

Nordfjordeid lectures on Positive Geometries and canonical forms

§5: Case studies

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Adjoints and canonical forms of polypols

Kathlén Kohn, Ragni Piene, Kristian Ranestad, Felix Rydell,
Boris Shapiro, Rainer Sinn, Miruna-Ştefana Sorea, and Simon Telen

Positive Geometries from Cubic Surfaces

Bernd Sturmfels and Simon Telen

Amplituhedra of genus 0

Positive Genus Pairs from Amplituhedra

Joris Koefer, Dmitrii Pavlov, Rainer Sinn

Theorem 3.1. *Let Z be a generic $n \times (k + m)$ matrix with positive maximal minors. Then, for any choice of integers k , m , and $n \geq k + m$, satisfying either $k = 1$, $k + m = n$, or $k = m = 2$, the boundary divisor $\partial_a \mathcal{A}_{n,k,m} \subset \text{Gr}(k, k + m)$ of the amplituhedron makes a genus zero pair with the ambient Grassmannian. That is,*

$$g(\text{Gr}(k, k + m), \partial_a \mathcal{A}_{n,k,m}) = 0$$

for the choices of k, n and m as above.

Theorem 4.1. *Let n and k be positive integers such that $n \geq 2(2k - 1)$. Further let Z be generic and have positive maximal minors, and $\partial_a \mathcal{A}_{n,k,2}(Z) \subset \text{Gr}(k, k + 2)$ be the algebraic boundary of the amplituhedron. Then, the genus of the pair $(\text{Gr}(k, k + 2), \partial_a \mathcal{A}_{n,k,2}(Z))$ is at least $1 + \frac{k-3}{2}C_k$, where C_k is the k -th Catalan number. In particular, for $k \geq 3$ it is positive.*

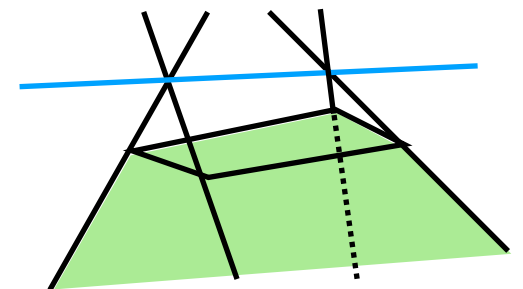
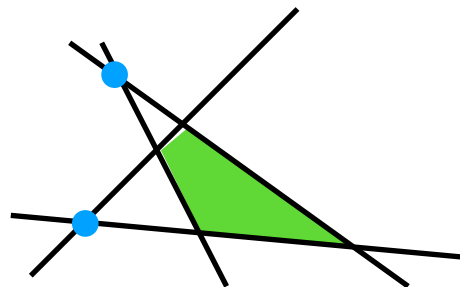
Adjoints of simple polytopes

$P \subset \mathbb{RP}^d$ simple convex polytope

$\overline{\partial P} = H_1 \cup \cdots \cup H_k$ simple hyperplane arrangement

$\mathcal{R}(P) = \{ \text{flats of } \overline{\partial P} \text{ which do not contain a face of } P \}$

“residual arrangement”

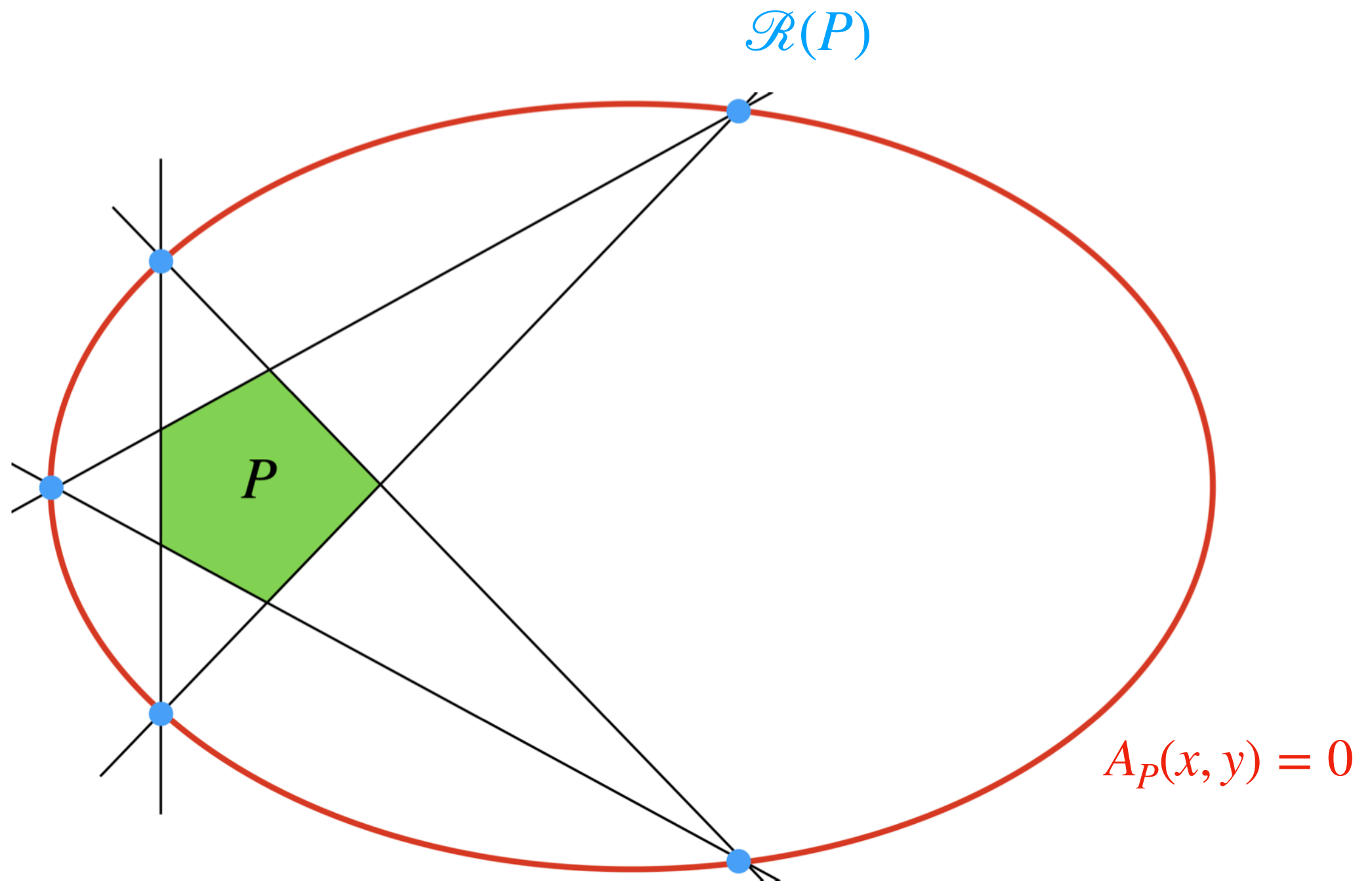


$A \in \mathbb{R}[x_0, \dots, x_d]_{k-d-1}$ is an **adjoint polynomial** of P if $A \in I(\mathcal{R}(P))$

$V(A) \subset \mathbb{RP}^d$ is an **adjoint hypersurface** of P

Theorem (Kohn-Ranestad '19) P has a unique adjoint hypersurface.

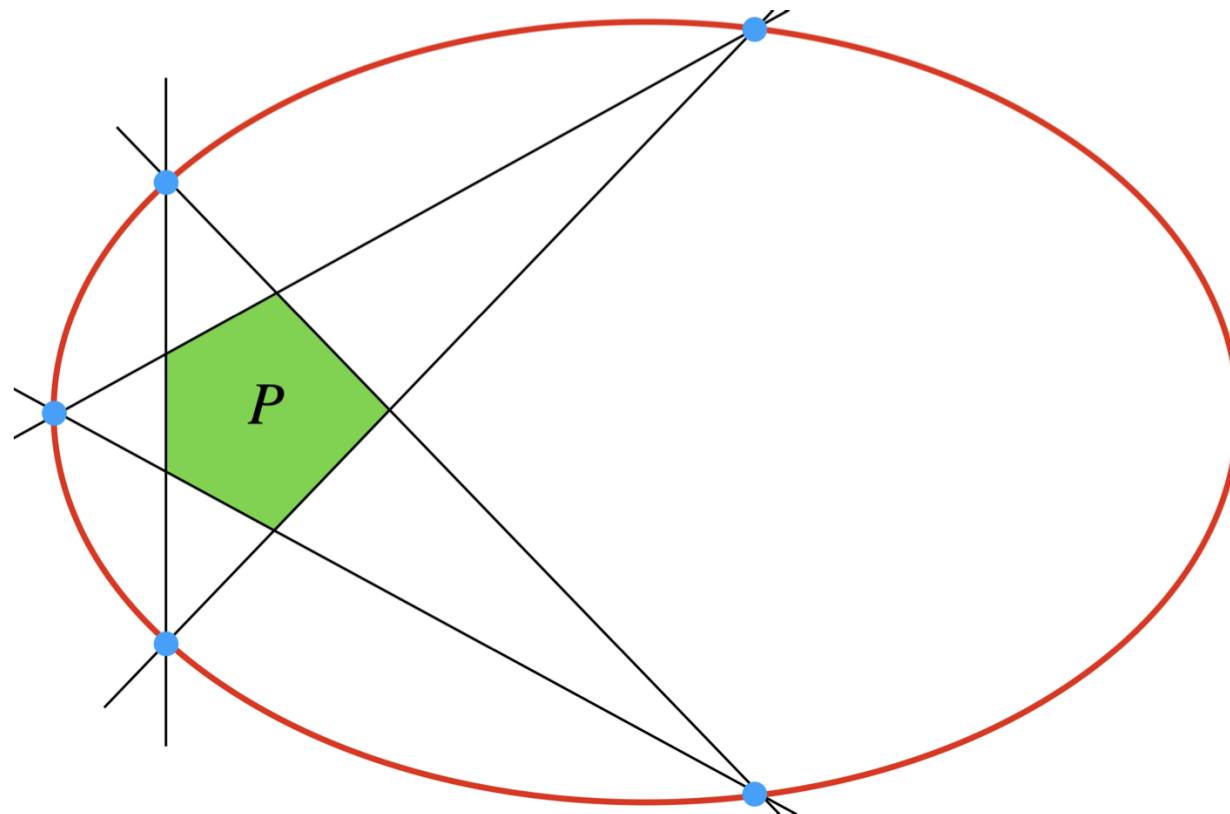
Adjoints of simple polytopes



Adjoints of polygons

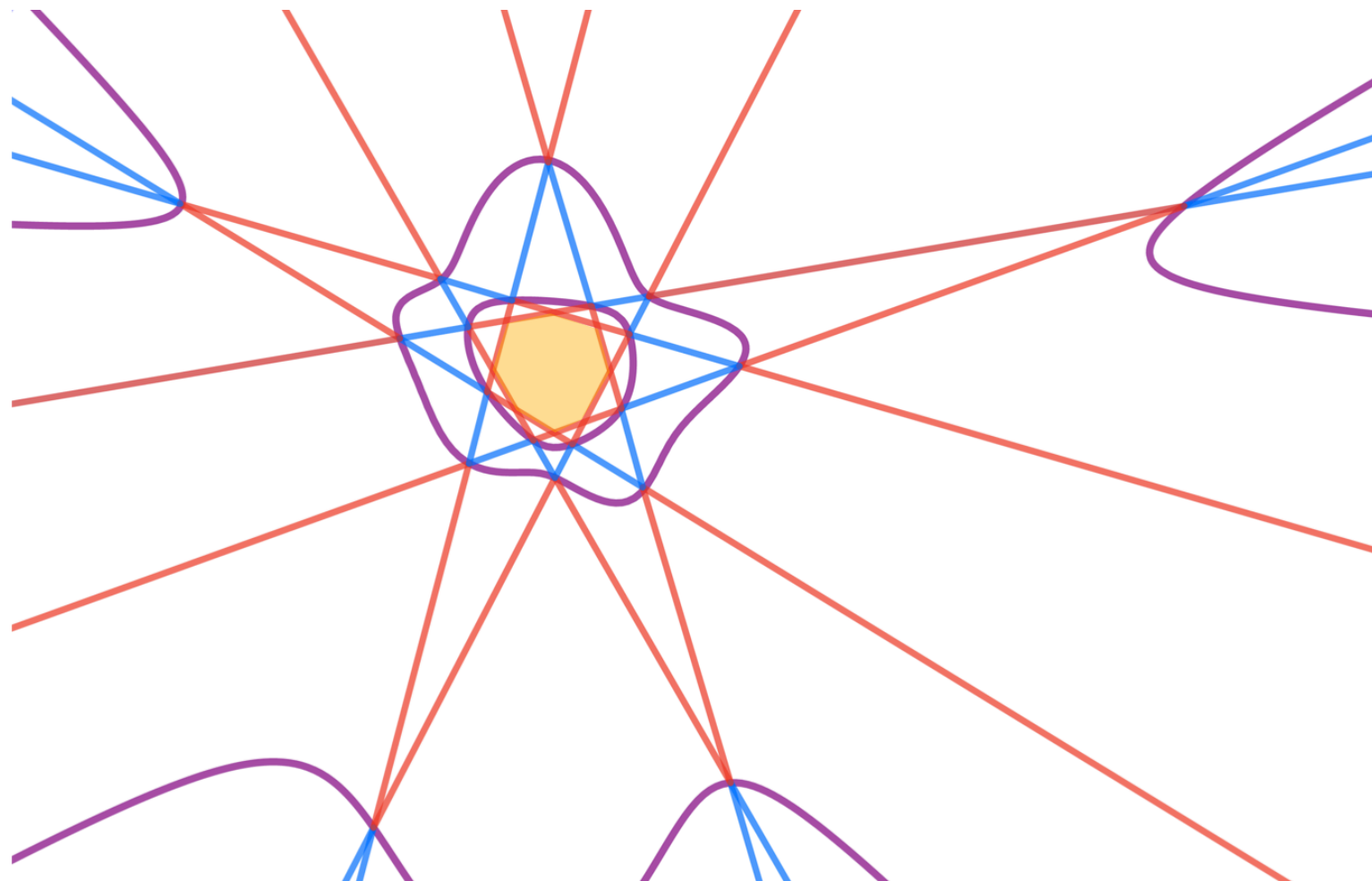
The adjoint of a k -gon $P \subset \mathbb{RP}^2$ is a curve of degree $k - 3$

$$|\mathcal{R}(P)| = \binom{k}{2} - k \quad h^0(\mathbb{P}^2, \mathcal{O}(k-3)) = \binom{k-1}{2} = \binom{k}{2} - k + 1$$



Adjoints of polygons

Theorem (KPRSSST '21) The adjoint curve $V(A_P)$ of a k -gon $P \subset \mathbb{RP}^2$ is hyperbolic with respect to any point in P . It has the maximal number of $\lfloor \frac{k-3}{2} \rfloor$ real nested ovals.



Adjoints of polygons

A complex k -gon P is a point in $\frac{(\mathbb{P}^2)^\vee \times \cdots \times (\mathbb{P}^2)^\vee}{D_k}$

The *adjoint map* sends P to its adjoint polynomial:

$$\alpha_k : \frac{(\mathbb{P}^2)^\vee \times \cdots \times (\mathbb{P}^2)^\vee}{D_k} \dashrightarrow \mathbb{P}\mathbb{C}[x, y, z]_{k-3} \quad \alpha_k(P) = A_P$$

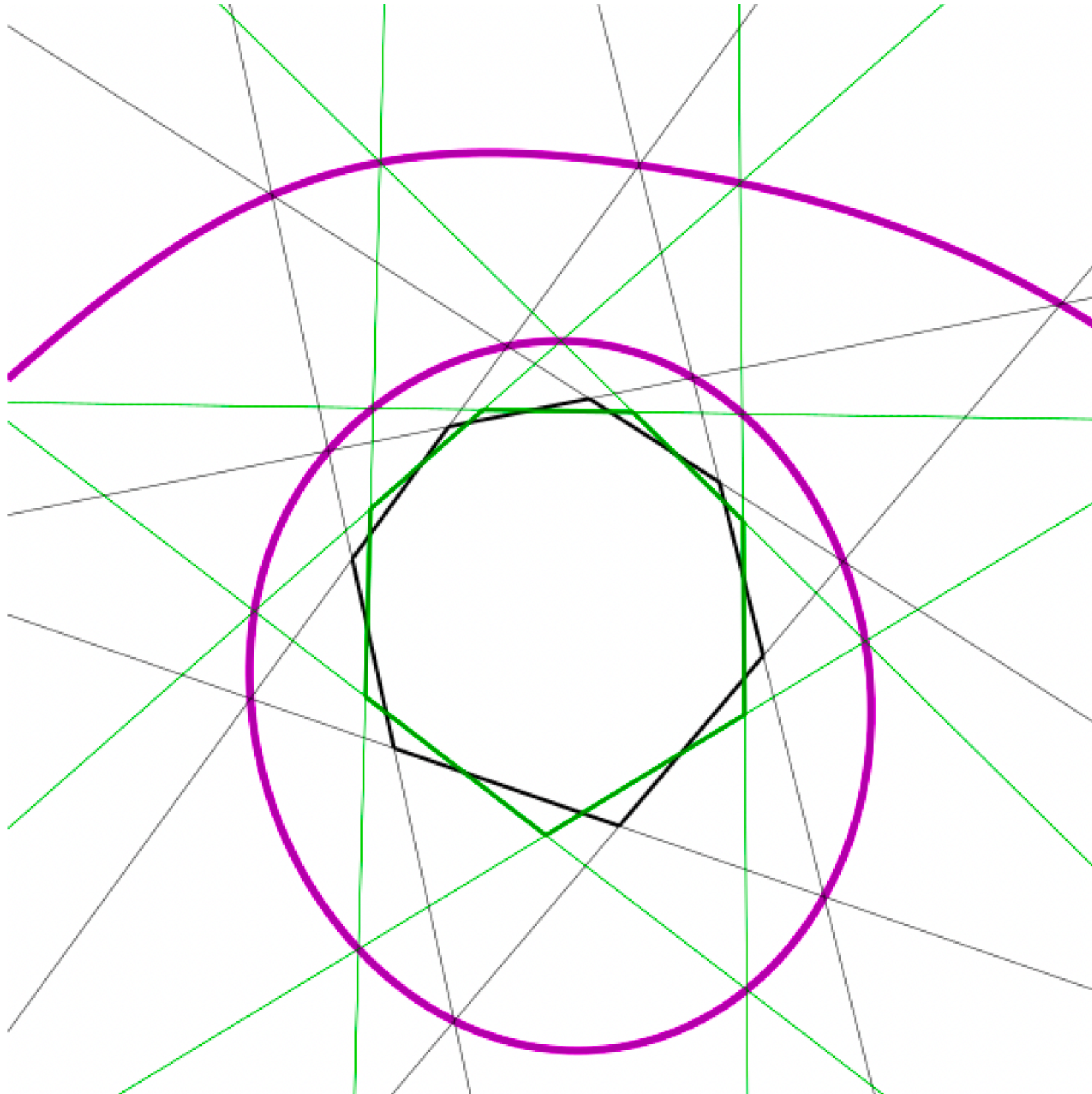
$\dim 2k$
 $\dim \binom{k-1}{2} - 1$

Theorem (KPRSSST '21) α_7 is dominant and generically finite of degree at least 864.

Theorem (Agostini, Plaumann, Sinn, Wesner '24) $\deg(\alpha_7) = 864$

We do not understand the image of α_8

Adjoints of polygons



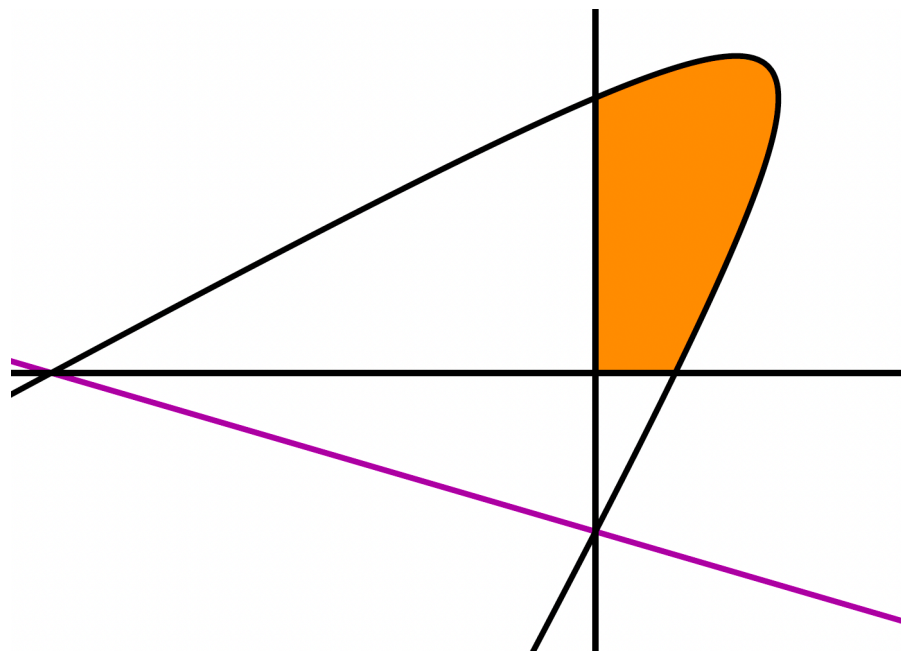
Adjoints of rational polypols

Data defining a **rational polypol** P in $\mathbb{P}^2$:

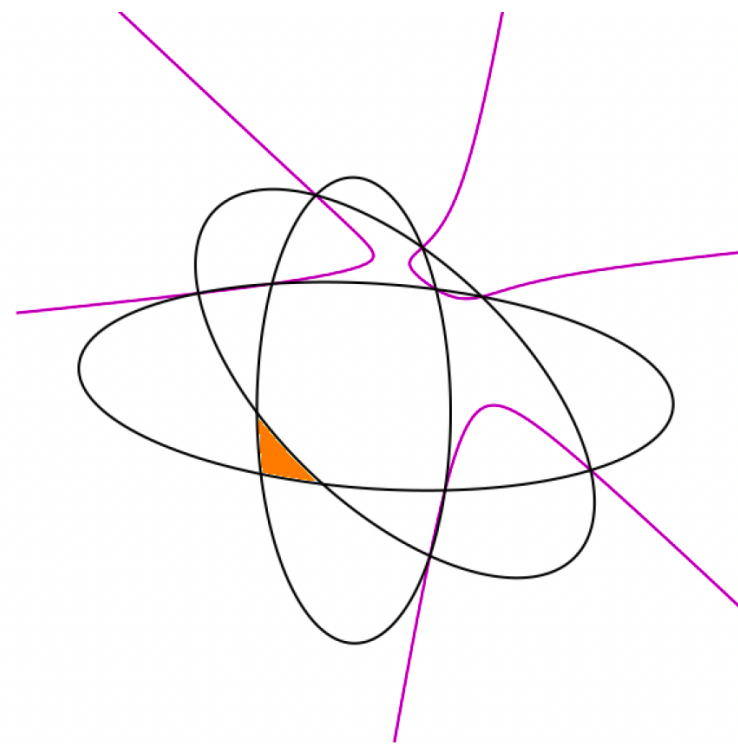
k distinct rational plane curves $C_1, \dots, C_k$ of degree $d_1, \dots, d_k$

k vertices $v_{i,i+1} \in C_i \cap C_{i+1}$, transverse intersections

Assume that $C = C_1 \cup C_2 \cup \dots \cup C_k$ has only nodal singularities



type (1,1,2)



type (2,2,2)

Adjoints of rational polypols

Data defining a **rational polypol** P in $\mathbb{P}^2$:

k distinct rational plane curves $C_1, \dots, C_k$ of degree $d_1, \dots, d_k$

k vertices $v_{i,i+1} \in C_i \cap C_{i+1}$, transverse intersections

Assume that $C = C_1 \cup C_2 \cup \dots \cup C_k$ has only nodal singularities

$\mathcal{R}(P) = \text{Sing}(C) \setminus \{v_{1,2}, \dots, v_{k-1,k}, v_{1,k}\}$ “**residual arrangement**”

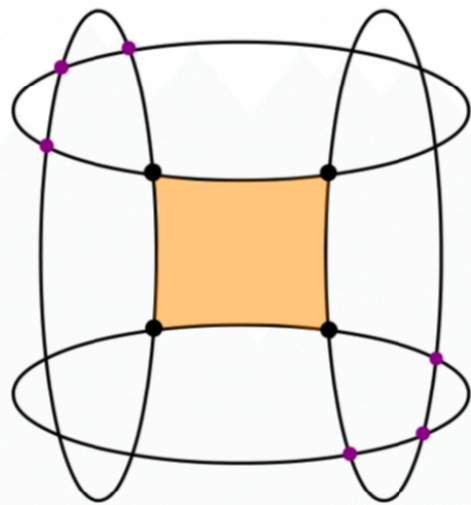
$A \in \mathbb{C}[x, y, z]_{d_1+\dots+d_k-3}$ is an **adjoint polynomial** of P if $A \in I(\mathcal{R}(P))$

$V(A) \subset \mathbb{P}^2$ is an **adjoint curve** of P

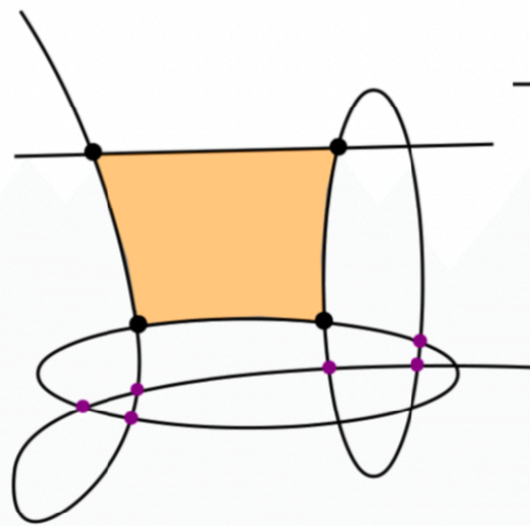
Theorem (KPRRSSST '21) P has a unique adjoint curve.

Adjoints of rational polypols

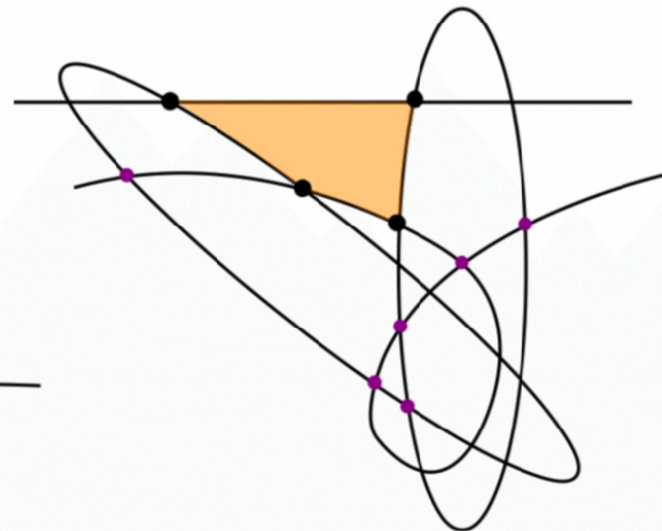
The adjoint map sends $(C_1, \dots, C_k, v_{1,2}, \dots, v_{1,k}) \mapsto A_P$



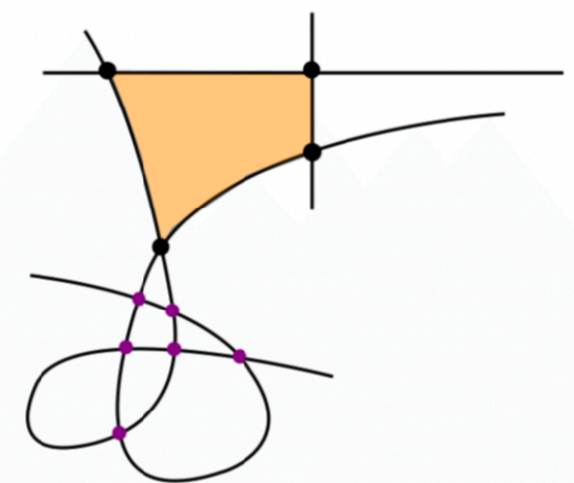
(2,2,2,2)



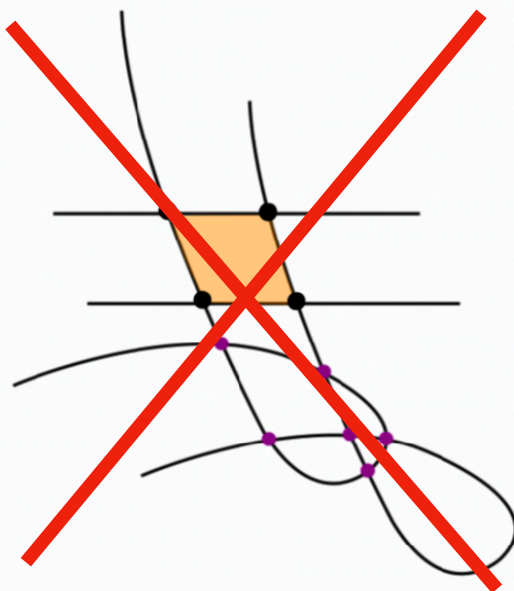
(1,2,2,3)



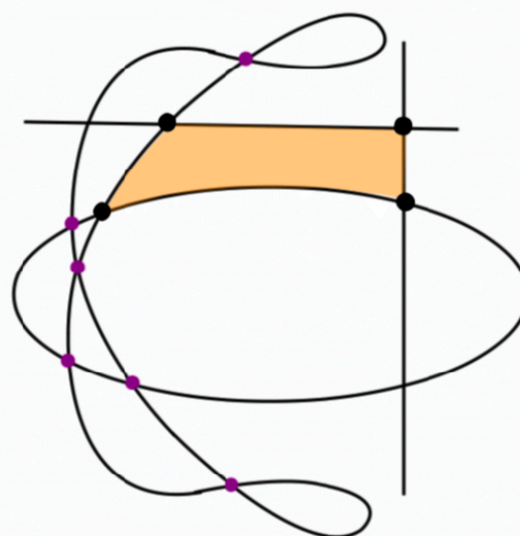
(1,2,3,2)



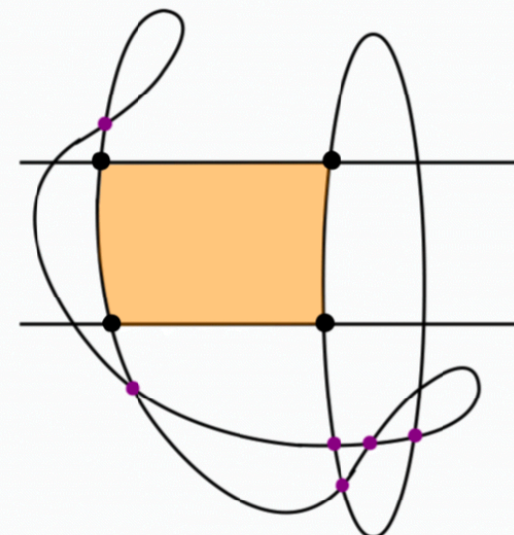
(1,1,3,3)



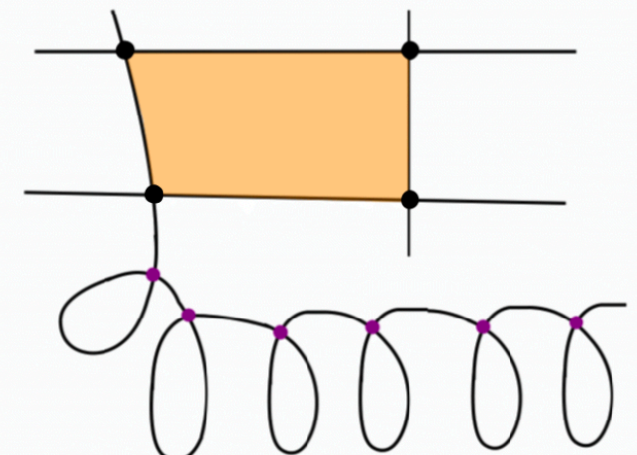
(1,3,1,3)



(1,1,2,4)

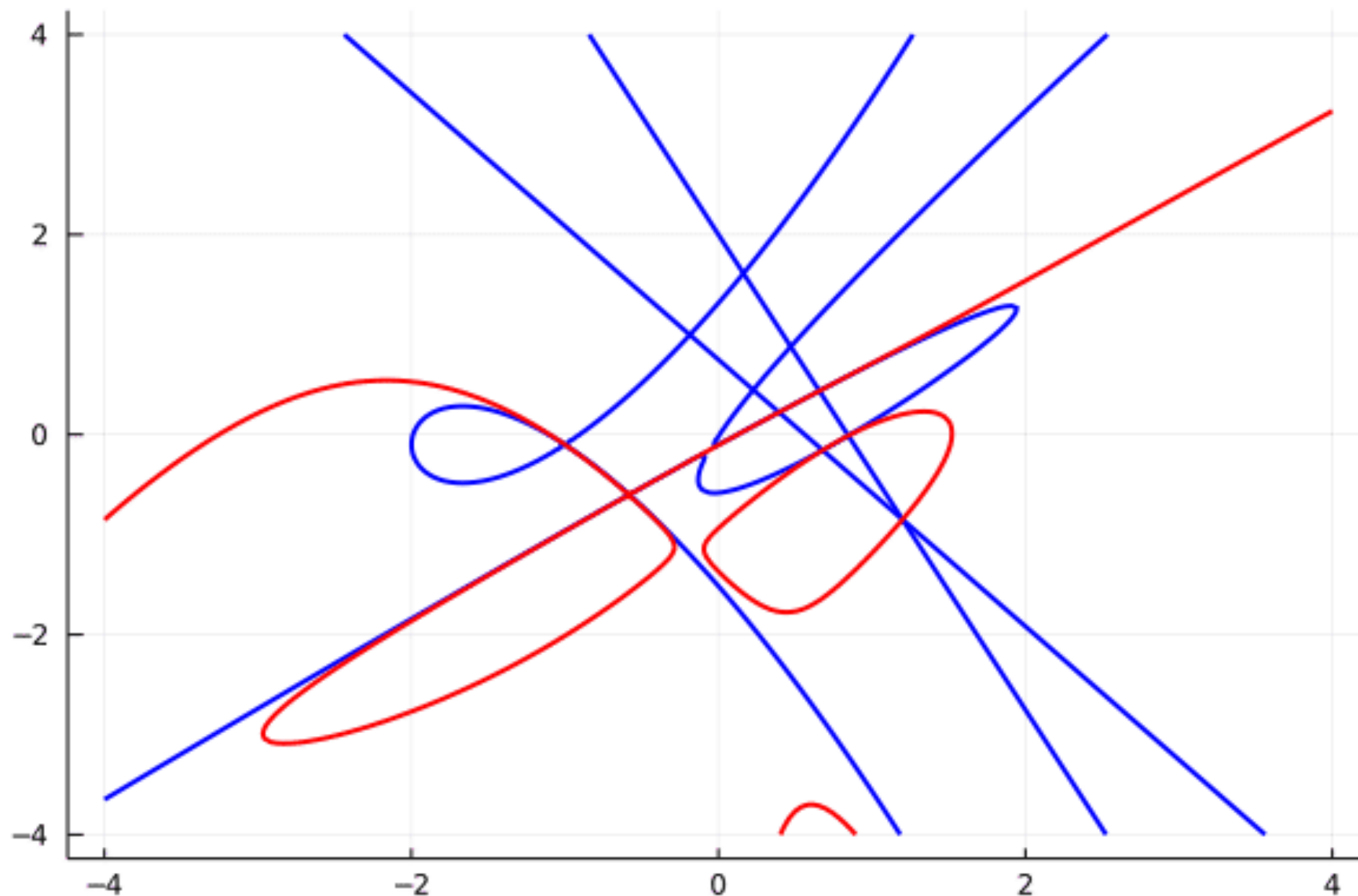


(1,2,1,4)



(1,1,1,5)

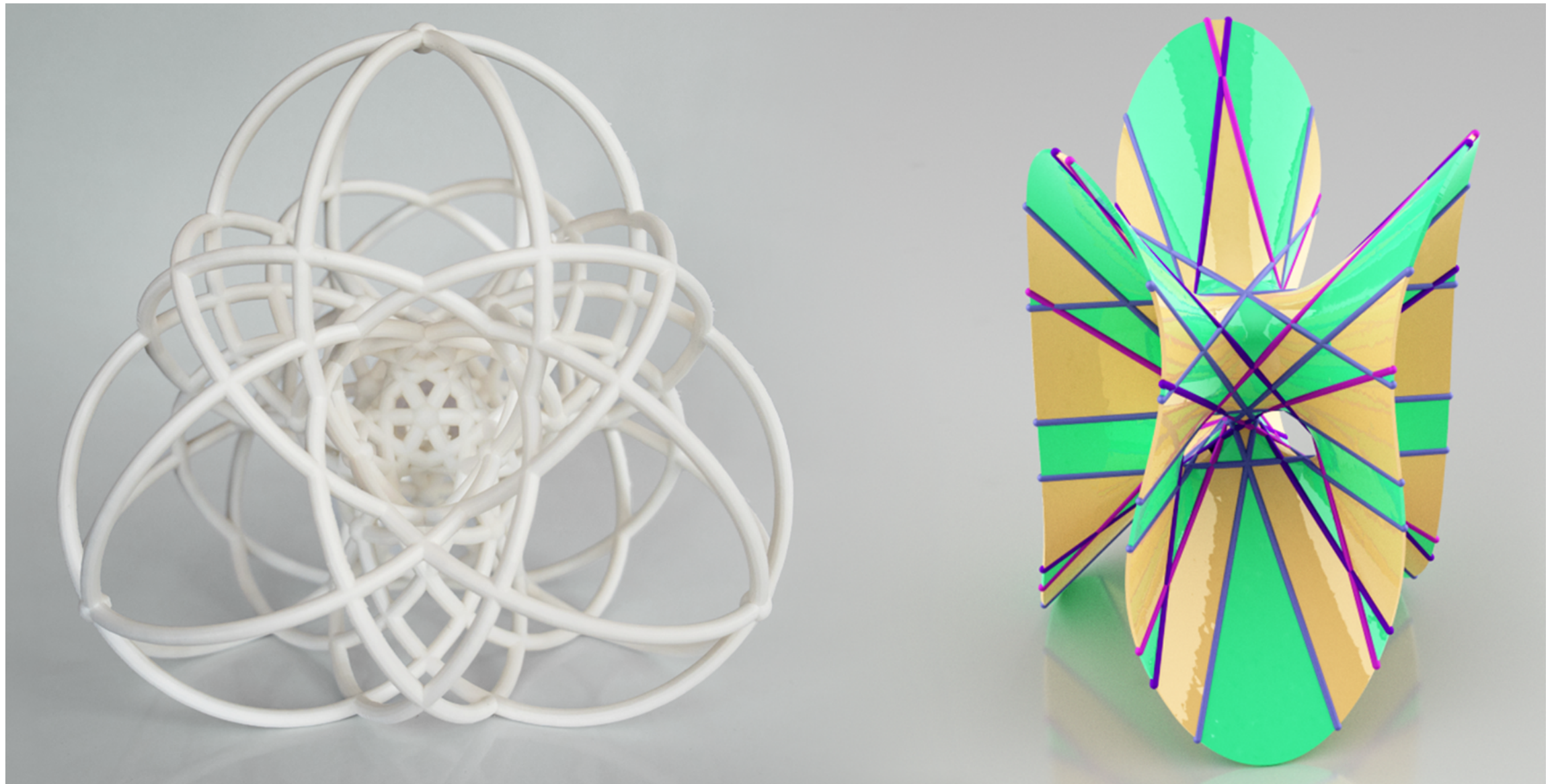
Adjoints of rational polypols



Positive geometries from cubic surfaces

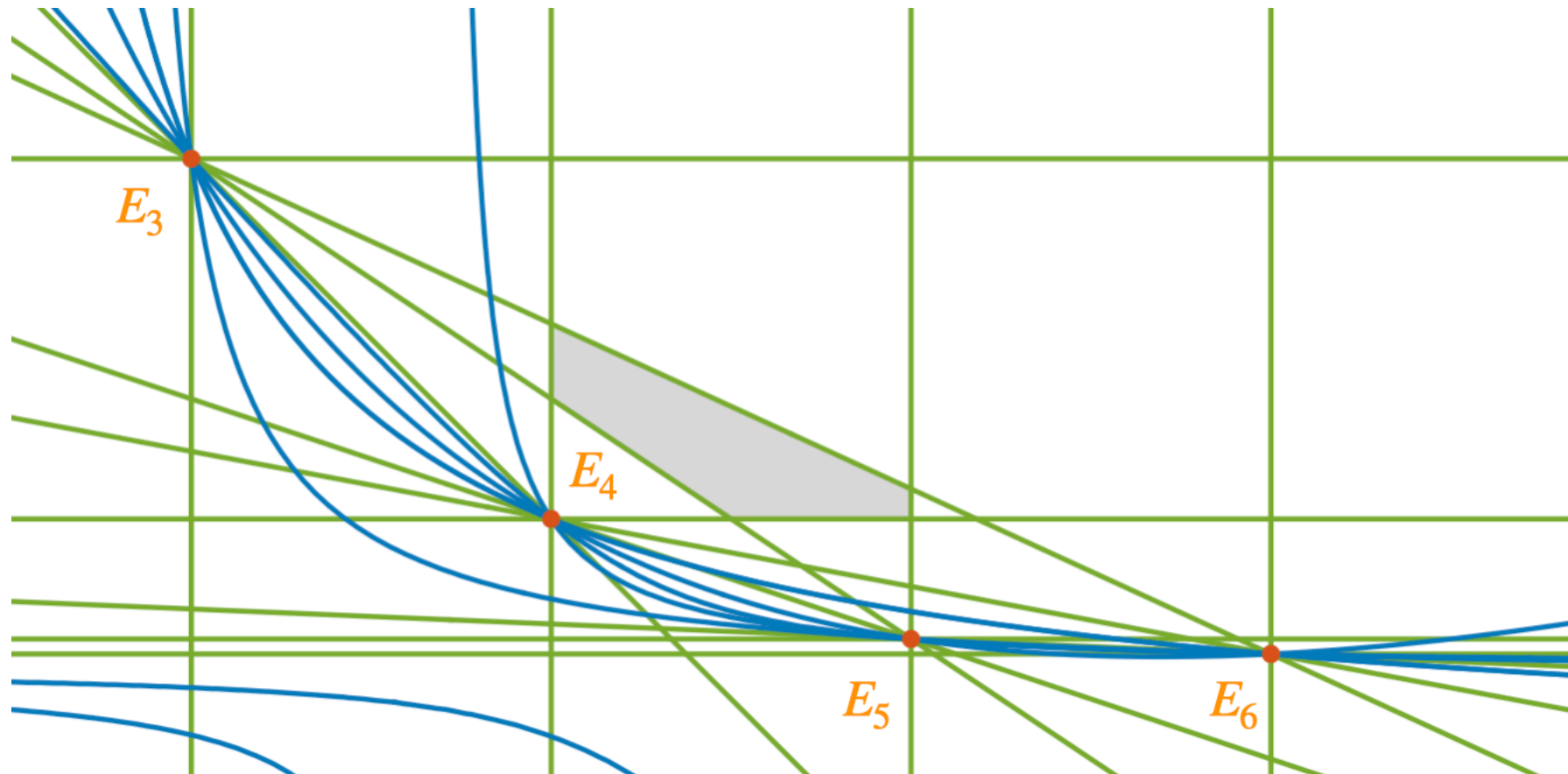
$$X = V(f) \subset \mathbb{P}^3$$

$$Y = \{ 27 \text{ lines on } X \}$$



Blowing up 6 points in $\mathbb{P}^2$

$$\begin{array}{lll} E_1 = (1 : 0 : 0), & E_2 = (0 : 1 : 0), & E_3 = (0 : 0 : 1), \\ E_4 = (1 : -1 : 1), & E_5 = (6 : -4 : 3), & E_6 = (24 : -11 : 8). \end{array}$$



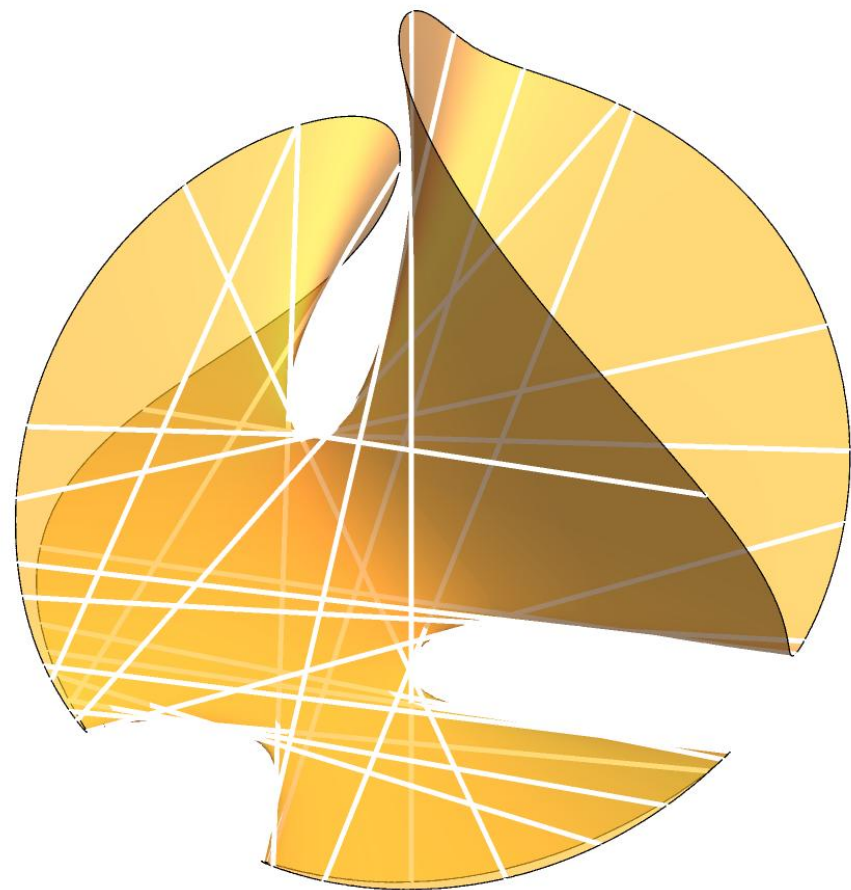
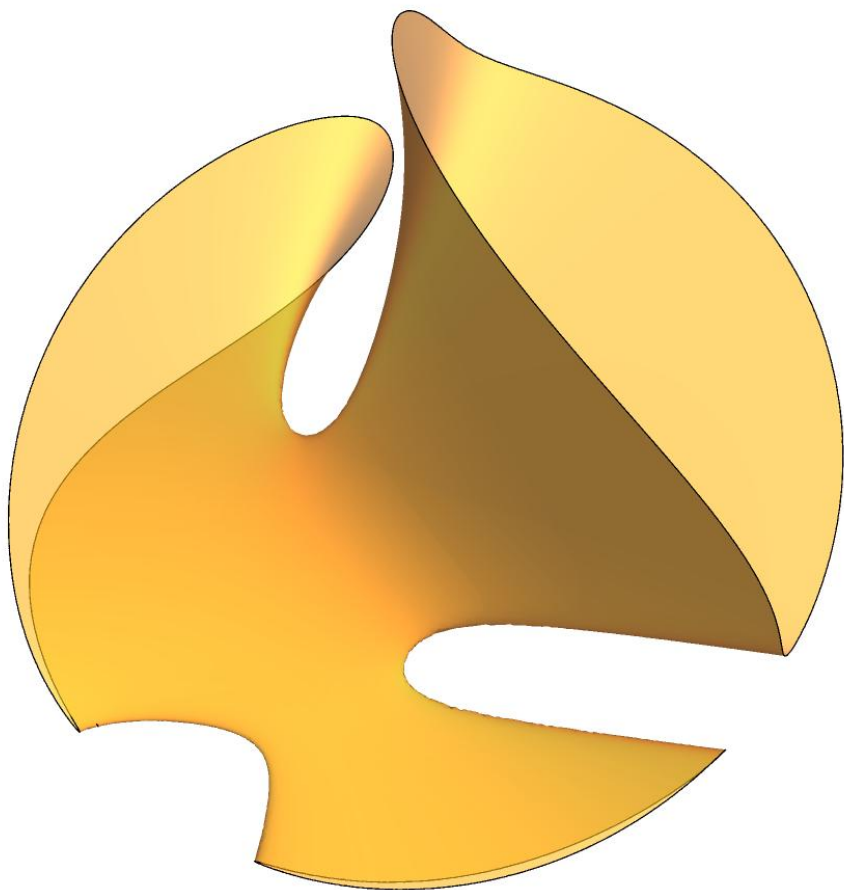
F_{ij} = line passing through E_i, E_j

G_i = conic through $E_j, j \neq i$

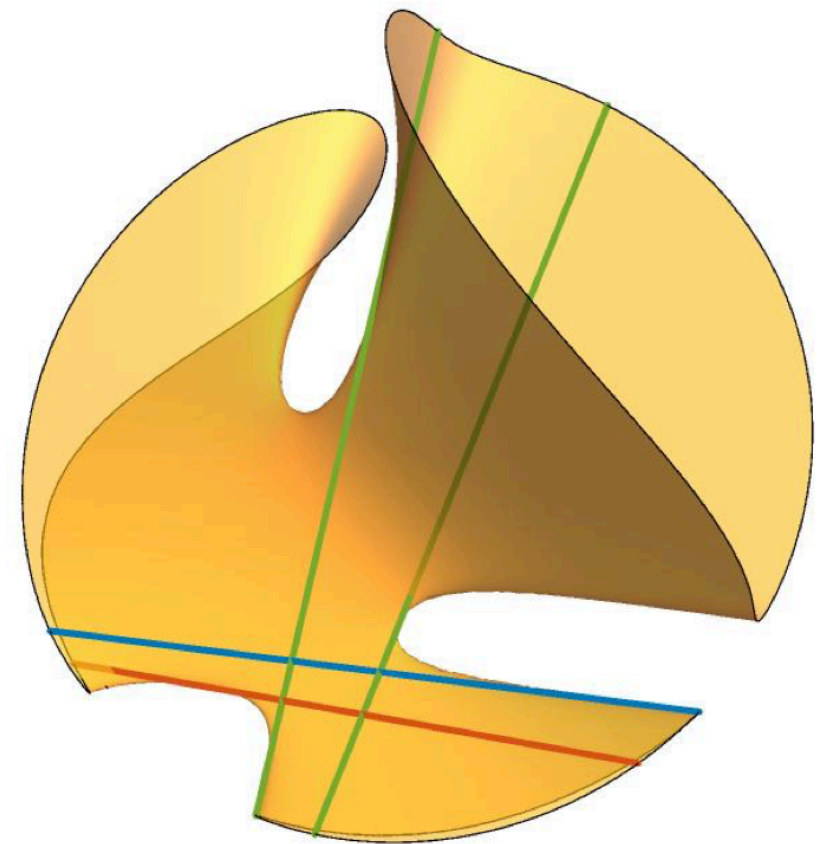
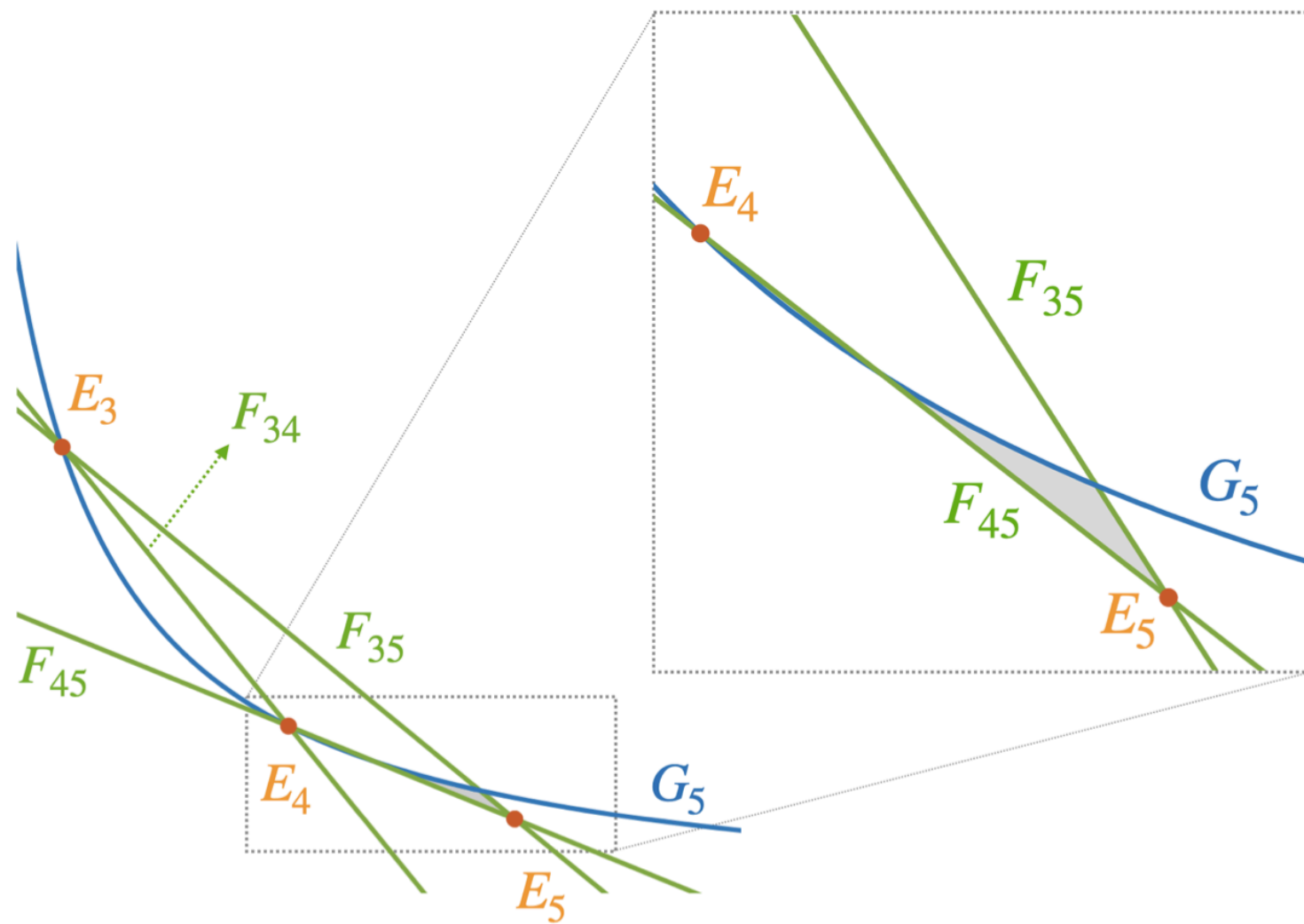
Blowing up 6 points in $\mathbb{P}^2$

$$y_0 = F_{12}F_{34}F_{56}, \quad y_1 = F_{13}F_{25}F_{46}, \quad y_2 = F_{12}F_{35}F_{46}, \quad y_3 = F_{13}F_{24}F_{56}.$$

$$31y_0y_1y_2 - 23y_0y_1y_3 + 19y_0y_2y_3 - 23y_1y_2y_3 - 32y_0^2y_1 + 45y_0y_1^2 - 16y_2^2y_3 + 13y_2y_3^2 = 0$$

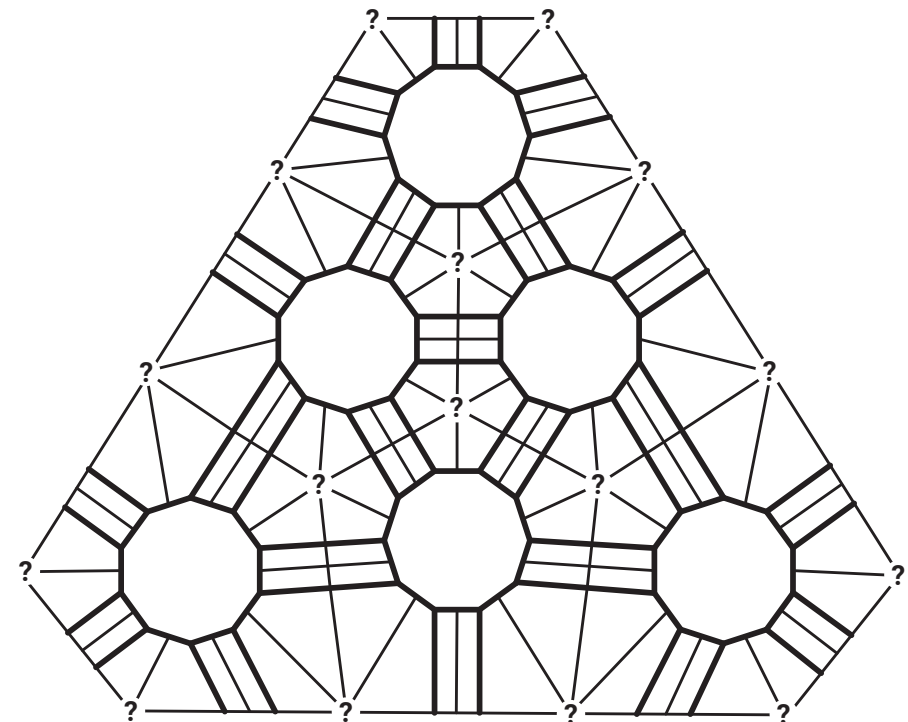
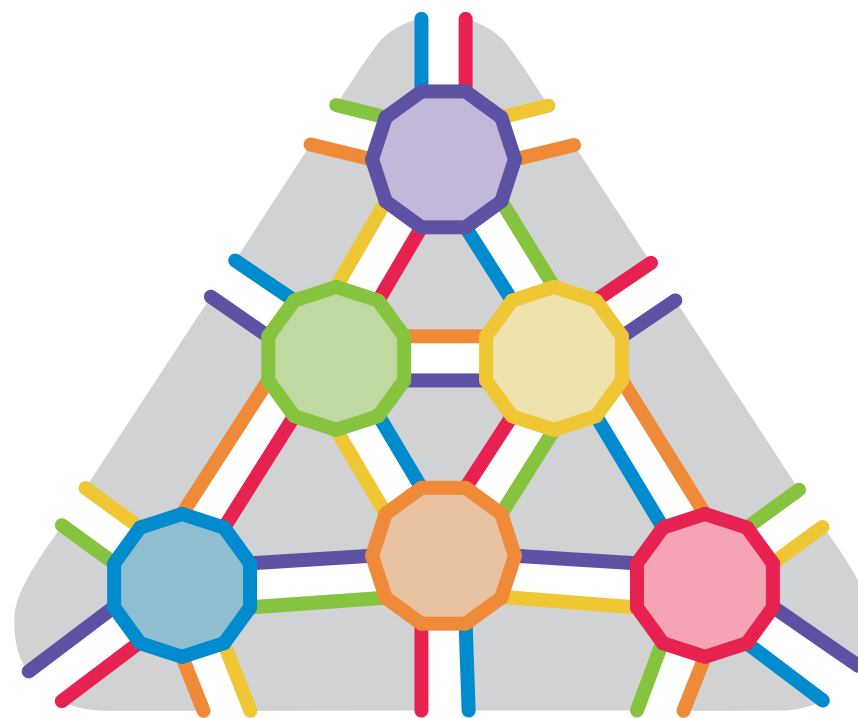
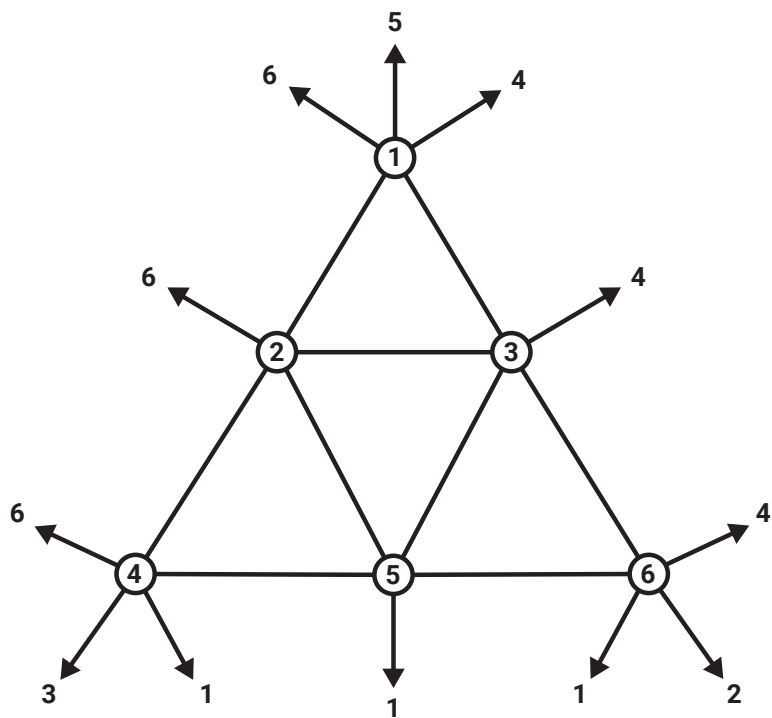


Blowing up 6 points in $\mathbb{P}^2$



130 regions

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface with 27 real lines and no Eckardt points. The complement $X_{\mathbb{R}} \setminus Y_{\mathbb{R}}$ consists of 130 connected components: 10 triangles, 90 quadrilaterals and 30 pentagons.



130 regions and 109 forms

Theorem (Sturmfels, T.).

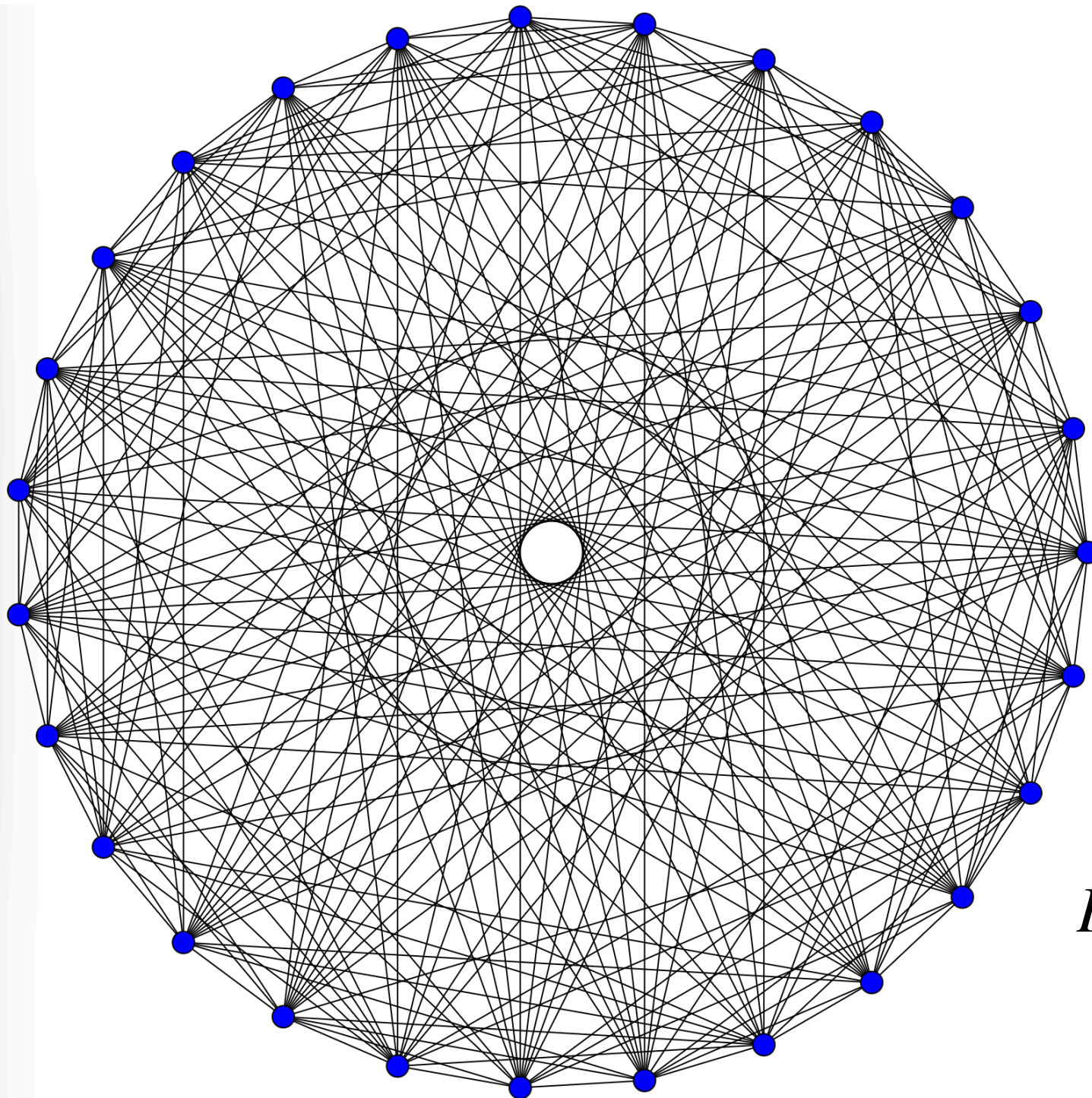
The arrangement (X, Y) is positive: $g(X, Y) = 0$. Each of the 130 curvy polygons is a positive geometry for (X, Y) . The combinatorial rank equals the real combinatorial rank: $\text{cr}(X, Y) = \text{cr}_{\mathbb{R}}(X, Y) = 109$.

Proof.
$$g(X) \leq g(X, Y) \leq g(X) + \sum_{i=1}^{27} g(L_i, L_i \cap \cup_{j \neq i} L_j)$$

(X, Y) is simple normal crossing:

$$\text{cr}(X, Y) = \dim_{\mathbb{C}} H^0(X, K_X + \sum_{i=1}^{27} L_i) = \dim_{\mathbb{C}} H^0(X, -8K_X) = \binom{8+3}{3} - \binom{5+3}{3} = 109.$$

130 regions and 109 forms



The dual complex $\Delta(Y)$ is the complement of the Schläfli graph

$$|V| = 27$$

$$|E| = 135$$

$$\tilde{H}_1(\Delta(Y)) = |E| - |V| + 1 = 109$$

Constructing the forms

Each of the 45 triangles corresponds to a tritangent plane of X in $\mathbb{P}^3$. We use the name of the triangle to label the corresponding linear form h . For instance, the triangle $S = F_{ij}F_{kl}F_{mn}$ gives the linear form $h_{(ij)(kl)(mn)}$ in y_0, y_1, y_2, y_3 . Its canonical form equals

$$\omega(S) = \operatorname{Res}_X \frac{1}{f h_{(ij)(kl)(mn)}} \Omega_{\mathbb{P}^3}. \quad (20)$$

We next write h_{ij} for the linear form given by $S = E_i F_{ij} G_j$. Then the canonical form equals

$$\omega(S) = \operatorname{Res}_X \frac{1}{f h_{ij}} \Omega_{\mathbb{P}^3}. \quad (21)$$

One sees from this formula that $\omega(S)$ has simple poles along each boundary line of S .

Constructing the forms

The canonical form of a 4-cycle is more interesting. We use the following observation.

Lemma 4.2. *In the Schläfli graph $\mathcal{S}_{27}^{10}$, every 4-cycle $L_1L_2L_3L_4$ has a unique vertex L that has distance two from each L_i . We call this vertex L the adjoint vertex of $L_1L_2L_3L_4$.*

The name *adjoint vertex* hints at the fact that the distance-two line consists of the zeros of the canonical form. For instance, for the 4-cycle $S = E_5F_{35}G_5F_{45}$ in Figure 4, the adjoint vertex is $L = F_{34}$. The canonical form has simple poles along the cycle, and zeros along L :

$$\omega(S) = \operatorname{Res}_X \frac{h_{43}}{f h_{53} h_{45}} \Omega_{\mathbb{P}^3} = \operatorname{Res}_X \frac{h_{34}}{f h_{35} h_{54}} \Omega_{\mathbb{P}^3}. \quad (22)$$

