METRIC ALGEBRAIC GEOMETRY
(EXERCISE SHEET)

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Quadratic forms and Symmetric matrices.

(1) Let $q : S^n \to \mathbb{R}$ be a smooth function. We say that the equation $q = 0$ is *regular*[1] if for every $x \in S^n$ such that $q(x) = 0$ we have[2] $D_x q \neq 0$. Let now $Q \in \mathrm{Sym}(n+1, \mathbb{R})$ and $q : S^n \to \mathbb{R}$ be given by

$$q(x) = x^T Q x.$$

Prove that $q = 0$ is regular if and only if $\det(Q) \neq 0$.

(2) Denote by $\Delta_{n,d}$ the set of polynomials $p \in \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ such that $p = 0$ *is not* regular on the sphere $S^n \subset \mathbb{R}^{n+1}$. This set is called the *real* discriminant. Denote also by $\Delta_{n,d}^{\mathbb{C}} \subset \mathbb{C}[z_0, \ldots, z_n]_{(d)}$ the *complex* discriminant, i.e. the set of polynomials p such that the projective algebraic set $\{p = 0\} \subset \mathbb{C}\mathrm{P}^n$ *is not* smooth. Show that

$$\Delta_{n,2} = \Delta_{n,2}^{\mathbb{C}} \cap \mathbb{R}[x_0, \ldots, x_n]_{(2)},$$

but in general $\Delta_{n,d} \neq \Delta_{n,d}^{\mathbb{C}} \cap \mathbb{R}[x_0, \ldots, x_n]_{(d)}$. (Hint: consider the case $n = 1$.)

(3) Prove that there are continuous semialgebraic functions[3]

$$\lambda_1 \leq \cdots \leq \lambda_n : \mathrm{Sym}(n, \mathbb{R}) \to \mathbb{R}$$

such that for every $Q \in \mathrm{Sym}(n, \mathbb{R})$ the list $\{\lambda_1(Q), \ldots, \lambda_n(Q)\}$ consists of the eigenevalues of Q; repetitions in the list means that the eigenvalue has multiplicity.

(4) Check that the set

$$\left\{ E_{ii}, \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}) \,\middle|\, i, j = 1, \ldots, n,\ i \neq j \right\}$$

form an orthonormal basis for space $\mathrm{Sym}(n, \mathbb{R})$ with the Frobenius scalar product.[4]

(5) Prove that if $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathrm{Sym}(n, \mathbb{R})$ such that for every $R \in O(n)$ and for every $Q \in \mathrm{Sym}(n, \mathbb{R})$ we have

$$\langle RQR^T, RQR^T \rangle = \langle Q, Q \rangle,$$

then the spaces $\mathbb{R} \cdot \{\mathbf{1}\}$ and $\{Q \,|\, \mathrm{tr}(Q) = 0\}$ are orthogonal with respect to this scalar product.

(6) Prove that for appropriate choices of $\beta_1, \beta_2 \in \mathbb{R}$ the quadratic form $\beta : \mathrm{Sym}(n, \mathbb{R}) \to \mathbb{R}$ defined by

$$\beta(Q) := \beta_1 \mathrm{tr}(Q)^2 + \beta_2 \mathrm{tr}(Q^2)$$

is positive definite.

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[1] Clearly, if $q = 0$ is a regular equation, $Z(q) := \{q = 0\}$ is a smooth submanifold.

[2] Here $D_x q : T_x S^n \to \mathbb{R}$ denotes the differential of q at x.

[3] I do not know of a "simple" proof of this result.

[4] Recall that this is defined as $\langle Q_1, Q_2 \rangle_F := \mathrm{tr}(Q_1 Q_2)$.

(7) Recall that the gradient, with respect to a scalar product $\langle \cdot, \cdot \rangle$, of a smooth function $f : \mathrm{Sym}(n, \mathbb{R}) \to \mathbb{R}$ at a point Q, is defined to be the unique vector $\nabla f(Q)$ such that for all $X \in \mathrm{Sym}(n, \mathbb{R})$ we have $D_Q f(X) = \langle \nabla f(Q), X \rangle$. Let now f be the determinant function, i.e.

$$f(Q) = \det(Q).$$

Prove that the grdient of f, with respect to the Frobenius scalar product, at a point Q equals $\alpha(Q)$, the adjoint of the cofactor matrix of Q.

(8) For every $0 \leq r \leq n$ let $S_r \subset \mathrm{Sym}(n, \mathbb{R})$ be the set of symmetric matrices with rank exactly r. It is a smooth set (can you prove it?). Let $Q \subset \mathrm{Sym}(n, \mathbb{R})$ be a fixed matrix and $f : S_r \to \mathbb{R}$ be the function defined by:

$$f(X) := \|Q - X\|_F^2, \quad X \in S_r.$$

Find the critical points of f on S_{n-1}. Can you guess how many critical points are there on S_r in general?

(9) Prove Eckart–Young Theorem for symmetric matrices, which states that for every $Q \in \mathrm{Sym}(n, \mathbb{R})$ we have

$$\mathrm{dist}_F(Q, \{\det = 0\}) = \min_i |\lambda_i(Q)|. \tag{1}$$

Moroever, show that, if $\det(Q) \neq 0$, then

$$\mathrm{dist}_F(Q, \{\det = 0\}) = \|Q^{-1}\|_{\mathrm{op}}^{-1}.$$

(This step is easy once you know (1)).

For the proof of (1), follow the following steps.

Start by proving that the minimum of $\|Q - X\|_F^2$, for $X \in \{\det = 0\}$ is reached at a matrix of rank $n - 1$.

Use the fact that S_{n-1} (defined above) is smooth to compute critical points of the function $f(Q) = \|Q - X\|_F^2$ on S_{n-1} using Lagrange multipliers rule. (You need to use the fact that $\nabla \det(X) = \alpha(X)$.)

Prove that the minimizer $X_0 \in S_{n-1}$ of $\|X - Q\|_F^2$ and Q are simoultaneously diagonalizable (you need to use the fact that $\alpha(X)X = \det(X)\mathbf{1}$.)

Conclude the argument.

Geometry in the space of polynomials. We denote by $V_{n,d} := \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ the space of real homogeneous polynomials of degree d in $n+1$ variables and by $\Delta_{n,d} \subset V_{n,d}$ the set of real polynomials p such that the equation $p = 0$ on $\mathbb{RP}^n$ (or S^n) *is not* regular.

(1) Show that $p \in \Delta_{n,d}$ if and only if there exists $x \in \mathbb{R}^{n+1} \setminus \{0\}$ such that

$$p(x) = \frac{\partial p}{\partial x_0}(x) = \cdots = \frac{\partial p}{\partial x_n}(x) = 0.$$

(2) Show that $V_{1,d}^{\mathbb{C}} \setminus \Delta_{1,d}^{\mathbb{C}}$ is connected.

(3) Consider the space of real, monic polynomials in one variable, of degree d and without multiple roots. How many components does this space have?

(4) Compute the length of the Veronese curve $\gamma : \mathbb{RP}^1 \to \mathbb{RP}^d$ given by:

$$\gamma([x_0, x_1]) := \left[x_0^d, \sqrt{d} x_0^{d-1} x_1, \ldots, \binom{d}{k}^{1/2} x_0^{d-k} x_1^k, \ldots, \sqrt{d} x_0 x_1^{d-k}, x_1^d \right].$$

Hint: compute instead the length of the "spherical" Veronese curve $\tilde{\gamma} : S^1 \to S^d$ given by:

$$\gamma(\theta) := \left[(\cos\theta)^d, \ldots, \binom{d}{k}^{1/2} (\cos\theta)^{d-k}(\sin\theta)^k, \ldots, (\sin\theta)^d\right],$$

and then use the fact that the double cover map $S^d \to \mathbb{R}\mathrm{P}^d$ is a local isometry.

Use the result you obtained (i.e. that the length of $\gamma(\mathbb{R}\mathrm{P}^1)$ is $\sqrt{d}\pi$) to run again the proof that $\mathbb{E}\#Z(p) = \sqrt{d}$.