TOPICS IN REAL AND TROPICAL ALGEBRAIC GEOMETRY
TROPICAL IDEALS

EXERCISE SHEET

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Exercise 1 (A tropical conic). Consider the polynomial
\[ f = x^2 + 4xy - 6x - 20y + 16, \]
where the field \( \mathbb{Q} \) has the 2-adic valuation.

1. Write down the tropical polynomial \( \text{trop}(f) \).
2. Draw a picture of the tropical hypersurface \( \text{trop}(V(f)) \) in \( \mathbb{R}^2 \), and compute the coordinates of all its vertices.
3. Consider the ideal \( I = \langle f, y - 2 \rangle \). What is \( \text{trop}(V(I)) \)? How does it relate to \( \text{trop}(V(f)) \) and \( \text{trop}(V(y - 2)) \)?
4. Consider the ideal \( I = \langle f, y + 4 \rangle \). What is \( \text{trop}(V(I)) \)? Is it related to \( \text{trop}(V(f)) \) and \( \text{trop}(V(y + 4)) \)?

Exercise 2 (A tropical surface). Consider the hypersurface \( X = V(x^2 + y + z) \subseteq \mathbb{C}^3 \).

1. Draw a picture of the tropical hypersurface \( \text{trop}(X) \) in \( \mathbb{R}^3 \), and indicate the multiplicities of the maximal cones.
2. Verify that \( \text{trop}(X) \) is a balanced polyhedral complex.
3. Check that the facet-ridge hypergraph of \( \text{trop}(X) \) is 2-connected.

Exercise 3 (Tropicalizing linear spaces under the trivial valuation). Consider the linear subspace \( L \) of \( \mathbb{C}^5 \) defined as
\[ L = \text{rowspace} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}. \]

1. Compute generators for the linear ideal \( I \subset \mathbb{C}[x_1, \ldots, x_5] \) of which \( L \) is the zero-locus (i.e., \( L = V(I) \)).
2. Compute the matroid \( M(L) \) on the ground set \( \{1, \ldots, 5\} \), describing it in terms of circuits and bases.
3. Is \( M(L) \) a graphical matroid?
4. Draw the lattice of flats of \( M(L) \).
(5) Write down a collection of tropical equations that describe the polyhedral complex \( \text{trop}(L) \) in tropical projective space \( \mathbb{R}^5/\mathbb{R} \cdot 1 \) (i.e., write down a tropical basis).

(6) What is the topology of \( \text{trop}(L) \cap S^3 \)?

Exercise 4 (Tropicalizing linear spaces under a non-trivial valuation). Consider the linear subspace \( L \) of \( \mathbb{C}\{\{t\}\}^4 \) defined as

\[
L = \text{rowspace} \begin{pmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 1 & t^2 \end{pmatrix}.
\]

(1) Compute generators for the ideal \( I \subset \mathbb{C}\{\{t\}\}[x_1, \ldots, x_4] \) of which \( L \) is the zero-locus (i.e., \( L = V(I) \)).

(2) Compute the (valuated) circuits of the valuated matroid \( M(L) \) on the ground set \( \{1, \ldots, 4\} \).

(3) Write down a collection of tropical equations that describe the polyhedral complex \( \text{trop}(L) \) in tropical projective space \( \mathbb{R}^4/\mathbb{R} \cdot 1 \) (i.e., write down a tropical basis).

(4) Compute all the faces of the polyhedral complex \( \text{trop}(L) \) explicitly.

Exercise 5 (Bend relations and tropical ideals). Consider the ideal \( J = \langle x^2 - 1 \rangle \subseteq \mathbb{C}[x] \), and let \( I = \text{trop}(J) \subseteq \mathbb{R}[x] \).

(1) Describe all polynomials in \( J \) of minimal support (with respect to inclusion).

(2) Find a (potentially infinite) set of tropical polynomials that generates \( I \).

(3) What is the tropical variety of \( I \)?

(4) Can \( I \) be finitely generated?

(5) Write down a collection of equivalences \( f \sim g \) that generates the congruence \( \text{bend}(I) \) on \( \mathbb{R}[x] \). Can you make this collection finite?

(6) Find a simple description of the coordinate semiring \( \mathbb{R}[x]/\text{bend}(I) \). What ‘dimension’ would you say it has?

Exercise 6 (Two tropical ideals with the same variety). Consider the ideals

\[
J_1 = \langle (x + y + 1)(xy + x + y) \rangle \quad \text{and} \quad J_2 = \langle (x + y)(x + 1)(y + 1) \rangle
\]

in \( \mathbb{C}[x, y] \), where \( \mathbb{C} \) has the trivial valuation. Let \( I_1 = \text{trop}(J_1) \) and \( I_2 = \text{trop}(J_2) \).

(1) Show that \( V(I_1) = V(I_2) \).

(2) Show that even though \( I_1 \) and \( I_2 \) contain the same tropical polynomials of degree at most 3, they contain different tropical polynomials of degree 4, and thus \( I_1 \neq I_2 \).

Exercise 7 (A non-realizable tropical ideal). Let \( I \subseteq \mathbb{R}[x, y, z] \) be the homogenization of the non-realizable tropical ideal discussed in the lecture (and in Exercise 12
below). Namely, $I$ is generated by polynomials of the form $f = \bigoplus_{x^u \in C} x^u$, where $C$ consists of exactly $k + 2$ monomials in a ‘standard triangle’ of size $k$, and $C$ is minimal with this property. (A ‘standard triangle’ of size $k$ in $\mathbb{R}[x, y, z]$ is a collection of monomials of the form
\[ \Delta = \{x^{n+i}y^{m+j}z^{l+k-i-j} : i, j \geq 0 \text{ and } i+j \leq k\}, \]
for fixed $k, l, n, m \in \mathbb{N}$.)

(1) Determine the Hilbert function of $I$.
(2) Compute the initial ideals in $(1, 1, 0)(I)$ and in $(0, 1, 0)(I)$.
(3) Expand on the previous item by computing the whole Gröbner complex of $I$ in $\mathbb{R}^3$, and determining the initial ideal corresponding to each cone.

Additional exercises:

Exercise 8 (Graphical matroids). Let $(V, E)$ be a graph. Show that the collection of subsets $C \subset E$ that form a non-self-intersecting cycle satisfy the circuit axioms of a matroid on the ground set $E$.

Exercise 9 (Bases of a matroid are equicardinal). Prove directly from the circuit axioms that all the bases of a matroid have the same cardinality.

Exercise 10 (Tropical linear spaces, tropical convexity, and tropical orthogonality). A subset $X \subset \mathbb{R}^n$ is called tropically convex if for any $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ we have $\lambda_1 x_1 \oplus \lambda_2 x_2 \in X$. Two points $x, y \in \mathbb{R}^n$ are called tropically orthogonal, denoted $x \perp y$, if $\min(x_1 + y_1, \ldots, x_n + y_n)$ is attained at least twice.

(1) Show that an intersection of tropically convex sets is a tropically convex set.
(2) Show that for any $a \in \mathbb{R}^n$, the set
\[ a^\perp := \{x \in \mathbb{R}^n : x \perp a\} \]
is tropically convex.
(3) Conclude that any tropical linear space is tropically convex.
(4) For a set $A \subset \mathbb{R}^n$, define its orthogonal set to be
\[ A^\perp := \{x \in \mathbb{R}^n : x \perp a \text{ for all } a \in A\}. \]
Show that a balanced polyhedral complex $L \subset \mathbb{R}^n$ is a tropical linear space if and only if $(L^\perp)^\perp = L$.

Exercise 11 (Maximal tropical ideals). Let $v \in \mathbb{R}^n$ and consider the ideal $I_v = \{f \in \mathbb{R}[x_1, \ldots, x_n] : v \in V(f)\}$.
(1) Show that $I_v$ is generated by binomials, and use this to show that $I_v$ is a tropical ideal.

(2) Use the weak Nullstellensatz for tropical ideals to show that if $I$ is a maximal tropical ideal of $\mathbb{R}[x_1, \ldots, x_n]$ and $I$ contains no monomials then $I = I_v$ for some $v \in \mathbb{R}^n$.

**Exercise 12 (A non-realizable tropical ideal).** A ‘standard triangle’ of size $k$ in the polynomial semiring $\mathbb{R}[x, y]$ is a collection of monomials of the form

$$\Delta = \{x^{n+i}y^{m+j} : i, j \geq 0 \text{ and } i + j \leq k\},$$

for fixed $k, n, m \in \mathbb{N}$. Consider the ideal $I \subseteq \mathbb{R}[x, y]$ generated by polynomials of the form $f = \bigoplus_{x^u \in C} x^u$, where $C$ consists of exactly $k + 2$ monomials in a ‘standard triangle’ of size $k$, and $C$ is minimal with this property.

(1) Show the given generators of $I$ satisfy the monomial elimination axiom, and conclude that $I$ is a tropical ideal.

(2) Fill all the details in the following proof that $I$ is a non-realizable tropical ideal.

Suppose $I = \text{trop}(J)$ for some $J \subseteq K[x, y]$. We have that $J$ contains a polynomial of the form $ax + by + c$. After scaling the variables, we can assume $a = b = c = 1$, so $x + y + 1 \in J$. It follows that $1 + x^3 + y^3 - 3xy \in J$. This leads to a contradiction.

(3) Show that the variety of $I$ is equal to the standard tropical line $V(x \oplus y \oplus 0)$.

**Exercise 13 (Irreducible tropical ideals).** A tropical ideal $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is **irreducible** if $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$. Use the ascending chain condition to show that every tropical ideal can be decomposed as a finite intersection of irreducible tropical ideals.

*Research problem:* Can you give interesting examples of irreducible tropical ideals? Can you find an interesting condition that implies irreducibility? What can you say about the varieties of irreducible tropical ideals?

**Exercise 14 (Sum and intersection of tropical ideals).** Give an example of two tropical ideals $I_1, I_2$ such that neither $I_1 + I_2$ nor $I_1 \cap I_2$ are tropical ideals.

*Research problem:* Can you define interesting notions of sum and intersection for tropical ideals? Desirable properties would be, for instance, that $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$, and $V(I_1 + I_2) = V(I_1) \cap_{st} V(I_2)$.

**Exercise 15 (Prime tropical ideals).** Say that a tropical ideal $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is **naively prime** if $f \circ g \in I$ implies $f \in I$ or $g \in I$. 
(a) Show that if \( \mathbf{v} \in \mathbb{R}^n \) then the tropical ideal \( I_{\mathbf{v}} := \{ f \in \mathbb{R}[x_1, \ldots, x_n] : \mathbf{v} \in V(f) \} \) is naively prime (see Exercise 11).

(b) Show that the ideal \( I \subseteq \mathbb{R}[x, y] \) defined as
\[
I = \left\{ \bigoplus_{i=0}^{n} f_i(x) \ast y^i : \text{for all } i, f_i(x) \in \mathbb{R}[x] \text{ satisfies } 0 \in V(f_i) \right\}
\]
is a tropical ideal, but it is not naively prime.

 Research problem: Can you find a useful notion of primality for tropical ideals (or their associated bend congruences / coordinate semirings)? For example, we probably want the trivial ideal \( I = \{ \infty \} \) to be prime, and if \( J \) is a linear ideal then \( \text{trop}(J) \) should be prime as well. (The ideals discussed in parts (a) and (b) of this exercise are all of this form.) A ‘good’ notion of primality should probably imply that the corresponding variety is irreducible.