Sums of Squares and Applications

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General notions and definitions for Lectures 1–2

Usually write $x = (x_1, \dots, x_n)$, and let $\mathbb{R}[x]_d$ denote the space of homogeneous polynomials (forms) of degree d. The forms in

$$P_{n,d} \;=\; \{f \in \mathbb{R}[x]_d \colon \forall\, \xi \in \mathbb{R}^n \;\, f(\xi) \ge 0\}$$

are called *positive semidefinite* (*psd*), the forms in

$$\Sigma_{n,d} \;=\; \Big\{f \in \mathbb{R}[x]_d \colon \exists\, r \ge 1,\; \exists\, p_1, \dots, p_r \in \mathbb{R}[x] \text{ with } f = \sum_{i=1}^r p_i^2\Big\}$$

are the *sums of squares*, or the *sos* forms. The sets $P_{n,d}$ and $\Sigma_{n,d}$ are also referred to as the *psd cone* and the *sos cone*, respectively.

Let k be a field. An *ordering* of k is a linear (total) ordering $\le$ of k that is compatible with $+$ and $\cdot$, in the sense that $a \le b$ implies $a + c \le b + c$, and also $ac \le bc$ if $c \ge 0$. Equivalently, $\le$ may be identified with the set $T_{\le} = \{a \in k \colon a \ge 0\}$ of its non-negative elements. The set $T = T_{\le}$ satisfies $T + T \subseteq T$, $TT \subseteq T$, $T \cup (-T) = k$ and $T \cap (-T) = \{0\}$. Such T is called a *positive cone*. Conversely, every positive cone corresponds to an ordering of k.

Fields that can be ordered are called real fields (they all have char $= 0$). A field R is *real closed* if it is real, but every proper finite extension of R is non-real. It is equivalent that $-1 \notin R$ and $R(\sqrt{-1})$ is algebraically closed. Real closed fields have "the same algebraic properties" as the field $\mathbb{R}$ of real numbers (the squares form the (unique) ordering, every univariate polynomial of odd degree has a root in R, etc).

Let $(k, \le)$ be an ordered field. A *real closure* of $(k, \le)$ is an algebraic field extension of k that is real closed and whose ordering extends the ordering $\le$ of k. Every ordered field has a real closure, which is unique in a strong sense: If R_1, R_2 are real closures of the ordered field $(k, \le)$, there exists a *unique* k-isomorphism $R_1 \xrightarrow{\sim} R_2$.

Let R be a real closed field. A subset $M \subseteq R^n$ is *semialgebraic* if M is a finite boolean combination of sets $\{\xi \in R^n \colon f_i(\xi) > 0\}$ with $f_i \in R[x_1, \dots, x_n]$. Every semialgebraic set M can be written (non-uniquely) in the form

$$M \;=\; \bigcup_{i=1}^r \{f_i = 0,\; g_{i1} > 0,\, \dots,\, g_{im_i} > 0\}$$

with polynomials f_i, g_{ij}.

Problem for Lecture 1

Exercise 1

(a) Prove that $\Sigma_{n,d} = P_{n,d}$ if $n = 2$ or $d = 2$. How many squares are needed in these cases to represent a general element of $\Sigma_{n,d}$?

Consider the (inhomogeneous) Motzkin polynomial $f = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ in $\mathbb{R}[x, y]$ and prove:

(b) $f(a, b) \geq 0$ for all $(a, b) \in \mathbb{R}^2$. (*Hint*: Arithmetic-geometric inequality)

(c) f is not a sum of squares of polynomials in $\mathbb{R}[x, y]$. (*Hint*: $f(x, 0) = f(0, y) = 1$)

(d) Find a representation of f as a sum of four squares of rational functions in $\mathbb{R}(x, y)$. (*Hint*: Multiply f with $1 + x^2$.)