

# Positive Geometries and Canonical Forms: Exercises

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This is an exercise sheet accompanying a minicourse taught in June 2026 at the Nordfjordeid Summer School on *Positive Geometry and Scattering Amplitudes*. The lectures are based on the following list of references.

## References

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- [2] F. Brown and C. Dupont. Positive geometries and canonical forms via mixed Hodge theory. *Communications in Mathematical Physics*, 406(11):267, 2025.
- [3] K. Kohn, R. Pien, K. Ranestad, F. Rydell, B. Shapiro, R. Sinn, M. Ștefana Sorea, and S. Telen. Adjoints and canonical forms of polypols. *Documenta Mathematica*, 30(2):275–346, 2025.
- [4] K. Kohn and K. Ranestad. Projective geometry of Wachspress coordinates. *Foundations of Computational Mathematics*, 20(5):1135–1173, 2020.
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- [7] B. Sturmfels and S. Telen. Positive geometries from cubic surfaces. 2026.
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## 1 Positive arrangements

**Exercise 1.** Let  $X$  be an irreducible complex projective variety, defined over  $\mathbb{R}$ . Show that  $X$  is totally real, i.e.,  $X_{\mathbb{R}}$  is Zariski dense in  $X$ , if and only if  $X$  has a smooth real point.  $\diamond$

**Exercise 2.** Is  $(X, Y)$  with  $X = \mathbb{P}^2$  and  $Y = V(x_0^2 + 2x_0x_1 + 2x_1^2 + 2x_1x_2 + x_2^2)$  a real arrangement?  $\diamond$

**Exercise 3.** Let  $X = \mathbb{P}^d$  and  $Y \subset X$  is a union of hyperplanes. Show that  $g(X, Y) = 0$ . Conclude that, if  $Y$  is defined over  $\mathbb{R}$ , then  $(X, Y)$  is a positive arrangement. The definition is given in [7, Definition 5.2].  $\diamond$

**Exercise 4.** Let  $f = x_0x_1x_2 + x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3$ . Show that the cubic surface  $V(f) \subset \mathbb{P}^3$  has four isolated singularities. This surface is called *Cayley's cubic surface*. Show that  $(X, Y)$  with  $X = \mathbb{P}^3$ ,  $Y = V(x_0x_1x_2x_3f)$  is a positive arrangement.  $\diamond$

**Exercise 5.** Let  $X$  be a nodal cubic plane curve and let  $Y$  be its singular point. Show that  $g(X, Y) = 0$  using [2, Proposition 3.5] and the results stated in the lecture.  $\diamond$

**Exercise 6.** Let  $X = \mathbb{P}^2$  and let  $Y$  be a union of rational curves. Show that  $g(X, Y) = 0$  using [2, Proposition 3.5] and the results stated in the lecture.  $\diamond$

## 2 Positive geometries and canonical forms

**Exercise 7.** Let  $X = V(-x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2$  and let  $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$ . Show that

$$\left( \frac{1}{1-x-y} \frac{dx}{y} \right)_{|X} = \frac{1}{t(t-1)} dt.$$

Here  $t$  is the local coordinate on  $X$  given by  $t \mapsto (t^2 + 1 : 2t : 1 - t^2)$ . See [8, Example 1.9].  $\diamond$

**Exercise 8.** Verify the following identity:

$$\text{Res}_{x=0} \text{Res}_{y=0} \frac{dx \wedge dy}{xy} = -\text{Res}_{y=0} \text{Res}_{x=0} \frac{dx \wedge dy}{xy}. \quad \diamond$$

**Exercise 9.** Compute the residue

$$\text{Res}_{x+2y-3=0} \left( \frac{-18 + x + 12y}{x(y-1)(x+2y-3)(x-3y-3)} dx \wedge dy \right). \quad \diamond$$

**Exercise 10.** Show that the following expression represents a 3-form on  $\mathbb{P}^3$ :

$$\omega = \frac{1}{x_3(x_0x_1x_2 + x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3)} \omega_{\mathbb{P}^3}.$$

Here  $\omega_{\mathbb{P}^3} = x_0 dx_1 \wedge dx_2 \wedge dx_3 - \dots - x_3 dx_0 \wedge dx_1 \wedge dx_2$ . Compute the residue of  $\omega$  along the Cayley cubic  $V(f)$ , using the following local coordinates:  $(u, v) \mapsto (1 : u : v : \frac{-uv}{u+v+uv})$ .  $\diamond$

**Exercise 11.** Show that the form in Exercise 9 is the canonical form of the quadrilateral  $\sigma$  with vertices  $(0, 1), (1, 1), (3, 0), (0, -1)$ , which is a region in a positive arrangement of four lines in  $\mathbb{P}^2$ .  $\diamond$

**Exercise 12.** Let  $X = \mathbb{P}^2$  and  $Y = V(x_1x_2(-x_0^2 + x_1^2 + x_2^2))$ . Show that  $(X, Y)$  is a positive arrangement. How many regions does this arrangement have? Can you find the canonical form of the region  $\sigma$  containing the point  $(x_0 : x_1 : x_2) = (1 : \frac{1}{2} : \frac{1}{2})$ ? The orientation is induced by the standard orientation of  $\mathbb{R}^2 = \mathbb{RP}^2 \setminus \{x_0 = 0\}$ . Same question for the region  $\sigma'$  containing  $(x_0 : x_1 : x_2) = (1 : -\frac{1}{2} : -\frac{1}{2})$ , and for the positive geometry  $\sigma + \sigma'$ .  $\diamond$

### 3 Polytopes

**Exercise 13.** Compute the canonical form of the pentagon  $P \subset \mathbb{R}^2$  with vertices  $(1, 0), (3, 0), (4, 2), (2, 4), (0, 2)$ .  $\diamond$

**Exercise 14.** The toric amplitude [9] of a simple polytope  $P$  is defined as

$$\text{Amp}_P(x) = \sum_{v \in \mathcal{V}(P)} \frac{|\det U_v|}{\prod_{v \in F} x_F},$$

where  $P = \{y \in \mathbb{R}^d : U \cdot y + z \geq 0\}$  is a minimal facet representation and the product in the denominator is over all facet variables  $x_F$  for which the vertex  $v$  belongs to the facet  $F$ . The polar dual of  $P$  is  $P^\circ = \{u \in (\mathbb{R}^d)^\vee : u \cdot y \geq -1 \text{ for all } y \in P\}$ . Prove that, if  $0 \in \text{int}(P)$ , then  $\text{Amp}_P(z) = \text{vol}(P^\circ)$ . More generally,  $\text{Amp}_P(Uy + z) = \text{vol}(P - y)^\circ$ . Here “vol” denotes the normalized volume, which is the Euclidean volume divided by  $d!$ .  $\diamond$

**Exercise 15.** What is the degree of the adjoint curve of an  $n$ -gon in terms of  $n$ ? Compute the adjoint curve of a hexagon of your choice via interpolation.  $\diamond$

**Exercise 16.** Compute the canonical form of the unit cube  $P = [0, 1]^3$ . What is its adjoint surface in  $\mathbb{P}^3$ ?  $\diamond$

### 4 Combinatorial rank

**Exercise 17.** Compute the (real) combinatorial rank of the positive arrangement  $(X, Y)$  from Exercise 11.  $\diamond$

**Exercise 18.** What is the combinatorial rank of a generic arrangement of  $m$  lines and  $n$  smooth conics in  $X = \mathbb{P}^2$ ? Compute this via the dual complex, and as the dimension of  $H^0(X, K_X + Y)$ .  $\diamond$