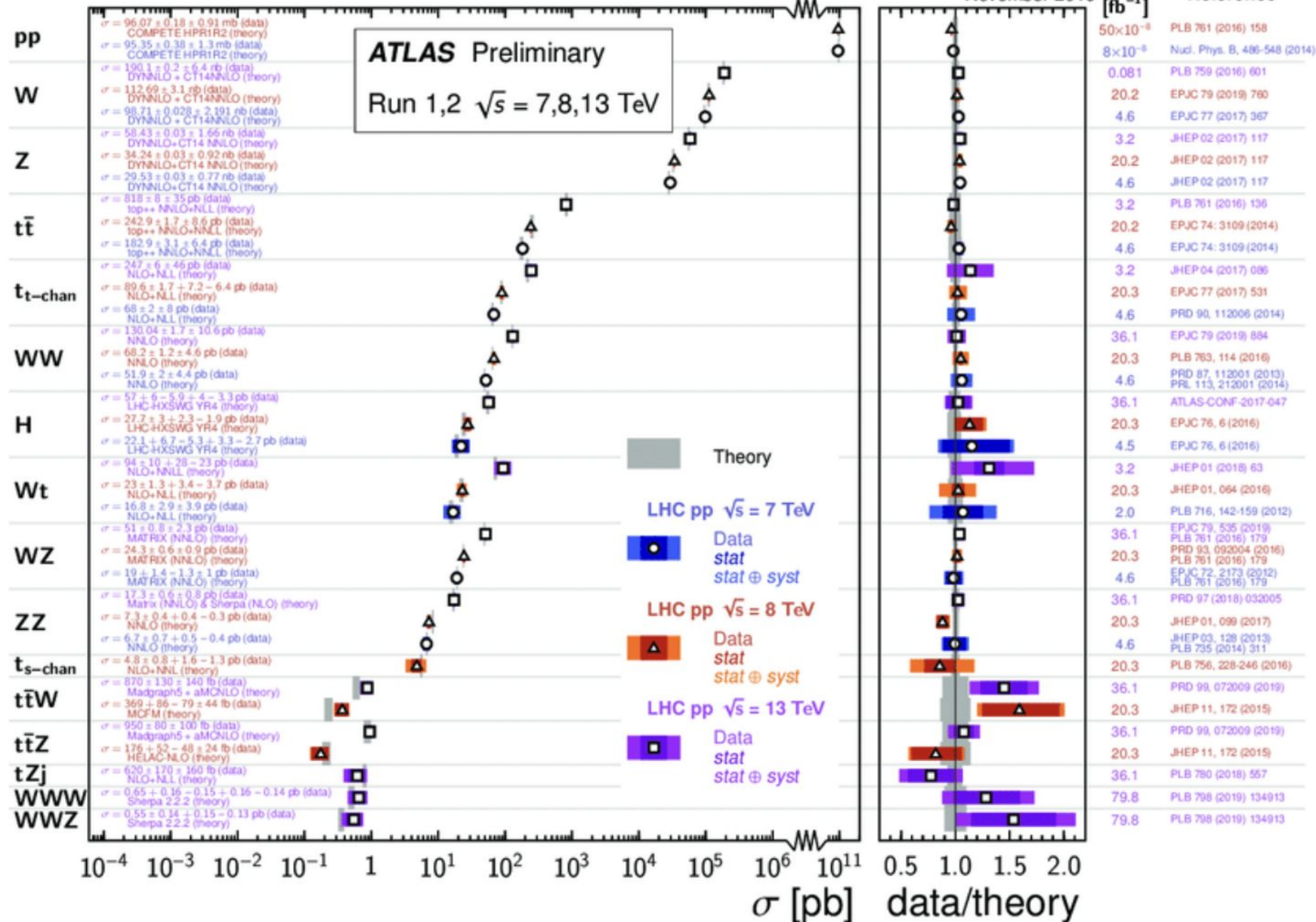


The LHC is the world's largest and most powerful collider.

Standard Model Total Production Cross Section Measurements

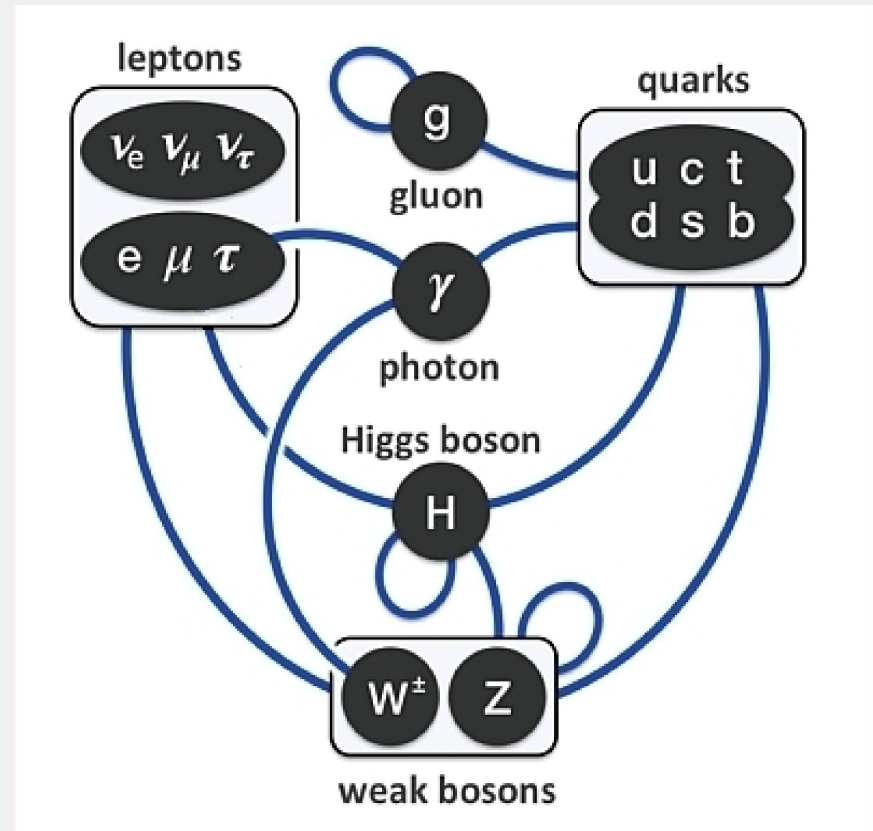
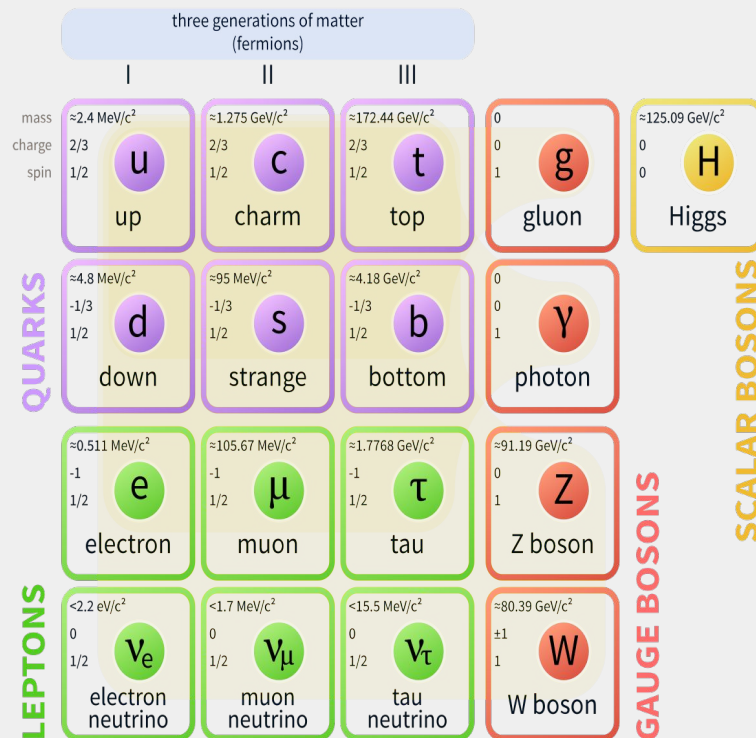
Status:
November 2019 $\int \mathcal{L} dt$
[fb⁻¹]

Reference

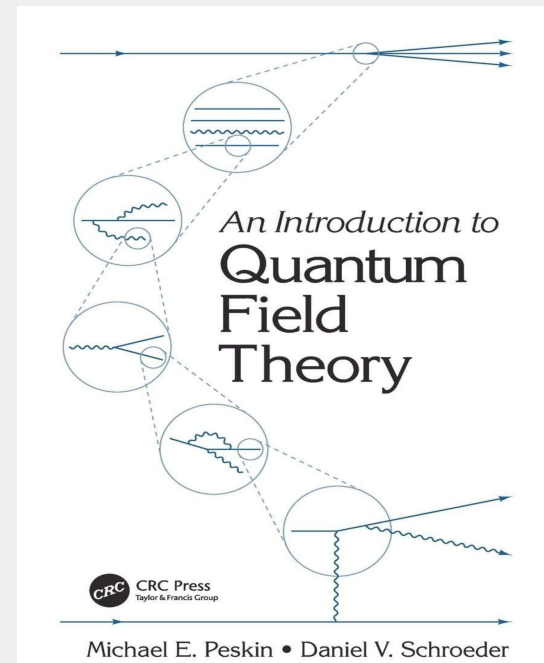
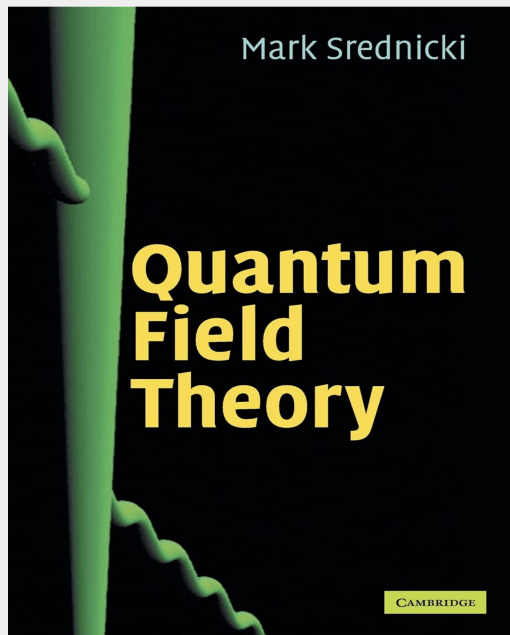


Particle interactions are described by the Standard Model

Standard Model of Elementary Particles

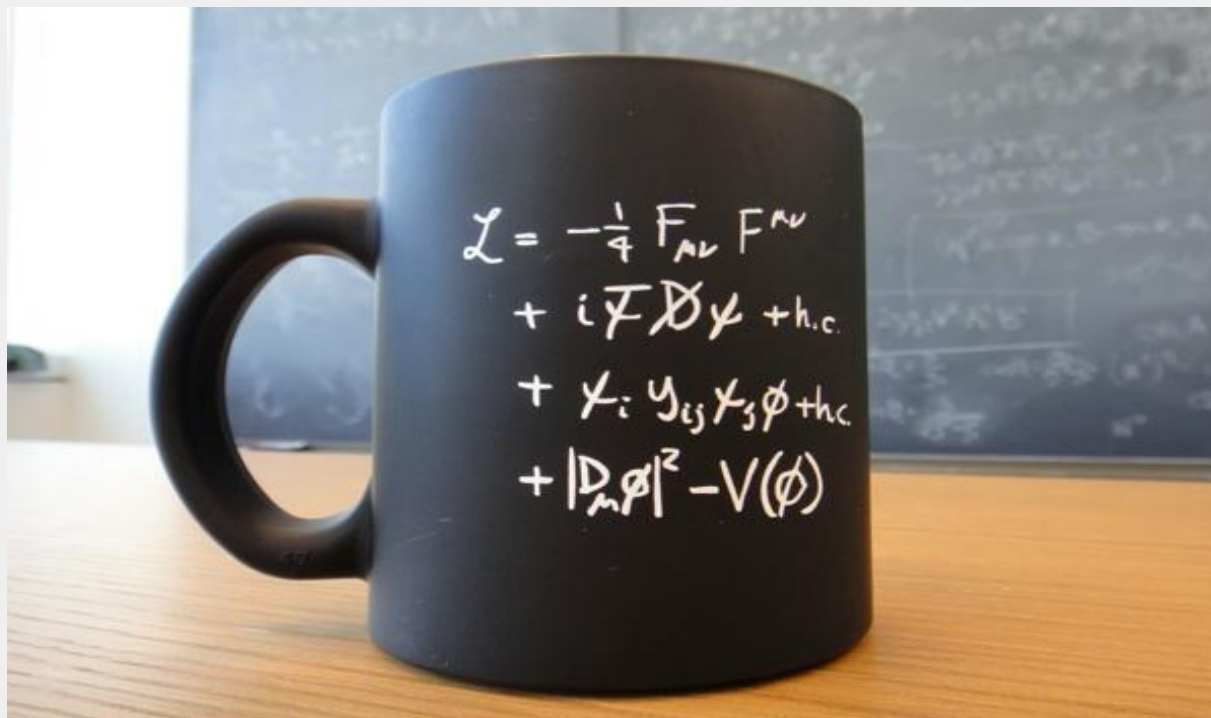


**Particle interactions
are described by
the Standard Model
using the language of
Quantum Field Theory**



Particle interactions are described by the Standard Model
using the language of Quantum Field Theory:

particles are excitations of quantum fields
described by
Standard Model Lagrangian



Standard Model Lagrangian

$$\begin{aligned}
 & \text{1} \quad -\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4}g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e + \\
 & \quad \frac{1}{2}ig_s^2 (\bar{q}_i^c \gamma^\mu q_j^c) g_\mu^a + \bar{G}^a \partial^2 G^a + g_s f^{abc} \partial_\mu \bar{G}^a G^b g_\mu^c - \partial_\nu W_\mu^+ \partial_\nu W_\mu^- - \\
 & \text{2} \quad M^2 W_\mu^+ W_\mu^- - \frac{1}{2}\partial_\nu Z_\mu^0 \partial_\nu Z_\mu^0 - \frac{1}{2c_w^2} M^2 Z_\mu^0 Z_\mu^0 - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \\
 & \quad \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h \left[\frac{2M^2}{g^2} + \right. \\
 & \quad \left. \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-) \right] + \frac{2M^4}{g^2} \alpha_h - igc_w [\partial_\nu Z_\mu^0 (W_\mu^+ W_\nu^- - \\
 & \quad W_\nu^+ W_\mu^-) - Z_\nu^0 (W_\mu^+ \partial_\nu W_\mu^- - W_\mu^- \partial_\nu W_\mu^+) + Z_\mu^0 (W_\nu^+ \partial_\nu W_\mu^- - \\
 & \quad W_\nu^- \partial_\nu W_\mu^+)] - igs_w [\partial_\nu A_\mu (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) - A_\nu (W_\mu^+ \partial_\nu W_\mu^- - \\
 & \quad W_\mu^- \partial_\nu W_\mu^+) + A_\mu (W_\nu^+ \partial_\nu W_\mu^- - W_\nu^- \partial_\nu W_\mu^+)] - \frac{1}{2}g^2 W_\mu^+ W_\mu^- W_\nu^+ W_\nu^- + \\
 & \quad \frac{1}{2}g^2 W_\mu^+ W_\nu^- W_\mu^+ W_\nu^- + g^2 c_w^2 (Z_\mu^0 W_\mu^+ Z_\nu^0 W_\nu^- - Z_\mu^0 Z_\mu^0 W_\nu^+ W_\nu^-) + \\
 & \quad g^2 s_w^2 (A_\mu W_\mu^+ A_\nu W_\nu^- - A_\mu A_\mu W_\nu^+ W_\nu^-) + g^2 s_w c_w [A_\mu Z_\nu^0 (W_\mu^+ W_\nu^- - \\
 & \quad W_\nu^+ W_\mu^-) - 2A_\mu Z_\mu^0 W_\nu^+ W_\nu^-] - g\alpha [H^3 + H\phi^0 \phi^0 + 2H\phi^+ \phi^-] - \\
 & \quad \frac{1}{8}g^2 \alpha_h [H^4 + (\phi^0)^4 + 4(\phi^+ \phi^-)^2 + 4(\phi^0)^2 \phi^+ \phi^- + 4H^2 \phi^+ \phi^- + 2(\phi^0)^2 H^2] - \\
 & \quad gMW_\mu^+ W_\mu^- H - \frac{1}{2}g \frac{M}{c_w^2} Z_\mu^0 Z_\mu^0 H - \frac{1}{2}ig[W_\mu^+ (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - \\
 & \quad W_\mu^- (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)] + \frac{1}{2}g[W_\mu^+ (H \partial_\mu \phi^- - \phi^- \partial_\mu H) - W_\mu^- (H \partial_\mu \phi^+ - \\
 & \quad \phi^+ \partial_\mu H)] + \frac{1}{2}g \frac{1}{c_w} (Z_\mu^0 (H \partial_\mu \phi^0 - \phi^0 \partial_\mu H) - ig \frac{s_w^2}{c_w} M Z_\mu^0 (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \\
 & \quad igs_w M A_\mu (W_\mu^+ \phi^- - W_\mu^- \phi^+) - ig \frac{1-2c_w^2}{2c_w} Z_\mu^0 (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + \\
 & \quad igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \frac{1}{4}g^2 W_\mu^+ W_\mu^- [H^2 + (\phi^0)^2 + 2\phi^+ \phi^-] - \\
 & \quad \frac{1}{4}g^2 \frac{1}{c_w^2} Z_\mu^0 Z_\mu^0 [H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+ \phi^-] - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z_\mu^0 \phi^0 (W_\mu^+ \phi^- + \\
 & \quad W_\mu^- \phi^+) - \frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z_\mu^0 H (W_\mu^+ \phi^- - W_\mu^- \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W_\mu^+ \phi^- + \\
 & \quad W_\mu^- \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W_\mu^+ \phi^- - W_\mu^- \phi^+) - g^2 \frac{s_w}{c_w} (2c_w^2 - 1) Z_\mu^0 A_\mu \phi^+ \phi^- - \\
 & \quad g^1 s_w^2 A_\mu A_\mu \phi^+ \phi^-] - \bar{e}^\lambda (\gamma \partial + m_e^\lambda) e^\lambda - \bar{\nu}^\lambda \gamma \partial \nu^\lambda - \bar{u}_j^\lambda (\gamma \partial + m_u^\lambda) u_j^\lambda - \\
 & \text{3} \quad \bar{d}_j^\lambda (\gamma \partial + m_d^\lambda) d_j^\lambda + igs_w A_\mu [-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}_j^\lambda \gamma^\mu u_j^\lambda) - \frac{1}{3}(\bar{d}_j^\lambda \gamma^\mu d_j^\lambda)] + \\
 & \quad \frac{ig}{4c_w} Z_\mu^0 [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{u}_j^\lambda \gamma^\mu (\frac{4}{3}s_w^2 - \\
 & \quad 1 - \gamma^5) u_j^\lambda) + (\bar{d}_j^\lambda \gamma^\mu (1 - \frac{8}{3}s_w^2 - \gamma^5) d_j^\lambda)] + \frac{ig}{2\sqrt{2}} W_\mu^+ [(\bar{\nu}^\lambda \gamma^\mu (1 + \gamma^5) e^\lambda) + \\
 & \quad (\bar{u}_j^\lambda \gamma^\mu (1 + \gamma^5) C_{\lambda\kappa} d_j^\kappa)] + \frac{ig}{2\sqrt{2}} W_\mu^- [(\bar{e}^\lambda \gamma^\mu (1 + \gamma^5) \nu^\lambda) + (\bar{d}_j^\kappa C_{\lambda\kappa}^\dagger \gamma^\mu (1 + \\
 & \quad \gamma^5) u_j^\lambda)] + \frac{ig}{2\sqrt{2}} \frac{m_\lambda^2}{M} [-\phi^+ (\bar{\nu}^\lambda (1 - \gamma^5) e^\lambda) + \phi^- (\bar{e}^\lambda (1 + \gamma^5) \nu^\lambda)] - \\
 & \text{4} \quad \frac{g}{2} \frac{m_\lambda^2}{M} [H (\bar{e}^\lambda e^\lambda) + i\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda)] + \frac{ig}{2M\sqrt{2}} \phi^+ [-m_\lambda^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 - \gamma^5) d_j^\kappa) + \\
 & \quad m_\lambda^\kappa (\bar{u}_j^\lambda C_{\lambda\kappa} (1 + \gamma^5) d_j^\kappa)] + \frac{ig}{2M\sqrt{2}} \phi^- [m_\lambda^\lambda (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 + \gamma^5) u_j^\kappa) - m_\lambda^\kappa (\bar{d}_j^\lambda C_{\lambda\kappa}^\dagger (1 - \\
 & \quad \gamma^5) u_j^\kappa)] - \frac{g}{2} \frac{m_\lambda^2}{M} H (\bar{u}_j^\lambda u_j^\lambda) - \frac{g}{2} \frac{m_\lambda^2}{M} H (\bar{d}_j^\lambda d_j^\lambda) + \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{u}_j^\lambda \gamma^5 u_j^\lambda) - \\
 & \quad \frac{ig}{2} \frac{m_\lambda^2}{M} \phi^0 (\bar{d}_j^\lambda \gamma^5 d_j^\lambda) + \bar{X}^+ (\partial^2 - M^2) X^+ + \bar{X}^- (\partial^2 - M^2) X^- + \bar{X}^0 (\partial^2 - \\
 & \text{5} \quad \frac{M^2}{c_w^2}) X^0 + \bar{Y} \partial^2 Y + igc_w W_\mu^+ (\partial_\mu \bar{X}^0 X^- - \partial_\mu \bar{X}^+ X^0) + igs_w W_\mu^+ (\partial_\mu \bar{Y} X^- - \\
 & \quad \partial_\mu \bar{X}^+ Y) + igc_w W_\mu^- (\partial_\mu \bar{X}^- X^0 - \partial_\mu \bar{X}^0 X^+) + igs_w W_\mu^- (\partial_\mu \bar{X}^- Y - \\
 & \quad \partial_\mu \bar{Y} X^+) + igc_w Z_\mu^0 (\partial_\mu \bar{X}^+ X^+ - \partial_\mu \bar{X}^- X^-) + igs_w A_\mu (\partial_\mu \bar{X}^+ X^+ - \\
 & \quad \partial_\mu \bar{X}^- X^-) - \frac{1}{2}gM [\bar{X}^+ X^+ H + \bar{X}^- X^- H + \frac{1}{c_w^2} \bar{X}^0 X^0 H] + \\
 & \quad \frac{1-2c_w^2}{2c_w} igM [\bar{X}^+ X^0 \phi^+ - \bar{X}^- X^0 \phi^-] + \frac{1}{2c_w} igM [\bar{X}^0 X^- \phi^+ - \bar{X}^0 X^+ \phi^-] + \\
 & \quad igM s_w [\bar{X}^0 X^- \phi^+ - \bar{X}^0 X^+ \phi^-] + \frac{1}{2}igM [\bar{X}^+ X^+ \phi^0 - \bar{X}^- X^- \phi^0]
 \end{aligned}$$

We will only focus on the first line:

$$-\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4} g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e .$$

**which describes gluon interactions
in Quantum Chromodynamics
(part of Standard Model)**

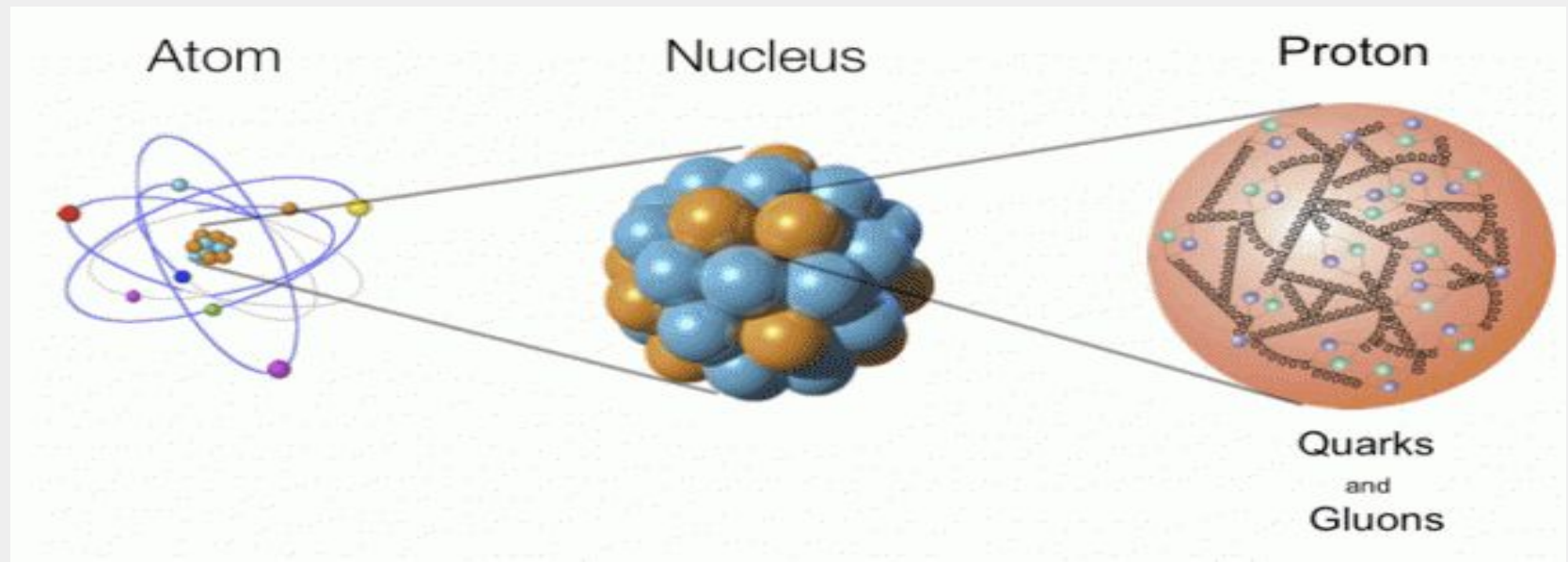
Quantum Chromodynamics (QCD)

The theory of strong interactions.

It describes the strong nuclear force which binds together

quarks inside protons and neutrons.

Gluons mediate the strong force.



We will only focus on the first line:

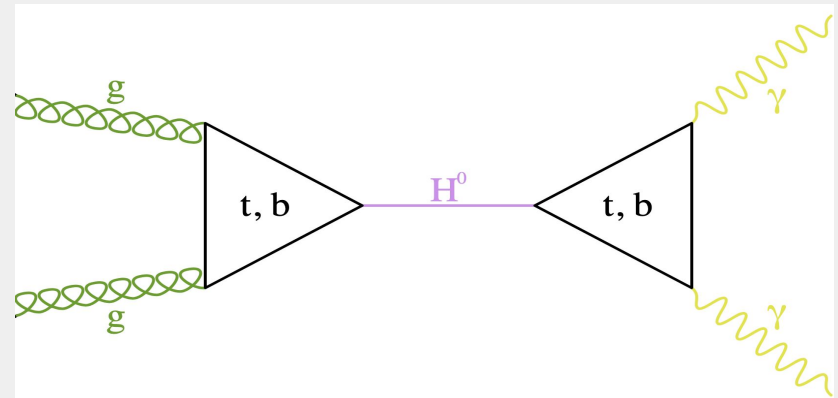
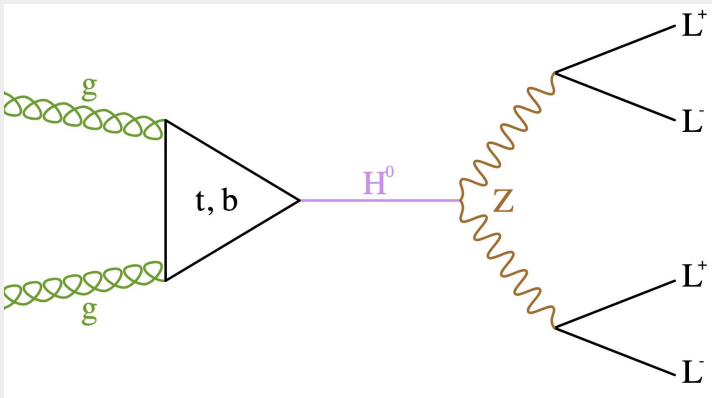
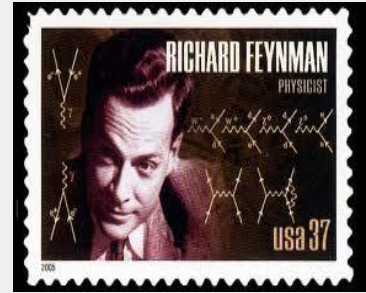
$$-\frac{1}{2}\partial_\nu g_\mu^a \partial_\nu g_\mu^a - g_s f^{abc} \partial_\mu g_\nu^a g_\mu^b g_\nu^c - \frac{1}{4} g_s^2 f^{abc} f^{ade} g_\mu^b g_\nu^c g_\mu^d g_\nu^e .$$

**which describes gluon interactions
in Quantum Chromodynamics
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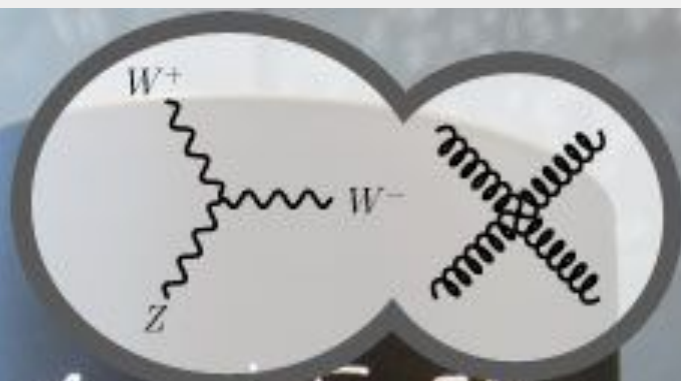
Scattering Amplitudes are computed via

Feynman diagrams:

pictorial representations of mathematical expressions







$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & + i \bar{\psi} \not{D} \psi + \text{h.c.} \\ & + \chi_i^\dagger y_{ij} \chi_j \phi + \text{h.c.} \\ & + |D_\mu \phi|^2 - V(\phi) \end{aligned}$$



Organizing Feynman diagrams

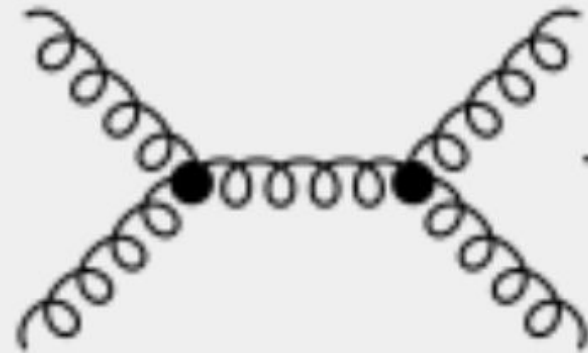
Amplitudes are indexed by

- $n = 4, 5, \dots$ = number of particles (4-point, 5-point, etc)
- $L = 0, 1, 2, 3$ = loop order (tree, one-loop, etc)

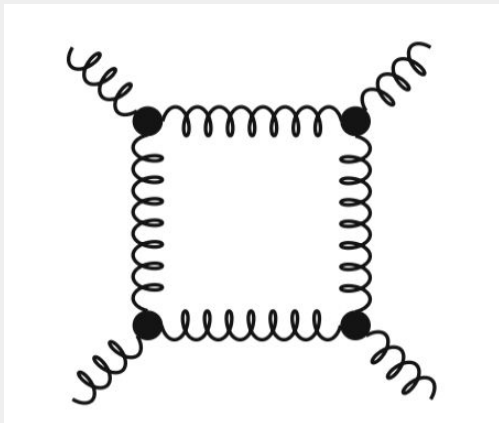
“the L -loop n -particle amplitude” is, in principle, computed by summing all Feynman diagrams with n external edges and L internal loops, and then **integrating** over all loops.

Organizing Feynman diagrams

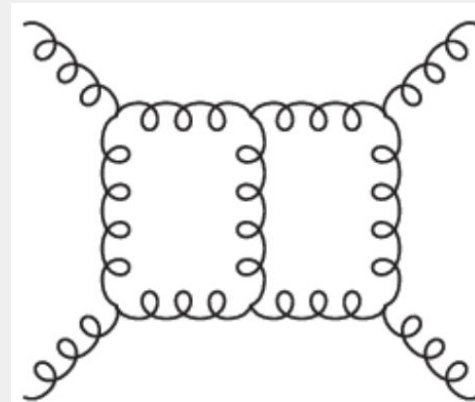
**Leading order
(LO), tree**



**Next-to-leading
order (NLO),
one-loop**



**Next-to-next-to
leading order
(NNLO), two-loops**



Feynman Diagrams: gluon scattering

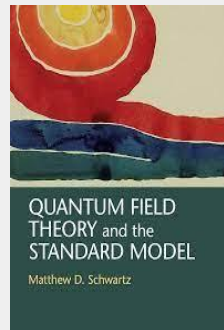
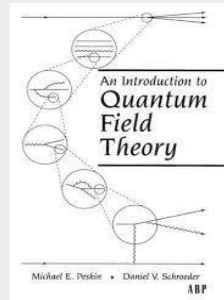
a, μ b, ν

$$= gf^{abc} [g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu]$$

a, μ c, ρ b, ν

$$= -ig^2 f^{abc} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + \text{permutations}$$

d, σ c, ρ



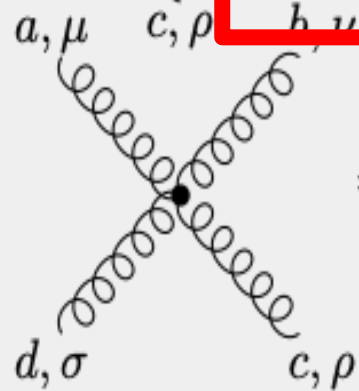
**To calculate any amplitude:
write down all Feynman diagrams,
sum them up**

Feynman Diagrams: gluon scattering



$$= gf^{abc} [g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu]$$

THIS WILL BE TOPIC 1 OF MY LECTURES



$$= -ig^2 f^{abc} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + \text{permutations}$$

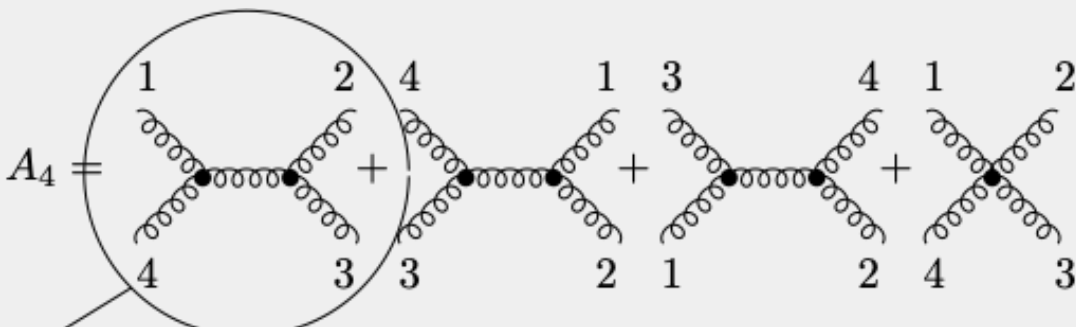
**To calculate any amplitude:
write down all Feynman diagrams,
sum them up**

Complications

Number of diagrams for $gg \rightarrow ng$ process:

n	2	3	4	5	6	7	8
#	4	25	220	2485	34300	559405	10525900

Each diagram is a complicated expression:



$$\begin{aligned}
 A_4 = & g^2 f^{abe} f^{cde} [g_{\mu\nu}(k - p_1)_\rho + g_{\nu\rho}(p_1 - p_4)_\mu + g_{\rho\mu}(p_4 - k)_\nu] \epsilon_1^\mu \epsilon_2^\lambda \epsilon_3^\sigma \epsilon_4^\nu \\
 & \times \frac{i}{k^2} [g_{\sigma\lambda}(k - p_2)^\rho + g_{\lambda}^\rho(p_2 - p_3)_\sigma + g^\rho{}_\sigma(p_3 - k)_\lambda], \quad k = p_1 + p_4
 \end{aligned}$$

Five Gluon Amplitude: $gg \rightarrow ggg$

[illegible]

1994-1995-95

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1999 年 10 月 1 日

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Abstract

 $\epsilon_1, \epsilon_2, \dots, \epsilon_n, \epsilon_{n+1}, \dots, \epsilon_N$ $\kappa_1 \in \{e_1, e_3, e_4, e_5\}$

+ 24 more pages

$$k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5$$

Six Gluon Amplitude: $gg \rightarrow gggg$



What to do?



Five Gluon Amplitude: $gg \rightarrow ggg$

$$\langle 12 \rangle^4$$

$$\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle$$

+ 24 more pages


$$k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5$$

Spinor Helicity Variables

Instead of labeling gluons by momentum and polarization, label gluons by a pair of spinors and a helicity plus or minus



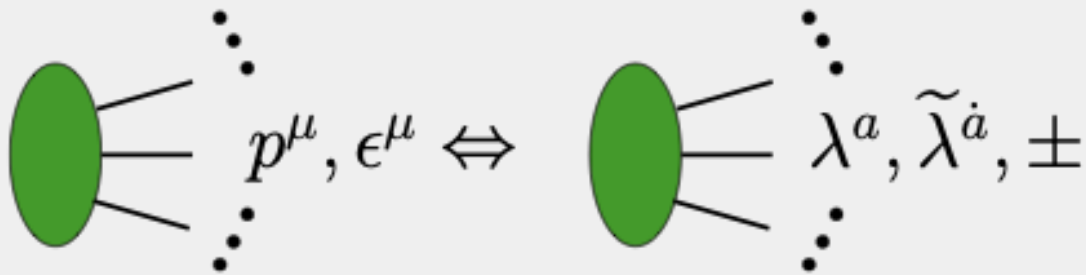
$$p_i^\mu = \sigma_{a\dot{a}}^\mu \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$$

$$\langle ij \rangle = \lambda_i^a \lambda_j^b \epsilon_{ab}$$

Spinor Helicity Variables

Instead of labeling gluons by momentum and polarization, label gluons by a pair of spinors and a helicity plus or minus

THIS WILL BE TOPIC 2 OF MY LECTURES



$$p_i^\mu = \sigma_{a\dot{a}}^\mu \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$$

$$\langle ij \rangle = \lambda_i^a \lambda_j^b \epsilon_{ab}$$

Five Gluon Amplitude: $gg \rightarrow ggg$

$$\langle 12 \rangle^4$$

$$\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle$$

+ 24 more pages


$$k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5$$

Parke-Taylor Formula

$$\begin{array}{c} + \\ + \\ + \\ \vdots \\ + \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} + \\ + \\ + \\ + \\ + \end{array} = \begin{array}{c} + \\ + \\ + \\ \vdots \\ + \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} + \\ + \\ + \\ + \\ - \end{array} = 0$$



$$\begin{array}{c} + \\ + \\ + \\ \vdots \\ + \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} + \\ + \\ + \\ + \\ + \end{array} = \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle n 1 \rangle}$$

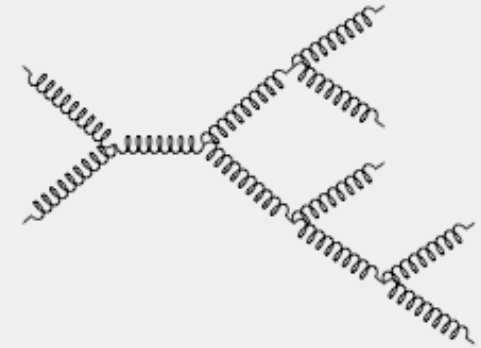
Parke-Taylor formula (1986)

Why is it important?

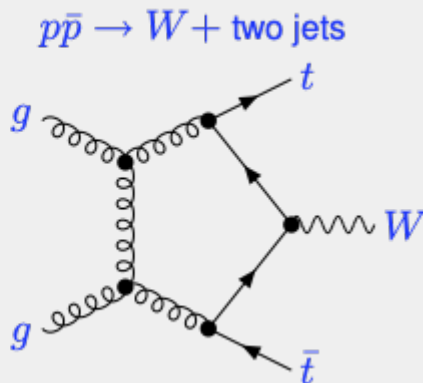
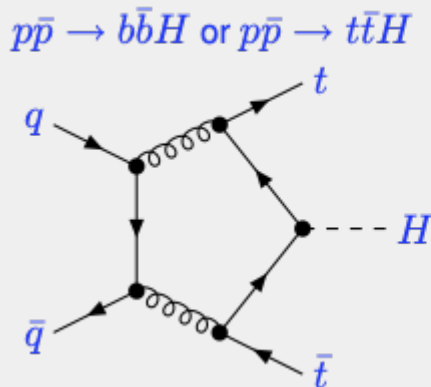
- It is astonishing that such an amazing formula can exist!
- The clear lesson is: Traditional Feynman rules are inefficient!
- The quest to figure out why leads to a new field of research.

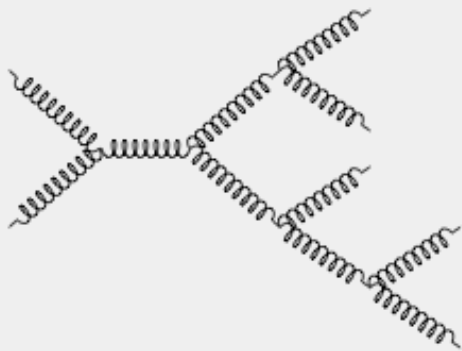
1986 – 2003

- **Berends-Giele recursion:**
prove Parke-Taylor formula

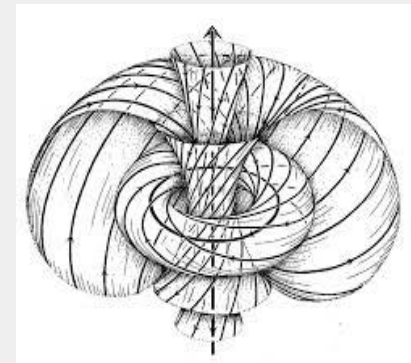


- **Bern-Dixon-Kosower methods:**
generalized unitarity for loops





Twistor Revolution



Commun. Math. Phys. 252, 189–258 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-004-1187-3

Communications in
**Mathematical
Physics**

Perturbative Gauge Theory as a String Theory in Twistor Space

Edward Witten

Institute for Advanced Study, Princeton, NJ 08540, USA

Received: 22 April 2004 / Accepted: 3 June 2004
Published online: 7 October 2004 – © Springer-Verlag 2004

Dedicated to Freeman Dyson for his 80th birthday

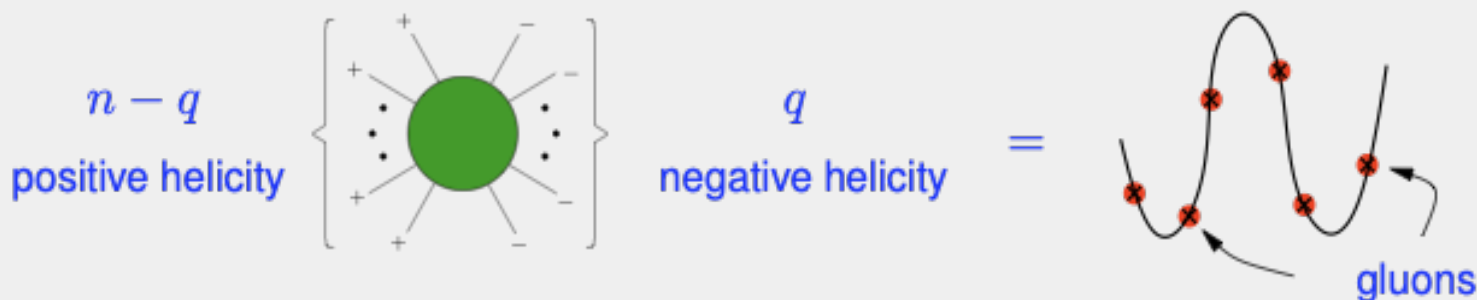
Abstract: Perturbative scattering amplitudes in Yang-Mills theory have many unexpected properties, such as holomorphy of the maximally helicity violating amplitudes. To interpret these results, we Fourier transform the scattering amplitudes from momentum space to twistor space, and argue that the transformed amplitudes are supported on certain holomorphic curves. This in turn is apparently a consequence of an equivalence between the perturbative expansion of $\mathcal{N} = 4$ super Yang-Mills theory and the D -instanton expansion of a certain string theory, namely the topological B model whose target space is the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$.

2003

Amplitudes and Twistors

Witten (2003): gluon amplitudes have rich mathematical structure when Penrose transformed to **twistor space** P^3

$$\mathcal{A}(\lambda, \mu, \pm) = \int d\tilde{\lambda} e^{i\mu\tilde{\lambda}} \mathcal{A}(\lambda, \tilde{\lambda}, \pm)$$



Amplitudes are supported on curves in twistor space.

degree = $q-1$

Curves in Twistor Space

$$\begin{array}{c} + \\ + \\ \vdots \\ + \end{array} \begin{array}{c} \text{---} A_n \text{---} \\ \text{---} \end{array} \begin{array}{c} + \\ + \\ \vdots \\ + \end{array} = \begin{array}{c} + \\ + \\ \vdots \\ + \\ - \end{array} \begin{array}{c} \text{---} A_n \text{---} \\ \text{---} \end{array} \begin{array}{c} + \\ + \\ \vdots \\ + \end{array} = 0$$

degree = $q-1$

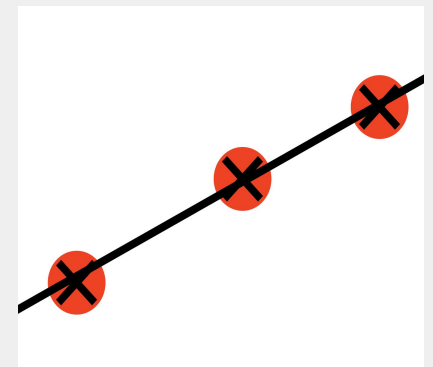
$d=-1$

$d=0$

$$\begin{array}{c} + \\ + \\ \vdots \\ + \end{array} \begin{array}{c} \text{---} A_n \text{---} \\ \text{---} \end{array} \begin{array}{c} + \\ + \\ \vdots \\ + \end{array} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

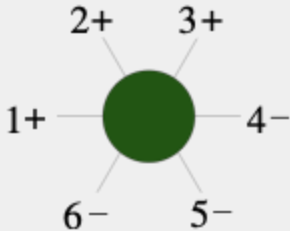
Parke-Taylor formula (1986)

$d=1$



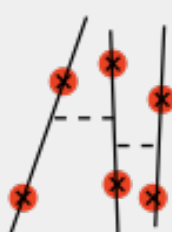
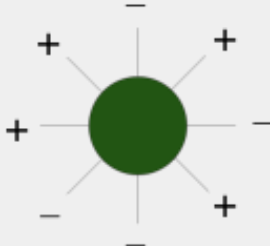
Twistor String Theory

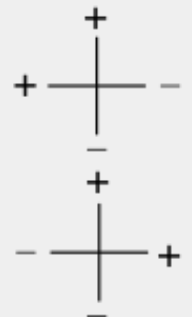
Calculating Amplitudes:
Solving Algebraic Equation (RSV)

$$\Leftrightarrow a_4 \sigma^4 + a_3 \sigma^3 + a_2 \sigma^2 + a_1 \sigma + a_0 = 0$$

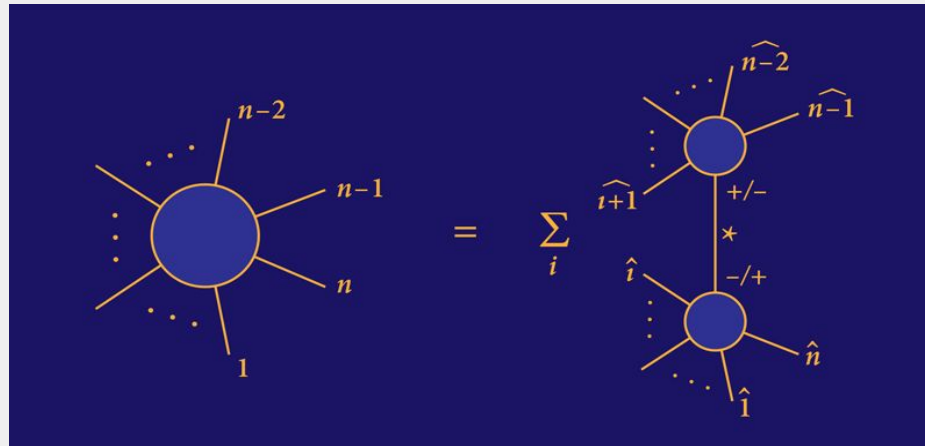
Calculating Amplitudes:
Using Simplest Building Blocks (CSW)

$$= \begin{array}{c} - \\ | \\ + \text{---} | \text{---} - \\ | \\ + \end{array} + \begin{array}{c} + \\ | \\ + \text{---} | \text{---} - \\ | \\ - \end{array} + \text{a few more}$$


Tree Diagrams

Thanks to these developments, there exist several simple and efficient algorithms for computing analytically any desired tree-amplitude.



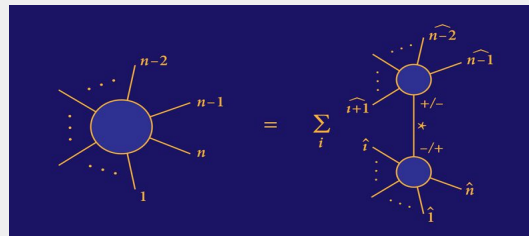
**Britto Cachazo
Feng Witten
(2005)**

$gg \rightarrow ng$	2	3	4	5	6	7	8
Feynman diagrams	4	25	220	2485	34300	559505	10525900
Modern Methods	1	1	2	3	6	10	20

Tree Diagrams

Thanks to these developments, there exist several simple and efficient algorithms for computing analytically any desired tree-amplitude.

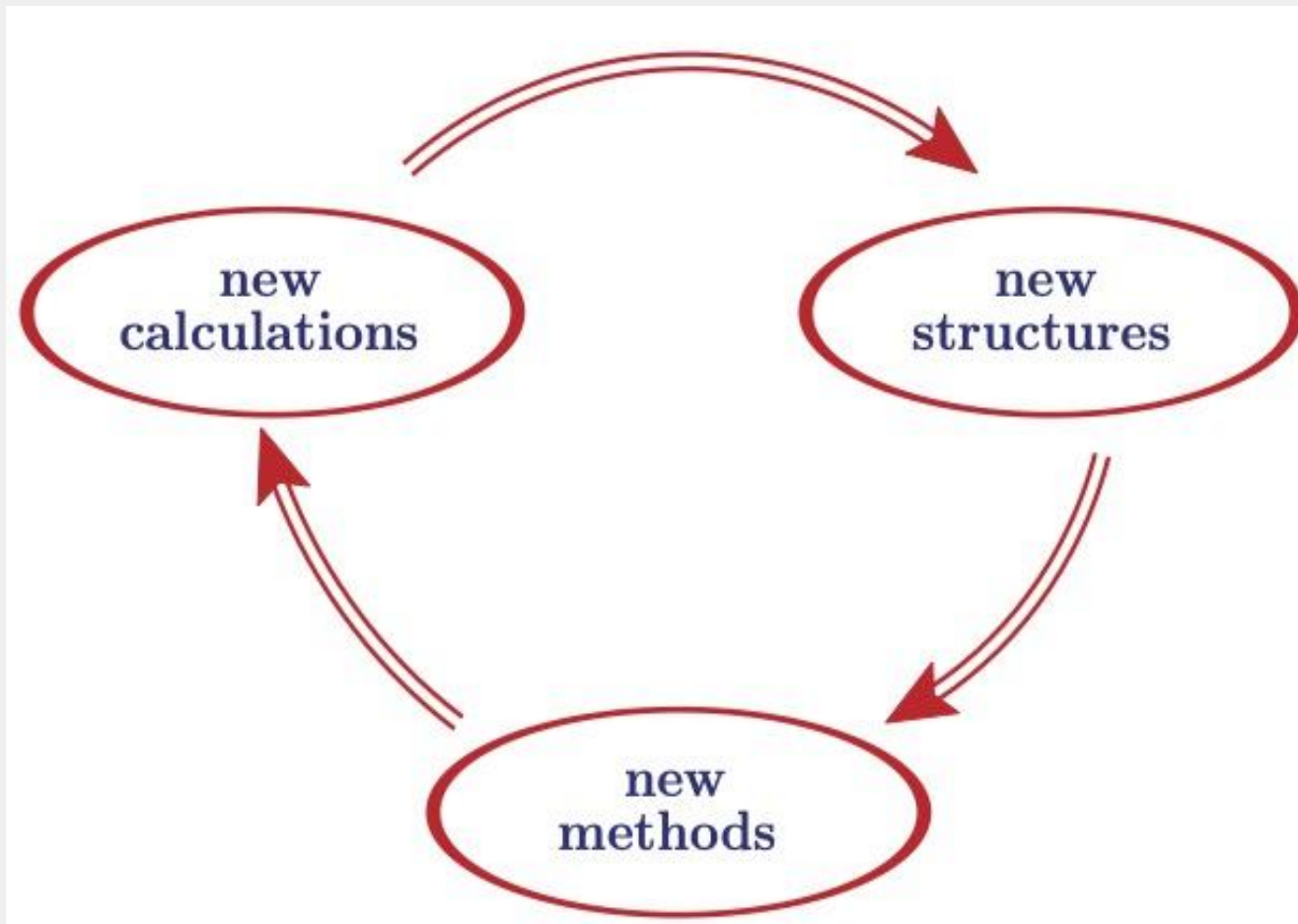
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**Britto Cachazo
Feng Witten
(2005)**

gg→ng	2	3	4	5	6	7	8
Feynman diagrams	4	25	220	2485	34300	559505	10525900
Modern Methods	1	1	2	3	6	10	20

Amplitudes Virtuous Circle

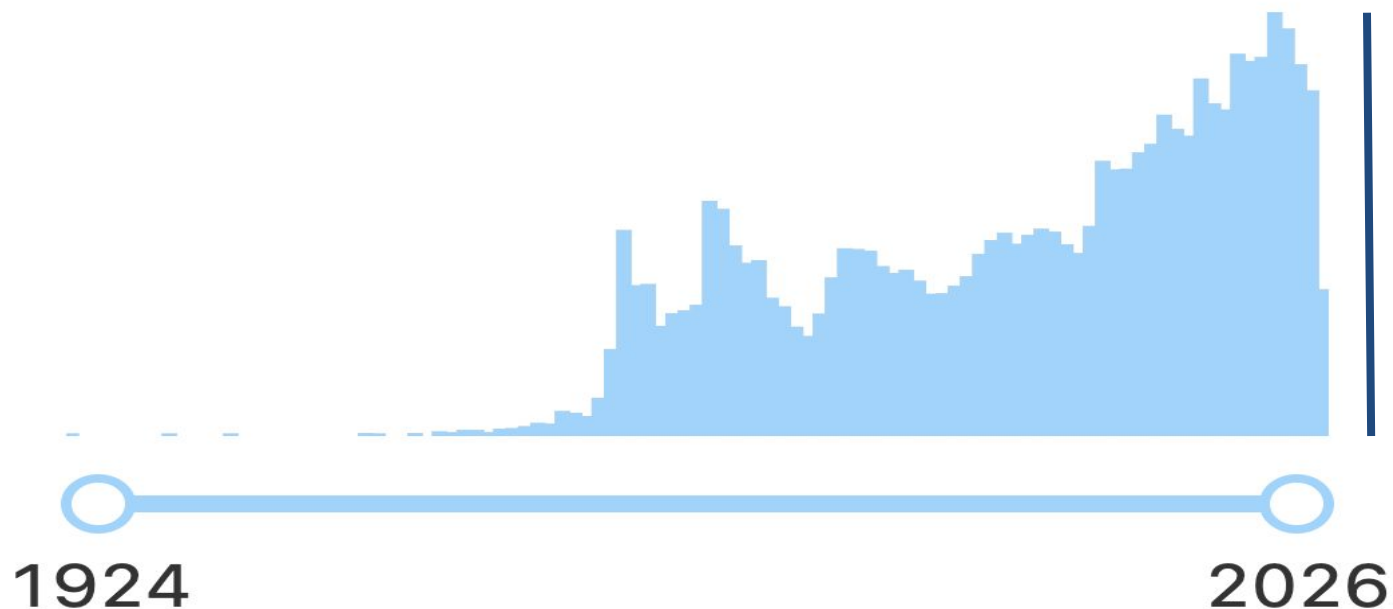


Scattering Amplitudes: rich and fast growing field

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 Amplitudes 2020, Zoom
 Amplitudes 2019, Dublin
 Amplitudes 2018, SLAC
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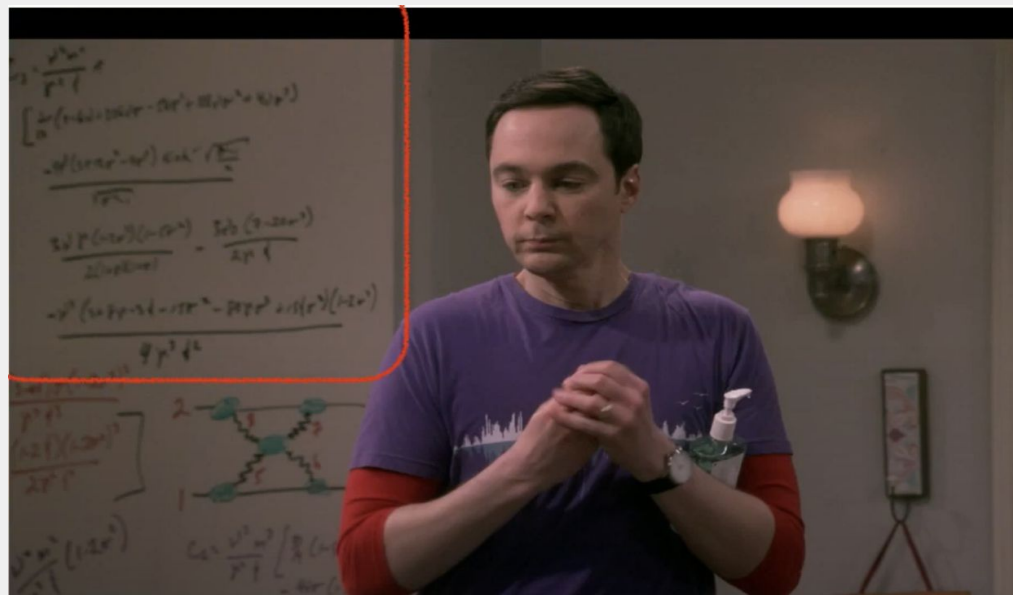


Maria O'Loughlin, Institute for Advanced Study

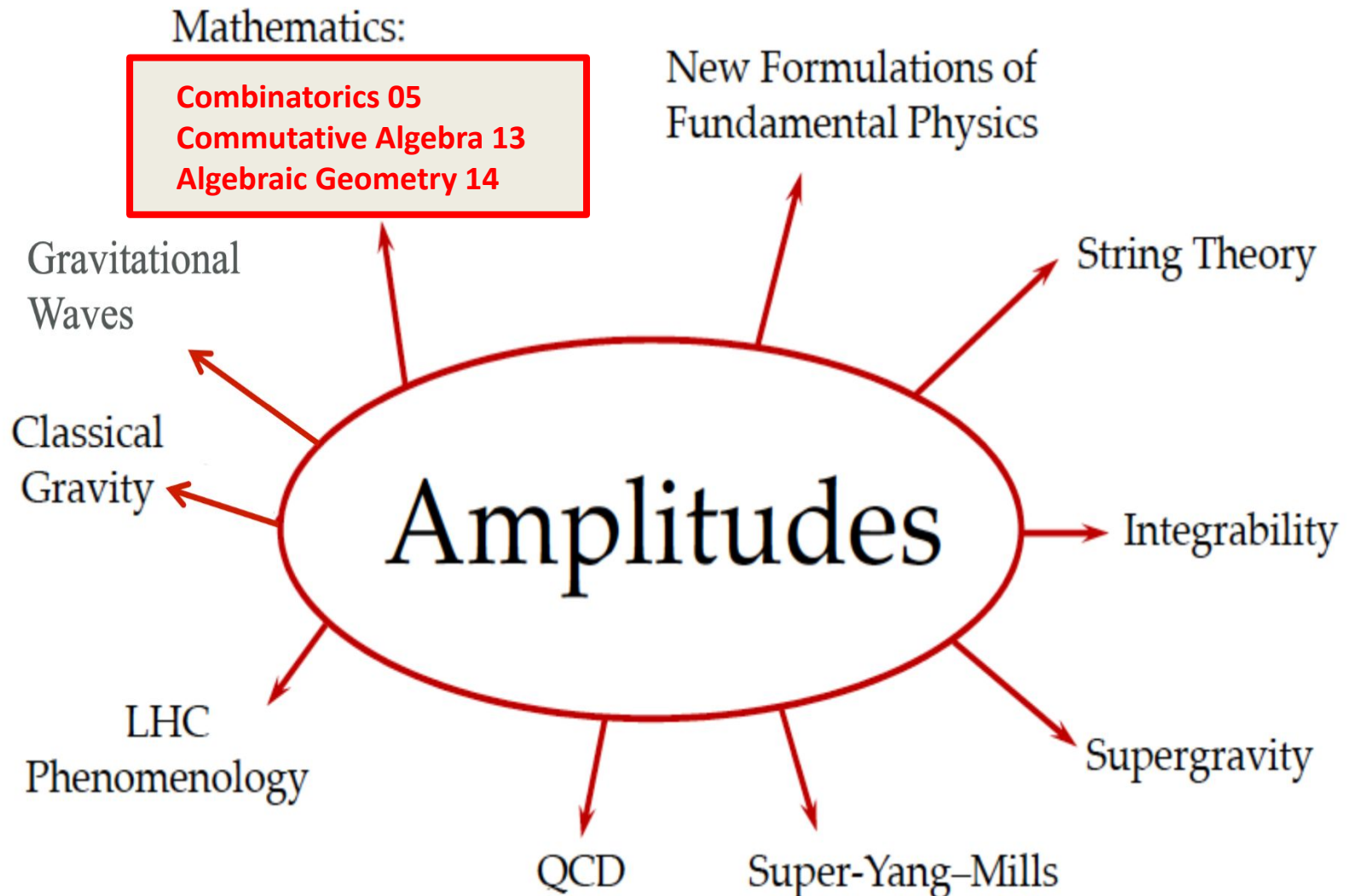
2012



2019



Amplitudes blossomed in many directions



Plan of Lectures: Anastasia

- **Motivation for Scattering Amplitudes**
- **Feynman Rules**
- **Helicity Amplitudes**
- **BCFW recursion relations**
- **Examples: n-point MHV, 6-point NMHV**