

Quadratic forms and Symmetric matrices.

- (1) Let $q : S^n \rightarrow \mathbb{R}$ be a smooth function. We say that the equation $q = 0$ is *regular*¹ if for every $x \in S^n$ such that $q(x) = 0$ we have² $D_x q \neq 0$. Let now $Q \in \text{Sym}(n+1, \mathbb{R})$ and $q : S^n \rightarrow \mathbb{R}$ be given by

$$q(x) = x^T Q x.$$

Prove that $q = 0$ is regular if and only if $\det(Q) \neq 0$.

- (2) Denote by $\Delta_{n,d}$ the set of polynomials $p \in \mathbb{R}[x_0, \dots, x_n]_{(d)}$ such that $p = 0$ is not regular on the sphere $S^n \subset \mathbb{R}^{n+1}$. This set is called the *real* discriminant. Denote also by $\Delta_{n,d}^{\mathbb{C}} \subset \mathbb{C}[z_0, \dots, z_n]_{(d)}$ the *complex* discriminant, i.e. the set of polynomials p such that the projective algebraic set $\{p = 0\} \subset \mathbb{C}P^n$ is not smooth. Show that

$$\Delta_{n,2} = \Delta_{n,2}^{\mathbb{C}} \cap \mathbb{R}[x_0, \dots, x_n]_{(2)},$$

but in general $\Delta_{n,d} \neq \Delta_{n,d}^{\mathbb{C}} \cap \mathbb{R}[x_0, \dots, x_n]_{(d)}$. (Hint: consider the case $n = 1$.)

- (3) Prove that there are continuous semialgebraic functions³

$$\lambda_1 \leq \dots \leq \lambda_n : \text{Sym}(n, \mathbb{R}) \rightarrow \mathbb{R}$$

such that for every $Q \in \text{Sym}(n, \mathbb{R})$ the list $\{\lambda_1(Q), \dots, \lambda_n(Q)\}$ consists of the eigenvalues of Q ; repetitions in the list means that the eigenvalue has multiplicity.

- (4) Check that the set

$$\left\{ E_{ii}, \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}) \mid i, j = 1, \dots, n, i \neq j \right\}$$

form an orthonormal basis for space $\text{Sym}(n, \mathbb{R})$ with the Frobenius scalar product.⁴

- (5) Prove that if $\langle \cdot, \cdot \rangle$ is a scalar product on $\text{Sym}(n, \mathbb{R})$ such that for every $R \in O(n)$ and for every $Q \in \text{Sym}(n, \mathbb{R})$ we have

$$\langle RQR^T, RQR^T \rangle = \langle Q, Q \rangle,$$

then the spaces $\mathbb{R} \cdot \{\mathbf{1}\}$ and $\{Q \mid \text{tr}(Q) = 0\}$ are orthogonal with respect to this scalar product.

- (6) Prove that for appropriate choices of $\beta_1, \beta_2 \in \mathbb{R}$ the quadratic form $\alpha : \text{Sym}(n, \mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\beta(Q) := \beta_1 \text{tr}(Q)^2 + \beta_2 \text{tr}(Q^2)$$

is positive definite.

- (7) Recall that the gradient, with respect to a scalar product $\langle \cdot, \cdot \rangle$, of a smooth function $f : \text{Sym}(n, \mathbb{R}) \rightarrow \mathbb{R}$ at a point Q , is defined to be the unique vector $\nabla f(Q)$ such that for all $X \in \text{Sym}(n, \mathbb{R})$ we have $D_Q f(X) = \langle Q, X \rangle$. Let now f be the determinant function, i.e.

$$f(Q) = \det(Q).$$

Prove that the gradient of f , with respect to the Frobenius scalar product, at a point Q equals $\alpha(Q)$, the adjoint of the cofactor matrix of Q .

¹Clearly, if $q = 0$ is a regular equation, $Z(q) := \{q = 0\}$ is a smooth submanifold.

²Here $D_x q : T_x S^n \rightarrow \mathbb{R}$ denotes the differential of q at x .

³I do not know of a "simple" proof of this result.

⁴Recall that this is defined as $\langle Q_1, Q_2 \rangle_F := \text{tr}(Q_1 Q_2)$.

- (8) For every $0 \leq r \leq n$ let $S_r \subset \text{Sym}(n, \mathbb{R})$ be the set of symmetric matrices with rank exactly r . It is a smooth set (can you prove it?). Let $A \in \text{Sym}(n, \mathbb{R})$ be a fixed matrix and $f : S_r \rightarrow \mathbb{R}$ be the function defined by:

$$f(Q) := \|A - Q\|_F^2.$$

Find the critical points of f on S_{n-1} . Can you guess how many critical points are there on S_r in general?

- (9) Prove Eckart–Young Theorem for symmetric matrices, which states that for every $Q \in \text{Sym}(n, \mathbb{R})$ we have

$$(1) \quad \text{dist}_F(Q, \{\det = 0\}) = \min_i |\lambda_i(Q)|.$$

Moreover, show that, if $\det(Q) \neq 0$, then

$$\text{dist}_F(Q, \{\det = 0\}) = \|Q^{-1}\|_{\text{op}}^{-1}.$$

(This step is easy once you know (1)).

For the proof of (1), follow the following steps.

Start by proving that the minimum of $\|X - Q\|_F^2$, for $X \in \{\det = 0\}$ is reached at a matrix of rank $n - 1$.

Use the fact that S_{n-1} (defined above) is smooth to compute critical points of the function $f(Q) = \|Q - X\|_F^2$ on S_{n-1} using Lagrange multipliers rule. (You need to use the fact that $\nabla \det(Q) = \alpha(Q)$.)

Prove that the minimizer $X_0 \in S_{n-1}$ of $\|X - Q\|_F^2$ and Q are simultaneously diagonalizable (you need to use the fact that $\alpha(Q)Q = \det(Q)\mathbf{1}$.)

Conclude the argument.