Sums of squares and applications

Exercise 3

Let f, $h \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ be positive definite forms, and assume that $\deg(h)$ divides $\deg(f)$. Show that, for some $m \ge 1$, the form fh^m is a sum of squares of forms. — *Hint*: The complement V of the projective hypersurface $h = 0$ in $\mathbb{P}^{n-1}$ is an affine variety with compact real points. If $\deg(f) = d \cdot \deg(h)$ then f/h^d is a regular function on V that is strictly positive.

Exercise 4

Show that the unit disk in $\mathbb{R}^2$ is a spectrahedron. Can you do this more generally for the unit ball B_n in $\mathbb{R}^n$? The task is to find a symmetric matrix $A = A(x)$ whose coefficients are polynomials of degree ≤ 1 in $x = (x_1, \dots, x_n)$, such that

$$B_n \;=\; \{\xi \in \mathbb{R}^n \colon A(\xi) \succeq 0\}.$$

Exercise 5

Let $\Sigma = \Sigma_{n,2d}$ be the cone of all homogeneous sums of squares of degree $2d$ in $\mathbb{R}[x]$. Prove that the dual cone Σ^* (consisting of all linear forms $\mathbb{R}[x]_{2d} \to \mathbb{R}$ that take non-negative values on Σ) is a spectrahedron.