# Exercises for Geometry and Combinatorics of Hyperbolic Polynomials

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Petter Brändén, KTH.

(1) A *circular domain* is a closed or open subset of $\mathbb{C}$ which is either a half-plane, disk or the exterior of a disk. Equivalently, a circular domain is the image of the open/closed lower half-plane under a Möbius transformation,

$$z \mapsto \frac{a+bz}{c+dz}, \quad a,b,c,d \in \mathbb{C}, ad-bc \neq 0.$$

Prove, using the material presented in the lectures, the
**Grace-Walsh-Szegő coincidence theorem**: If $f(z_1,\ldots,z_n)$ is a multiaffine and symmetric polynomial, $C$ is a circular domain (where $C$ is required to be convex if $f$ does not have degree $n$) and $\zeta_1,\ldots,\zeta_n \in C$, then there is a $\zeta \in C$ such that

$$f(\zeta_1,\ldots,\zeta_n) = f(\zeta,\ldots,\zeta).$$

(*Hint:* Prove it first when $C$ is the closed lower half-plane.)

(2) Two degree $d$ polynomials $p(x) = a_0 + a_1x + \cdots + a_dx^d$ and $q(x) = b_0 + b_1x + \cdots + b_dx^d$ are *apolar* if

$$\sum_{k=0}^{d} (-1)^k \frac{a_k b_{d-k}}{\binom{d}{k}} = 0.$$

Prove
**Grace's theorem**: If $p$ and $q$ are apolar and $C$ is a circular domain containing all zeros of $p$, then $C$ contains at least one zero of $q$.

(3) Prove the
**Schur-Szegő composition theorem**: Let $p(x) = a_0 + a_1x + \cdots + a_dx^d$ and $q(x) = b_0 + b_1x + \cdots + b_dx^d$ be two degree $d$ polynomials. If the circular region $C$ contains all the zeros of $p$, then each zero $\zeta$ of the polynomial

$$\sum_{k=0}^{d} \frac{a_k b_k}{\binom{d}{k}} x^k$$

is of the form $\zeta = -\alpha\beta$, where $\alpha \in C$ and $q(\beta) = 0$.

(4) For $\sigma \in \mathfrak{S}_n$, let

$$\mathcal{X}(\sigma) = \{i : \sigma(i) > i\} \quad \text{and} \quad \mathcal{Y}(\sigma) = \{\sigma(i) : \sigma(i) \leq i\}.$$

Prove that the polynomial

$$A_n = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in \mathcal{X}(\sigma)} x_i \prod_{j \in \mathcal{Y}(\sigma)} y_j$$

is stable. *Hint:* Find a recursion.

(5) Let $X = \operatorname{diag}(x_1,\ldots,x_n)$. Characterize all normal matrices $A$ such that $\det(X+A)$ is stable.

(6) Prove that

$$\det(X+A) = \sum_{S\subseteq[n]} \det(A(S,S)) \prod_{i\in[n]\setminus S} x_i,$$

where $A(S,S)$ is the submatrix of $A$ whose rows and columns are indexed by $S$.

(7) Let $A$ be a matrix with complex entries. Prove that the polynomial

$$\sum_{|S|=|R|} (-1)^{|S|} |\det(A(R,S))|^2 \mathbf{x}^R \mathbf{y}^S$$

is stable.

(8) Let $U = (U_{ij})_{i,j=1}^m$ be a unitary matrix, and define an element in $T \in \mathbb{C}\mathfrak{S}_m$ by

$$T = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sign}(\sigma) \left( \prod_{i=1}^m U_{i\sigma(i)} \right) \sigma.$$

Prove that $T : \mathbb{C}_{\mathbf{1}}[x_1,\dots,x_m] \to \mathbb{C}_{\mathbf{1}}[x_1,\dots,x_m]$ preserves stability.

(9) Prove that a polynomial $a+bx+cy+dxy \in \mathbb{R}[x,y]$ is stable if and only if $bc \geq ad$.

(10) Let MAP : $\mathbb{C}[x_1,\dots,x_m] \to \mathbb{C}[x_1,\dots,x_m]$ (*multiaffine part*) be the linear operator defined by

$$\mathrm{MAP}(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}) = \begin{cases} x_1^{\alpha_1}\cdots x_m^{\alpha_m}, & \text{if } \alpha_j > 1 \text{ for some } 1 \leq j \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that MAP preserves stability.

(11) Let $G=(V,E)$ be a simple graph. A *matching* in $G$ is a set $M \subseteq E$ such that if $e$ and $e'$ are different edges in $M$, then $e \cap e' = \emptyset$. Let $\mu_e$, $e \in E$ be nonnegative real numbers. Prove that the *multivariate matching polynomial*

$$P_{G,\mu}(\mathbf{x}) = \sum_{M\in\mathcal{M}} (-1)^{|M|} \prod_{e=\{i,j\}\in M} \mu_e x_i x_j,$$

where $\mathcal{M}$ is the set of matchings in $G$, is stable.

(12) Suppose $P \in \mathbb{R}[x_1,\dots,x_m]$ is stable. Prove that

$$\frac{\partial P}{\partial x_i}(\mathbf{x}) \cdot \frac{\partial P}{\partial x_i}(\mathbf{x}) \geq P(\mathbf{x}) \cdot \frac{\partial^2 P}{\partial x_i \partial x_j}(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^m$. *Hint:* Reduce the statement to polynomials of the form $a+bx+cy+dxy \in \mathbb{R}[x,y]$.

(13) Let $A$ be a hermitian $n\times n$ matrix, $S \subset [n]$ and $i,j \in [n]\setminus S$. Prove

$$\det(A(S\cup\{i\},S\cup\{i\}))\cdot\det(A(S\cup\{j\},S\cup\{j\})) \geq \det(A(S\cup\{i,j\},S\cup\{i,j\}))\cdot\det(A(S,S)).$$

(14) Suppose $P(\mathbf{x}) \in \mathbb{R}_\kappa[\mathbf{x}]$ is stable and that all its coefficients are nonnegative. Write $P(\mathbf{x})$ as

$$P(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^m} \binom{\kappa}{\alpha} a(\alpha)\mathbf{x}^\alpha, \text{ where } \binom{\kappa}{\alpha} = \binom{\kappa_1}{\alpha_1}\cdots\binom{\kappa_m}{\alpha_m}.$$

Prove that

$$a(\alpha)a(\beta) \geq a(\alpha\wedge\beta)a(\alpha\vee\beta)$$

for all $\alpha,\beta \in \mathbb{N}^n$.

(15) Suppose $Q, R \in \mathbb{C}[x_1, \ldots, x_m]$ are such that the polynomial $Q + \alpha R$ is stable for all $\alpha \in \mathbb{R}$. Prove that either $Q + x_{m+1}R$ or $Q - x_{m+1}R$ is stable.

(16) Suppose $P \in \mathbb{R}[x_1, \ldots, x_m]$ is multiaffine. Prove that $P$ is stable if and only if

$$\frac{\partial P}{\partial x_i}(\mathbf{x}) \cdot \frac{\partial P}{\partial x_i}(\mathbf{x}) \geq P(\mathbf{x}) \cdot \frac{\partial^2 P}{\partial x_i \partial x_j}(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^m$ and $1 \leq i < j \leq m$.

(17) Suppose $h \in \mathbb{R}[x_1, \ldots, x_m]$ is hyperbolic and $\mathbf{u}, \mathbf{v} \in \Lambda_+$. Prove

$$D_{\mathbf{u}} D_{\mathbf{v}} h(\mathbf{x}) \cdot h(\mathbf{x}) \leq D_{\mathbf{u}} h(\mathbf{x}) \cdot D_{\mathbf{v}} h(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^m$.

(18) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \Lambda_+$. Prove

$$h(\mathbf{w} + \mathbf{u} + \mathbf{v})h(\mathbf{w}) \leq h(\mathbf{w} + \mathbf{u})h(\mathbf{w} + \mathbf{v}). \tag{1}$$

*Hint:* Prove that

$$\frac{d}{dt}\left(\frac{h(\mathbf{w} + \mathbf{u} + t\mathbf{v})}{h(\mathbf{w} + t\mathbf{v})}\right)\Big|_{t=0} \leq 0.$$

(19) Define a *rank function* (associated to $h$) as follows. If $\mathbf{x} \in \mathbb{R}^n$, then $\mathrm{rk}(\mathbf{x})$ is defined to be the number of eigenvalues of $\mathbf{x}$ (counted with multiplicities) which are nonzero. Prove

$$\mathrm{rk}(\mathbf{x}) = \deg h(\mathbf{e} + t\mathbf{x}).$$

(20) Use (1) to prove that

$$\mathrm{rk}(\mathbf{w} + \mathbf{u} + \mathbf{v}) + \mathrm{rk}(\mathbf{w}) \leq \mathrm{rk}(\mathbf{w} + \mathbf{u}) + \mathrm{rk}(\mathbf{w} + \mathbf{v}).$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \Lambda_+$.

(21) Let $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \Lambda_+$ and define a function $r : 2^{[m]} \to \mathbb{N}$ by

$$r(S) = \mathrm{rk}\left(\sum_{i \in S} \mathbf{u}_i\right).$$

Deduce that $r$ is a *polymatroid* i.e.,

(a) $r(\emptyset) = 0$,

(b) $r(S) \leq r(T)$, whenever $S \subseteq T$,

(c) $r$ is *submodular*, i.e.,

$$r(S \cup T) + r(S \cap T) \leq r(S) + r(T).$$

(22) If $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \Lambda_+$ all have rank at most one, deduce that $r$ is the rank function of a matroid. Such matroids are called *hyperbolic matroids*.

(23) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{C}^d$. Prove that for all $S \subseteq [m]$,

$$\mathrm{rk}\left(\sum_{j \in S} \mathbf{v}_j \mathbf{v}_j^*\right) = \dim \mathrm{span}\{\mathbf{v}_j : j \in S\},$$

and deduce that all matroids which are representable over $\mathbb{C}$ are hyperbolic.

(24) Suppose $P \in \mathbb{R}[x_1, \ldots, x_m]$ has degree $d$, and let

$$h(x_1, \ldots, x_{m+1}) = x_{m+1}^d P(x_1/x_{m+1}, \ldots, x_m/x_{m+1}),$$

be its homogenization. Prove that $P$ is stable if and only if $h$ is hyperbolic with respect $(1, 1, \ldots, 1, 0)$ and its hyperbolicity cone contains $(0, \infty)^m \times \{0\}$.

(25) Suppose $P \in \mathbb{R}[x_1, \ldots, x_m]$ is stable and let

$$\mathcal{C}_P = \{\mathbf{x} \in \mathbb{R}^m : P(\mathbf{y}) \neq 0 \text{ for all } \mathbf{y} \geq \mathbf{x}\},$$

where $\mathbf{y} \geq \mathbf{x}$ if $y_j \geq x_j$ for all $1 \leq j \leq m$. Prove that $\mathcal{C}_P$ is convex.

(26) Suppose $P \in \mathbb{R}[x_1, \ldots, x_m]$ is stable and homogeneous. Prove that all its nonzero coefficients have the same sign.

(27) Suppose

$$P = \sum_{S \subseteq [m]} a(S) \prod_{i \in S} x_i \in \mathbb{R}[x_1, \ldots, x_m]$$

is stable, multiaffine and homogeneous. Thus the standard basis vectors $e_1, \ldots, e_m \in \Lambda_+$ defines a hyperbolic matroid. Prove that the set of bases of this matroid is

$$\{S \subseteq [m] : a(S) \neq 0\}.$$

(28) Prove that the Fano matroid $F_7$ is not hyperbolic.

(29) Prove that the Vámos matroid $V_8$ is hyperbolic.

(30) Suppose $P \in \mathbb{R}[x_1, \ldots, x_m]$ is stable and all its coefficients nonnegative. Prove that its homogenization

$$x_{m+1}^d P(x_1/x_{m+1}, \ldots, x_m/x_{m+1}),$$

where $d$ is the degree, is stable.

(31) Suppose $h \in \mathbb{R}[x_1, \ldots, x_m]$ is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^m$ and that $\Lambda_+ \cap (-\Lambda_+) = \{0\}$. Prove that

$$\|\mathbf{x}\| = \max\{|\lambda_1(\mathbf{x})|, \ldots, |\lambda_d(\mathbf{x})|\}$$

defines a norm on $\mathbb{R}^m$. Prove also that the bilinear form

$$\langle \mathbf{u}, \mathbf{v} \rangle = D_{\mathbf{u}} h(\mathbf{e}) \cdot D_{\mathbf{v}} h(\mathbf{e}) - D_{\mathbf{u}} D_{\mathbf{v}} h(\mathbf{e}) \cdot h(\mathbf{e})$$

is positive definite.

(32) Prove that the characteristic polynomial of a bipartite graph $G$ is an odd or even polynomial, i.e., $\mathrm{Spec}(G) = -\mathrm{Spec}(G)$.

(33) Let $G = (V, E)$ be a simple graph where $V = \{1, \ldots, n\}$, and let $U = (U_{ie})$ be the $V \times E$ matrix defined by

$$U_{ie} = \begin{cases} -1 & \text{if } e = \{i, j\} \text{ and } i < j, \\ 1 & \text{if } e = \{i, j\} \text{ and } j < i, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the minors of $U$ are either $0, 1$ or $-1$. Let $W \subseteq V$ and $F \subseteq E$, and $|W| = |F|$. Prove that $\det(U(W, F)) = \pm 1$ if and only if $[F, V \setminus W]$ is a *rooted forest*[1].

(34) Let $X = \mathrm{diag}(x_1, \ldots, x_n)$ and $W = \mathrm{diag}(\{w_e\}_{e \in E})$, where the variables in $W$ appear in the same order as in $U$. Prove that

$$\det(X + UWU^T) = \sum_{\mathcal{F} = [F, R]} \prod_{i \in R} x_i \prod_{e \in F} w_e,$$

where the sum is over all rooted forests $\mathcal{F}$ in $G$. *Hint:* Use the Binet-Cauchy formula.

(35) Prove that the above polynomial is stable.

[1] The pair $[F, R]$ is called a rooted forest if the graph $(V, F)$ contains no cycle, and each connected component of $(V, F)$ contains a unique $v \in R$.

(36) Deduce that if $G$ is connected, then

$$\det\big((UWU)([n]\setminus\{i\},[n]\setminus\{i\})\big)=\sum_T\prod_{e\in T}w_e,$$

where the sum is over all spanning trees $F$ in $G$.

(37) If $A=(a_{ij})_{i,j=1}^n$ is an $n\times n$ matrix over $\mathbb{C}$, let $[A]$ be the following element in the group algebra $\mathbb{C}\mathfrak{S}_n$

$$[A]=\sum_{w\in\mathfrak{S}_n}a_{1,w(1)}a_{2,w(2)}\cdots a_{n,w(n)}\cdot w^{-1}.$$

If $A$ is $p\times r$ and $\alpha\in\mathbb{N}^p,\beta\in\mathbb{N}^q$ let $A_{\alpha,\beta}$ be the matrix obtained from $A$ by keeping $\alpha_i$ copies of the $i$th row and $\beta_j$ copies of the $j$th column of $A$ for each $i,j$. Let $A$ be $p\times q$, $B$ be $q\times r$ and $C=AB$. If $|\alpha|=\alpha_1+\cdots+\alpha_p=|\beta|=m$, prove that

$$[C_{\alpha,\beta}]=\sum_{|\gamma|=m}\frac{1}{\gamma!}[A_{\alpha,\gamma}][B_{\gamma,\beta}].$$

(38) If $A$ is $p\times r$ and $\alpha\in\mathbb{N}^p,\beta\in\mathbb{N}^q$, define a formal series

$$F_A(\mathbf{x},\mathbf{y})=\sum_{\alpha,\beta}(-1)^{|\beta|}\frac{[A_{\alpha,\beta}]}{\alpha!\beta!}\mathbf{x}^\alpha\mathbf{y}^\beta,$$

where the sum is over all $\alpha\in\mathbb{N}^p,\beta\in\mathbb{N}^q$ for which $|\alpha|=|\beta|$. Let $A$ be $p\times q$, $B$ be $q\times r$. Prove that

$$F_{AB}(\mathbf{x},\mathbf{w})=F_A(\mathbf{x},-\partial_{\mathbf{z}})F_B(\mathbf{z},\mathbf{w})\Big|_{\mathbf{z}=0},\quad\text{where } \partial_{\mathbf{z}}=(\partial/\partial z_1,\partial/\partial z_2,\ldots).$$

(39) If $A$ is a *totally positive matrix*[2], prove that the polynomial

$$\sum_{S,T}(-1)^{|T|}\det(A(S,T))\mathbf{x}^S\mathbf{y}^T$$

is stable. *Hint:* Google the Loewner-Whitney theorem and hit $F_A$ with a suitable algebra homomorphism $\phi:\mathbb{C}\mathfrak{S}_n\to\mathbb{C}$.

[2] A matrix is totally nonnegative if all its minors are nonnegative.