TOPICS IN REAL AND TROPICAL ALGEBRAIC GEOMETRY TROPICAL IDEALS

EXERCISE SHEET

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Exercise 1 (*A tropical conic*)**.** Consider the polynomial

$$f = x^2 + 4xy - 6x - 20y + 16,$$

where the field $\mathbb{Q}$ has the 2-adic valuation.

(1) Write down the tropical polynomial $\operatorname{trop}(f)$.
(2) Draw a picture of the tropical hypersurface $\operatorname{trop}(V(f))$ in $\mathbb{R}^2$, and compute the coordinates of all its vertices.
(3) Consider the ideal $I = \langle f, y - 2 \rangle$. What is $\operatorname{trop}(V(I))$? How does it relate to $\operatorname{trop}(V(f))$ and $\operatorname{trop}(V(y-2))$?
(4) Consider the ideal $I = \langle f, y + 4 \rangle$. What is $\operatorname{trop}(V(I))$? Is it related to $\operatorname{trop}(V(f))$ and $\operatorname{trop}(V(y+4))$?

Exercise 2 (*A tropical surface*)**.** Consider the hypersurface $X = V(x^2+y+z) \subseteq \mathbb{C}^3$.

(1) Draw a picture of the tropical hypersurface $\operatorname{trop}(X)$ in $\mathbb{R}^3$, and indicate the multiplicities of the maximal cones.
(2) Verify that $\operatorname{trop}(X)$ is a balanced polyhedral complex.
(3) Check that the facet-ridge hypergraph of $\operatorname{trop}(X)$ is 2-connected.

Exercise 3 (*Tropicalizing linear spaces under the trivial valuation*)**.** Consider the linear subspace L of $\mathbb{C}^5$ defined as

$$L = \text{rowspace} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}.$$

(1) Compute generators for the linear ideal $I \subset \mathbb{C}[x_1, \ldots, x_5]$ of which L is the zero-locus (i.e., $L = V(I)$).
(2) Compute the matroid $M(L)$ on the ground set $\{1, \ldots, 5\}$, describing it in terms of circuits and bases.
(3) Is $M(L)$ a graphical matroid?
(4) Draw the lattice of flats of $M(L)$.

(5) Write down a collection of tropical equations that describe the polyhedral complex $\operatorname{trop}(L)$ in tropical projective space $\mathbb{R}^5 / \mathbb{R}\cdot\mathbf{1}$ (i.e., write down a tropical basis).
(6) What is the topology of $\operatorname{trop}(L) \cap S^3$?

Exercise 4 (*Tropicalizing linear spaces under a non-trivial valuation)***.** Consider the linear subspace L of $\mathbb{C}\{\{t\}\}^4$ defined as

$$L = \operatorname{rowspace}\begin{pmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 1 & t^2 \end{pmatrix}.$$

(1) Compute generators for the ideal $I \subset \mathbb{C}\{\{t\}\}[x_1, \ldots, x_4]$ of which L is the zero-locus (i.e., $L = V(I)$).
(2) Compute the (valuated) circuits of the valuated matroid $\mathcal{M}(L)$ on the ground set $\{1, \ldots, 4\}$.
(3) Write down a collection of tropical equations that describe the polyhedral complex $\operatorname{trop}(L)$ in tropical projective space $\mathbb{R}^4 / \mathbb{R}\cdot\mathbf{1}$ (i.e., write down a tropical basis).
(4) Compute all the faces of the polyhedral complex $\operatorname{trop}(L)$ explicitly.

Exercise 5 (*Bend relations and tropical ideals)***.** Consider the ideal $J = \langle x^2 - 1\rangle \subseteq \mathbb{C}[x]$, and let $I = \operatorname{trop}(J) \subseteq \overline{\mathbb{R}}[x]$.

(1) Describe all polynomials in J of minimal support (with respect to inclusion).
(2) Find a (potentially infinite) set of tropical polynomials that generates I.
(3) What is the tropical variety of I?
(4) Can I be finitely generated?
(5) Write down a collection of equivalences $f \sim g$ that generates the congruence $\operatorname{bend}(I)$ on $\overline{\mathbb{R}}[x]$. Can you make this collection finite?
(6) Find a simple description of the coordinate semiring $\overline{\mathbb{R}}[x]/\operatorname{bend}(I)$. What 'dimension' would you say it has?

Exercise 6 (*Two tropical ideals with the same variety)***.** Consider the ideals

$$J_1 = \langle (x+y+1)(xy+x+y)\rangle \quad \text{and} \quad J_2 = \langle (x+y)(x+1)(y+1)\rangle$$

in $\mathbb{C}[x, y]$, where $\mathbb{C}$ has the trivial valuation. Let $I_1 = \operatorname{trop}(J_1)$ and $I_2 = \operatorname{trop}(J_2)$.

(1) Show that $V(I_1) = V(I_2)$.
(2) Show that even though I_1 and I_2 contain the same tropical polynomials of degree at most 3, they contain different tropical polynomials of degree 4, and thus $I_1 \neq I_2$.

Exercise 7 (*A non-realizable tropical ideal)***.** Let $I \subseteq \overline{\mathbb{R}}[x, y, z]$ be the homogenization of the non-realizable tropical ideal discussed in the lecture (and in Exercise 12 below). Namely, I is generated by polynomials of the form $f = \bigoplus_{\mathbf{x^u} \in C} \mathbf{x^u}$, where C consists of exactly $k+2$ monomials in a 'standard triangle' of size k, and C is minimal with this property. (A 'standard triangle' of size k in $\overline{\mathbb{R}}[x,y,z]$ is a collection of monomials of the form

$$\Delta = \{x^{n+i}y^{m+j}z^{l+k-i-j} : i,j \geq 0 \text{ and } i+j \leq k\},$$

for fixed $k,l,n,m \in \mathbb{N}$.)

(1) Determine the Hilbert function of I.
(2) Compute the initial ideals $\mathrm{in}_{(1,1,0)}(I)$ and $\mathrm{in}_{(0,1,0)}(I)$.
(3) Expand on the previous item by computing the whole Gröbner complex of I in $\mathbb{R}^3$, and determining the initial ideal corresponding to each cone.

Additional exercises:

Exercise 8 (*Graphical matroids)***.** Let (V,E) be a graph. Show that the collection of subsets $C \subset E$ that form a non-self-intersecting cycle satisfy the circuit axioms of a matroid on the ground set E.

Exercise 9 (*Bases of a matroid are equicardinal)***.** Prove directly from the circuit axioms that all the bases of a matroid have the same cardinality.

Exercise 10 (*Tropical linear spaces, tropical convexity, and tropical orthogonality)***.** A subset $X \subset \overline{\mathbb{R}}^n$ is called **tropically convex** if for any $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \overline{\mathbb{R}}$ we have $\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 \in X$. Two points $x, y \in \overline{\mathbb{R}}^n$ are called **tropically orthogonal**, denoted $x \perp y$, if $\min(x_1+y_1, \ldots, x_n+y_n)$ is attained at least twice.

(1) Show that an intersection of tropically convex sets is a tropically convex set.
(2) Show that for any $a \in \overline{\mathbb{R}}^n$, the set

$$a^\perp := \{x \in \overline{\mathbb{R}}^n : x \perp a\}$$

is tropically convex.
(3) Conclude that any tropical linear space is tropically convex.
(4) For a set $A \subset \mathbb{R}^n$, define its **orthogonal set** to be

$$A^\perp := \{x \in \mathbb{R}^n : x \perp a \text{ for all } a \in A\}.$$

Show that a balanced polyhedral complex $L \subset \mathbb{R}^n$ is a tropical linear space if and only if $(L^\perp)^\perp = L$.

Exercise 11 (*Maximal tropical ideals)***.** Let $\mathbf{v} \in \mathbb{R}^n$ and consider the ideal $I_\mathbf{v} = \{f \in \overline{\mathbb{R}}[x_1, \ldots, x_n] : \mathbf{v} \in V(f)\}$.

(1) Show that $I_{\mathbf{v}}$ is generated by binomials, and use this to show that $I_{\mathbf{v}}$ is a tropical ideal.
(2) Use the weak Nullstellensatz for tropical ideals to show that if I is a maximal tropical ideal of $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ and I contains no monomials then $I = I_{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{R}^n$.

Exercise 12 (*A non-realizable tropical ideal)*. A 'standard triangle' of size k in the polynomial semiring $\overline{\mathbb{R}}[x, y]$ is a collection of monomials of the form

$$\Delta = \{x^{n+i}y^{m+j} : i, j \geq 0 \text{ and } i + j \leq k\},$$

for fixed $k, n, m \in \mathbb{N}$. Consider the ideal $I \subseteq \overline{\mathbb{R}}[x, y]$ generated by polynomials of the form $f = \bigoplus_{\mathbf{x}^{\mathbf{u}} \in C} \mathbf{x}^{\mathbf{u}}$, where C consists of exactly $k+2$ monomials in a 'standard triangle' of size k, and C is minimal with this property.

(1) Show the given generators of I satisfy the monomial elimination axiom, and conclude that I is a tropical ideal.
(2) Fill all the details in the following proof that I is a non-realizable tropical ideal.

Suppose $I = \operatorname{trop}(J)$ for some $J \subseteq K[x, y]$. We have that J contains a polynomial of the form $ax+by+c$. After scaling the variables, we can assume $a = b = c = 1$, so $x + y + 1 \in J$. It follows that $1 + x^3 + y^3 - 3xy \in J$. This leads to a contradiction.

(3) Show that the variety of I is equal to the standard tropical line $V(x \oplus y \oplus 0)$.

Exercise 13 (*Irreducible tropical ideals)*. A tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is **irreducible** if $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$. Use the ascending chain condition to show that every tropical ideal can be decomposed as a finite intersection of irreducible tropical ideals.

<u>*Research problem:*</u> Can you give interesting examples of irreducible tropical ideals? Can you find an interesting condition that implies irreducibility? What can you say about the varieties of irreducible tropical ideals?

Exercise 14 (*Sum and intersection of tropical ideals)*. Give an example of two tropical ideals I_1, I_2 such that neither $I_1 + I_2$ nor $I_1 \cap I_2$ are tropical ideals.

<u>*Research problem:*</u> Can you define interesting notions of sum and intersection for tropical ideals? Desirable properties would be, for instance, that $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$, and $V(I_1 + I_2) = V(I_1) \cap_{\text{st}} V(I_2)$.

Exercise 15 (*Prime tropical ideals)*. Say that a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is **naively prime** if $f \odot g \in I$ implies $f \in I$ or $g \in I$.

(a) Show that if $\mathbf{v} \in \mathbb{R}^n$ then the tropical ideal $I_{\mathbf{v}} := \{f \in \overline{\mathbb{R}}[x_1, \dots, x_n] : \mathbf{v} \in V(f)\}$ is naively prime (see Exercise 11).

(b) Show that the ideal $I \subseteq \overline{\mathbb{R}}[x, y]$ defined as

$$I = \left\{ \bigoplus_{i=0}^{n} f_i(x) \odot y^i : \text{for all } i, f_i(x) \in \overline{\mathbb{R}}[x] \text{ satisfies } 0 \in V(f_i) \right\}$$

is a tropical ideal, but it is not naively prime.

Research problem: Can you find a useful notion of primality for tropical ideals (or their associated bend congruences / coordinate semirings)? For example, we probably want the trivial ideal $I = \{\infty\}$ to be prime, and if J is a linear ideal then $\mathrm{trop}(J)$ should be prime as well. (The ideals discussed in parts (a) and (b) of this exercise are all of this form.) A 'good' notion of primality should probably imply that the corresponding variety is irreducible.