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Preliminaries on Amplitudes

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We work in complexified Minkowski space-time $\mathbb{C}^{1,3}$ with mostly minus metric, denoting inner product of two vectors (a, b) by $a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$.

Scattering amplitudes of $n > 2$ gluons denoted by $A_n(p_i, \epsilon_i)$ is a function of n momenta $p_1, \dots, p_n \in \mathbb{C}^{1,3}$ and n polarizations $\epsilon_1, \dots, \epsilon_n \in \mathbb{C}^{1,3}$ subject to:

① each $p_i \neq 0$

② $p_i \cdot p_i = \epsilon_i \cdot p_i = \epsilon_i \cdot \epsilon_i = 0$ for all i

③ momentum conservation $\sum_{i=1}^n p_i = 0$

Amplitudes must satisfy additional mathematical and physical constraints, such as

① linear in each ϵ_i : $A(p_i, z_i \epsilon_i) = A(p_i, \epsilon_i) \prod_{i=1}^n z_i$, $z_i \in \mathbb{C}$

② invariant under gauge transformation

$$A(p_i, \epsilon_i + z_i p_i) = A(p_i, \epsilon_i) \quad \forall z_i \in \mathbb{C}$$

③ Lorentz invariant

$$A(R p_i, R \epsilon_i) = A(p_i, \epsilon_i) \quad \forall R \in SO(1, 3, \mathbb{C})$$

Amplitudes are computed by Feynman Rules

Feynman Rules

 for Yang-Mills theory
(QCD, gluons)

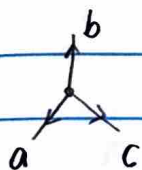
A Feynman diagram is a connected, plane tree graph in which $n > 2$ vertices (called external) have degree 1, and the remaining vertices (called internal) have degree 3 or 4, together w/ labeling of external vertices $1, \dots, n$ in clockwise order as one encircles the diagram from the outside.

- (1) We assign edge labels $1, 2, \dots, n$ on the external edges and $n+1, n+2, \dots$ on the internal edges arbitrarily.

Next, we assign a pair (p_e, ϵ_e) of elements of $\mathbb{C}^{1,3}$ to each edge

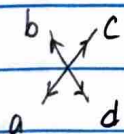
We also assign orientation: external edge - outward
internal edge - inward

- (2) To each vertex we assign the following expression



$$V_3(a, b, c) = \frac{i}{\sqrt{2}} (\epsilon_a \cdot \epsilon_b) (\epsilon_c \cdot (p_a - p_b)) + \text{cyclic}(a, b, c)$$

$(abc) \rightarrow (cab), (bca)$



$$V_4(a, b, c, d) = i (\epsilon_a \cdot \epsilon_c) (\epsilon_b \cdot \epsilon_d) - \frac{i}{2} (\epsilon_a \cdot \epsilon_b) (\epsilon_c \cdot \epsilon_d) - \frac{i}{2} (\epsilon_b \cdot \epsilon_c) (\epsilon_d \cdot \epsilon_a)$$

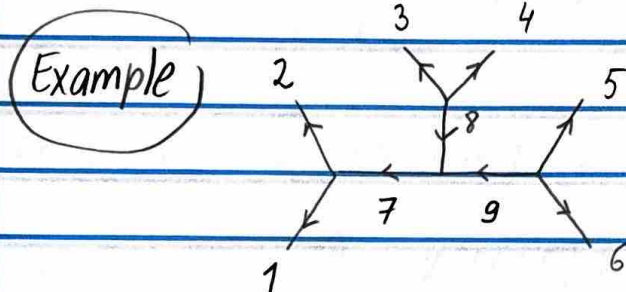
Here all edges are directed outward.

For every inward directed edge: reverse sign of p_e

use ϵ_a^* instead of ϵ_a

We use negative indices for ingoing edges: $\epsilon_{-a} = \epsilon_a^*$; $p_{-a} = -p_a$

③ Act w/ $-i \prod_{e \text{ internal edges}} \frac{-i}{p_e \cdot p_e} \frac{\partial}{\partial \epsilon_e} \frac{\partial}{\partial \epsilon_e^*}$ on product of vertex factors



$$-i \left(\prod_{e=7}^9 \frac{-i}{p_e \cdot p_e} \frac{\partial}{\partial \epsilon_e} \frac{\partial}{\partial \epsilon_e^*} \right) V_3(1, 2, -7) V_3(3, 4, 8) V_3(5, 6, 9) V_3(7, -8, -9)$$

④ Finally, eliminate internal p_e using momentum conservation

$$p_7 = p_1 + p_2$$

$$p_8 = -p_3 - p_4$$

$$p_9 = -p_5 - p_6$$

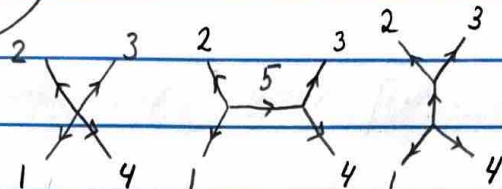
Note that internal ϵ_e have been already eliminated. this is because product of vertex factors is manifestly linear in each ϵ_e and $\epsilon_e^* \Rightarrow$ when we act w/ $\frac{\partial}{\partial \epsilon_e}$ & $\frac{\partial}{\partial \epsilon_e^*}$ they are eliminated

The result of this procedure is quantity

$$A_n(p_e, \epsilon_e)$$

only external p_e & ϵ_e

Example

 $n=4$ 

Let's evaluate the second diagram:

$$-i \frac{-i}{p_5 \cdot p_5} \frac{\partial}{\partial \epsilon_5} \cdot \frac{\partial}{\partial \epsilon_5^*} V_3(1, 2, 5) V_3(-5, 3, 4) =$$

$$\epsilon_1 \cdot \epsilon_2 (\cancel{\epsilon_5} \cdot (p_1 - p_2)) + \epsilon_2 \cdot \cancel{\epsilon_5} (\epsilon_1 \cdot (p_2 - p_5)) + \cancel{\epsilon_5} \cdot \epsilon_1 (\epsilon_2 \cdot (p_5 - p_1))$$

$$= \frac{1}{2} \frac{1}{p_5 \cdot p_5} (\epsilon_1 \cdot \epsilon_2 (p_1 - p_2) + \epsilon_2 (\epsilon_1 \cdot (p_2 - p_5)) + \epsilon_1 (\epsilon_2 \cdot (p_5 - p_1))) \cdot$$

$$\cdot (\epsilon_3 (\epsilon_4 \cdot (-p_5 - p_3)) + \epsilon_3 \cdot \epsilon_4 (p_3 - p_4) + \epsilon_4 (\epsilon_3 \cdot (p_4 + p_5))) =$$

$$\cancel{\epsilon_5} \cdot \epsilon_3 (\epsilon_4 \cdot (-p_5 - p_3)) + \epsilon_3 \cdot \epsilon_4 (\cancel{\epsilon_5} \cdot (p_3 - p_4)) + \epsilon_4 \cdot \cancel{\epsilon_5} (\epsilon_3 \cdot (p_4 + p_5))$$

$$p_5 = -p_1 - p_2 \quad (1^{st} \text{ line})$$

$$= \frac{1}{2} \frac{1}{(p_1 + p_2)^2} [\epsilon_1 \cdot \epsilon_2 (p_1 - p_2) + 2\epsilon_2 (\epsilon_1 \cdot p_2) - 2\epsilon_1 (\epsilon_2 \cdot p_1)] \cdot$$

$$p_5 = p_3 + p_4 \quad (2^{nd} \text{ line})$$

$$[-2\epsilon_3 (\epsilon_4 \cdot p_3) + \epsilon_3 \cdot \epsilon_4 (p_3 - p_4) + 2\epsilon_4 (\epsilon_3 \cdot p_4)]$$

Amplitude = sum over all graphs

$$n=4 \quad 3$$

$$n=5 \quad 10$$

$$n=6 \quad 38 \leftarrow \text{see figure 3 of our lectures. (9 topologies)}$$

used "color-ordered", not "color-ordered" $\Rightarrow$
 "cyclically ordered" or

220)

slides

for $n=5$ 25 not color ordered

Helicity Amplitudes

- So far amplitudes were functions of (p_i, ϵ_i) denoted by

$$A_n(p_i, \epsilon_i)$$

- Now let us introduce new variables $(\lambda_i, \tilde{\lambda}_i, h_i)$ called spinor helicity variables

- Helicity Amplitudes are functions of $(\lambda_i, \tilde{\lambda}_i, h_i)$, denoted by

$$A_n(\lambda_i, \tilde{\lambda}_i, h_i)$$

- * $(\lambda_i, \tilde{\lambda}_i)$ are called spinors. They are non-zero elements of $\mathbb{C}^2$

Think of them as 2×1 matrices $\lambda_i = (\lambda_i^1, \lambda_i^2)^T$ and

column

$$1 \times 2 \text{ matrices } \tilde{\lambda}_i = (\tilde{\lambda}_i^1, \tilde{\lambda}_i^2)$$

row

subject to $\sum_{i=1}^n \lambda_i \tilde{\lambda}_i = 0$

- * (h_i) are called helicities. They are elements of $\mathbb{Z}/2$

For gluons, take $h_i = + \text{ or } -$

* Helicity amplitudes must satisfy:

(1) homogeneity in spinor helicity variables

$$A_n(z_i \lambda_i, \tilde{z}_i \tilde{\lambda}_i, h_i) = A_n(\lambda_i, \tilde{\lambda}_i, h_i) \prod_{i=1}^n z_i^{-2h_i} \quad \forall z_i \in \mathbb{C}^*$$

(2) Lorentz invariance

$$A_n(R\lambda_i, \tilde{R}\tilde{\lambda}_i, h_i) = A_n(\lambda_i, \tilde{\lambda}_i, h_i) \quad \forall R, \tilde{R} \in SL(2, \mathbb{C})$$

(recall: $so_0(3,1) \simeq sl_2(2) \oplus sl_2(2)$)

* We can think of any helicity amplitude as function of Lorentz invariant quantities defined by

$$\langle ij \rangle \equiv \det \begin{pmatrix} \lambda_i & \lambda_j \\ \epsilon_{ab} \lambda_i^a \lambda_j^b \end{pmatrix} \quad ; \quad [ij] \equiv \det \begin{pmatrix} \tilde{\lambda}_i & \tilde{\lambda}_j \\ \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}} \end{pmatrix}$$

* Helicity amplitudes contain the same information as amplitudes we discussed before.

$$A(p_i, \epsilon_i) \leftrightarrow A(\lambda_i, \tilde{\lambda}_i, h_i)$$

$\rightarrow h_i = + \text{ or } - \text{ for gluons}$

* The identification is

$$p_i = \lambda_i \tilde{\lambda}_i \quad (1^-) = \dots$$

(see over)

$$\epsilon_i^+ = \sqrt{2} \frac{\tilde{\lambda}_i \tilde{\lambda}_i}{\langle i i \rangle} \quad \text{arbitrary spinors}$$

$$\epsilon_i^- = -\sqrt{2} \frac{\lambda_i \tilde{\lambda}_i}{[i i]}$$

$$\langle ir \rangle = \det(\lambda_i; r_i); \quad [ir] = \det(\tilde{\lambda}_i^T; \tilde{r}_i^T)$$

(*)

In $(p_i = \lambda_i \tilde{\lambda}_i)$ we adopt the identification

$$a = (a^0, a^1, a^2, a^3) \leftrightarrow \begin{pmatrix} a^0 - a^3 & a^1 + ia^2 \\ a^1 - ia^2 & a^0 + a^3 \end{pmatrix} \quad 4\text{-vector} \leftrightarrow \overset{2 \times 2}{\text{matrix}}$$

Note 1

$$a \cdot a = \det \begin{pmatrix} a^0 - a^3 & a^1 + ia^2 \\ a^1 - ia^2 & a^0 + a^3 \end{pmatrix} = (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 \quad \checkmark$$

Note 2

$$p_i \cdot p_j = \frac{1}{2} \langle ij \rangle [ij]$$

Comment 1

dependence on $r_i / \tilde{r}_i$ drops out because if I change say $\tilde{r}_i \rightarrow \tilde{r}_i + \epsilon \tilde{r}_i + \epsilon' \tilde{\lambda}_i$
 (space of possible $\tilde{r}$ is 2 dim, so any variation is of form $\uparrow$) changes by $\tilde{\lambda}_i \cdot \tilde{\lambda}_i$
drops out since $\frac{\lambda_i \tilde{r}_i}{[ir]}$ is invariant

Comment 2

if we rescale $\lambda_i \rightarrow t_i \lambda_i$ and $\tilde{\lambda}_i \rightarrow t_i^{-1} \tilde{\lambda}_i$ for any $t_i \in \mathbb{C}^*$,

then p_i is not changed
 $\epsilon_i^{h_i}$ scales as $t_i^{-2h_i}$ w/ $h_i = \pm 1$

this explains the relation between p.6 formula (1) p.1 formula (1) (nearby)

Comment 3

For simplicity, choose $r_i \equiv r$ and $\tilde{r}_i = \tilde{r} \Rightarrow$

$$\epsilon_i^+ \cdot \epsilon_j^- = - \frac{\langle jr \rangle [ir]}{\langle ir \rangle [jr]} ; p_i \cdot \epsilon_j^+ = \frac{\langle ir \rangle [ji]}{\sqrt{2} \langle jr \rangle} ; p_i \cdot \epsilon_j^- = \frac{[ir] \langle jk \rangle}{\sqrt{2} [jr]}$$

$$A_3 = \frac{1}{\sqrt{2}} (\epsilon_1 \cdot \epsilon_2) (\epsilon_3 \cdot (p_1 - p_2)) + \text{cyclic}(1,2,3)$$

(8)

3-point amplitudes

Three-point helicity amplitudes are special because spinor helicity variables are highly constrained.

Specifically, note that

$$\langle 12 \rangle [12] = 2 p_1 \cdot p_2 = (p_1 + p_2)^2 = p_3^2 = 0$$

$$(\text{we used } p_i \cdot p_j = \frac{1}{2} \langle ij \rangle [ij]; p_1 + p_2 + p_3 = 0; p_1^2 = p_2^2 = p_3^2 = 0)$$

$\Rightarrow$ we must have either $\langle 12 \rangle = 0$ or $[12] = 0$

• if we repeat the same for 183; 283 $\Rightarrow$

$\Rightarrow$ either all $\langle ij \rangle = 0$ or all $[ij] = 0$ if $n=3$

• Straightforward (tedious) to check that

- if all $\langle ij \rangle = 0 \Rightarrow$ only $A_3(++-) \neq 0$

- if all $[ij] = 0 \Rightarrow$ only $A_3(--+) \neq 0$

• Let's workout $A(1^- 2^- 3^+)$ ($[ij] = 0$)

(see lectures and exercises for details)

$$\text{Start with } A(1^-, 2^-, 3^+) = \frac{1}{\sqrt{2}} (\epsilon_2^- \cdot \epsilon_3^+) (\epsilon_1^- \cdot (p_2 - p_3)) + \epsilon_1^+ \cdot p_i \sim [ij] = 0 \left(\begin{array}{l} \text{only 2 terms} \\ (\epsilon_1^+ \cdot \epsilon_2^-) \sim \epsilon_3^+ \cdot (p_1 - p_2) = 0 \end{array} \right) + (\epsilon_3^+ \cdot \epsilon_1^-) (\epsilon_2^- \cdot (p_3 - p_1)) \Rightarrow$$

$$(\epsilon_2^- \cdot \epsilon_3^+) (\epsilon_1^- \cdot (p_2 - p_3)) = 2 (\epsilon_2^- \cdot \epsilon_3^+) (\epsilon_1^- \cdot p_2)$$

$\downarrow p_3 = -p_1 - p_2$ $(p_1 \cdot \epsilon_1^- = 0)$

(9)

$$\Rightarrow \sqrt{2} (\epsilon_2^- \cdot \epsilon_3^+) (\epsilon_1^- \cdot p_2) - \sqrt{2} (\epsilon_3^+ \cdot \epsilon_1^-) (\epsilon_2^- \cdot p_1) =$$

see notes/exercises

$$= \frac{\langle 12 \rangle [3r]^2}{[1r][2r]} = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

see notes/exercises

$$\frac{[3r]}{[1r]} = \frac{\langle 12 \rangle}{\langle 23 \rangle}$$

and

$$\frac{[3r]}{[2r]} = \frac{\langle 12 \rangle}{\langle 31 \rangle}$$

because $\langle 23 \rangle [3r] = -\langle 21 \rangle [1r]$
due to momentum
conservation

$$\langle 211+2+31r \rangle = 0$$

So

$$\boxed{\begin{aligned} A(1^-, 2^-, 3^+) &= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \\ A(1^+, 2^+, 3^-) &= \frac{[12]^3}{[23][31]} \end{aligned}}$$

MHV amplitudes

Let me state w/o proof for now.

$$n > 3 \quad A_n(++ \dots +) = A_n(- \dots -) = 0$$

$$A_n(-+ \dots +) = A_n(+---) = 0$$

(mention
one-minus A1 hype)

$$\boxed{A_n(1^+ \dots i^-, \dots j^-, \dots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}}$$

6-point NMHV amplitudes

MHV	2 minuses	$\overline{\text{MHV}}$	2 pluses
NMHV	3 minuses		
$N^k \text{MHV}$	$(k+2)$ minuses		
$(n=4)$	$++--$, $--++$	(see exercise 2)	MHV
$(n=5)$	$--+++$ or $+++--$		$\overline{\text{MHV}}$
	MHV		$\overline{\text{MHV}}$
$(n=6)$	$--++++$		MHV

3 types NMHV

$\left\{ \begin{array}{l} ---+++ \\ +-+--+ \\ ++-+- - \end{array} \right.$

$$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = \frac{[4|P_{32}|1\rangle^3}{[23][34][2|P_{34}|5\rangle\langle 56\rangle\langle 61\rangle S_{234}} + \frac{[6|P_{54}|3\rangle^3}{[61][12][2|P_{34}|5\rangle\langle 34\rangle\langle 45\rangle S_{612}}$$

$(P_3+P_2)^2$
 $[23]\langle 35\rangle + [24]\langle 45\rangle$
 $S_{234} \rightarrow (P_2+P_3+P_4)^2$
 $\langle 23\rangle[23] + \langle 34\rangle[34] + \langle 24\rangle[24]$

All other $n=6$ NMHV amplitudes can be obtained from these by some cyclic rotation of indices possibly accompanied by parity conjugation.

How to derive these formulas? BCFW recursion relations.

BCFW recursion relations

Britto, Cachazo, Feng hep-th/0412328

choice without loss of generality

Consider $A_n(1, 2, \dots, (n-1)^-, n^+)$, then they claim that the following recursion relation is valid:

$$A_n(1, 2, \dots, (n-1)^-, n^+) = \sum_{i=1}^{n-3} \sum_{n=i+1} (A_{i+2}(\hat{n}, 1, 2, \dots, i, -\hat{p}_{n,i}^h) \frac{1}{p_{n,i}^2} A_{n-i}(\hat{p}_{n,i}^{-h}, i+1, \dots, n-2, \hat{n-1})),$$

where

$$p_{n,i} = p_n + p_i + p_i$$

$$\hat{p}_{n,i} = p_{n,i} + \frac{p_{n,i}^2}{\langle n-1 | p_{n,i} | n \rangle} \tilde{\lambda}_{n-1} \tilde{\lambda}_n$$

$$\hat{p}_{n-1} = p_{n-1} - \frac{p_{n,i}^2}{\langle n-1 | p_{n,i} | n \rangle} \tilde{\lambda}_{n-1} \tilde{\lambda}_n$$

$$\hat{p}_n = p_n + \frac{p_{n,i}^2}{\langle n-1 | p_{n,i} | n \rangle} \tilde{\lambda}_{n-1} \tilde{\lambda}_n$$

$$\langle i | \sum_r p_r | j \rangle = \sum_r \langle i | r \rangle \langle r | j \rangle$$

(Note)

- same shift

$$-\hat{p}_{n,i}^2 = \hat{p}_n^2 = \hat{p}_{n-1}^2 = 0$$

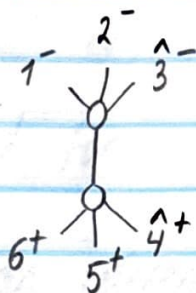
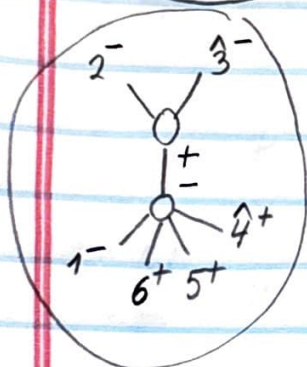
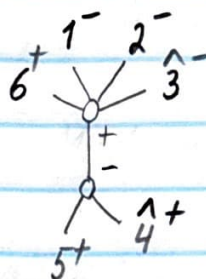
$$\rightarrow p_{n,i}^2 + 2 p_{n,i}^2 \frac{\langle n-1 | p_{n,i} | n \rangle}{\langle n-1 | p_{n,i} | n \rangle} \left(-\frac{1}{2}\right)$$

$$-\lambda_{\hat{n-1}} = \lambda_{n-1} \quad \tilde{\lambda}_{\hat{n-1}} = \tilde{\lambda}_{n-1} - \frac{p_{n,i}^2}{\langle n-1 | p_{n,i} | n \rangle} \tilde{\lambda}_n$$

$$-\lambda_{\hat{n}} = \lambda_n + \frac{p_{n,i}^2}{\langle n-1 | p_{n,i} | n \rangle} \lambda_{n-1} \quad \tilde{\lambda}_{\hat{n}} = \tilde{\lambda}_n$$

(obvious from $p_{n-1} = \lambda_{n-1} \tilde{\lambda}_{n-1}$; $p_n = \lambda_n \tilde{\lambda}_n$)

Example 1

 $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ ||
zeroflip indices & conjugate
 $i \rightarrow i+3$ $\langle \rangle \rightarrow []$

let's compute this diagram

$$\frac{\langle 23 \rangle^3}{\langle 3\hat{P} \rangle \langle \hat{P}2 \rangle} \frac{1}{(p_2 + p_3)^2} \frac{\langle 1\hat{P} \rangle^3}{\langle \hat{P}4 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} =$$

Note: $\lambda_3 = \lambda_3$, $\lambda_4 = \lambda_4 - \frac{(p_2 + p_3)^2}{\langle 32 \rangle \langle 24 \rangle} \lambda_3$

$$\langle \cdot \hat{P} \rangle = - \frac{\langle \cdot | 2+3 | 4 \rangle}{[\hat{P}4]} \quad (\langle \cdot \hat{P}_{n,i} \rangle [\hat{P}_{n,i} n] = \langle \cdot | P_{n,i} | n \rangle)$$

$$= \frac{\langle 1 | 2+3 | 4 \rangle^3}{[23][34] \langle 56 \rangle \langle 61 \rangle (p_2 + p_3 + p_4)^2 \langle 5 | 3+4 | 2 \rangle}$$

exercise

Adding the second term gives formula on page (10)

Example 2

MHV n-point

$$A(1^-, 2^+, \dots, (j-1)^+, (j)^-, (j+1)^+, \dots, (n-1)^+, n^+)$$

Assume its valid for all MHV w/ less than n gluons.

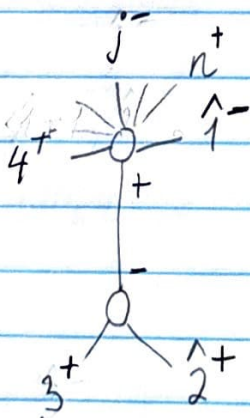
$\hat{1}^-, \hat{2}^+$ chose as reference gluons $\Rightarrow$ only one term non-zero.

$$A(4, 5, \dots, n, \hat{1}^-, \hat{2}^+, 3) \Rightarrow \frac{\langle j \hat{1} \rangle^4}{\langle 45 \rangle \dots \langle n-1, n \rangle \langle n \hat{1} \rangle \langle \hat{1} \hat{p} \rangle \langle \hat{p} 4 \rangle} \cdot \frac{1}{\langle 23 \rangle [23]} \cdot \frac{[\hat{2} 3]^3}{[3 \hat{p}] [\hat{p} 2]} \quad (p = p_2 + p_3)$$

$$\Rightarrow \text{we get } A(1^-, 2^+, \dots, (j-1)^+, j^-, (j+1)^+, \dots, (n-1)^+, n^+) =$$

$$= \frac{\langle 1j \rangle^4}{\langle 45 \rangle \langle 56 \rangle \dots \langle n-1, n \rangle \langle n \hat{1} \rangle \langle \hat{1} 2 \rangle \langle 23 \rangle \langle 34 \rangle} \quad \checkmark$$

(see exercise)



$$\frac{1}{\langle \hat{1} \hat{p} \rangle \langle \hat{p} 4 \rangle} \frac{1}{\langle 23 \rangle [23]} \frac{[\hat{2} 3]^3}{[3 \hat{p}] [\hat{p} 2]} \rightarrow$$

$$\rightarrow \frac{1}{\langle 1 \hat{1} \hat{p} 3 \rangle} \cdot \frac{1}{\langle 4 \hat{1} \hat{p} 2 \rangle} \frac{[\hat{2} 3]^2}{\langle 23 \rangle} \rightarrow \frac{1}{\langle 12 \rangle [23]} \frac{1}{\langle 43 \rangle [32]} \quad \otimes$$

$$\otimes \frac{[\hat{2} 3]^2}{\langle 23 \rangle} \rightarrow \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle}$$