

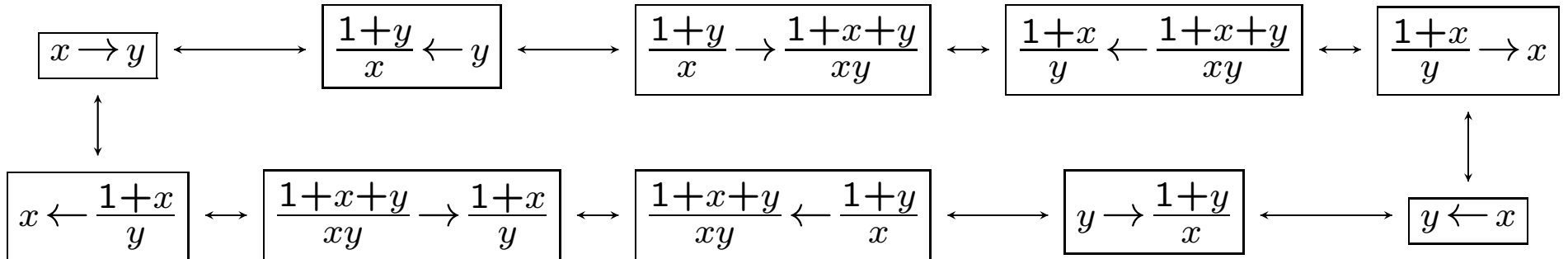
EXERCISES

The pentagon relation

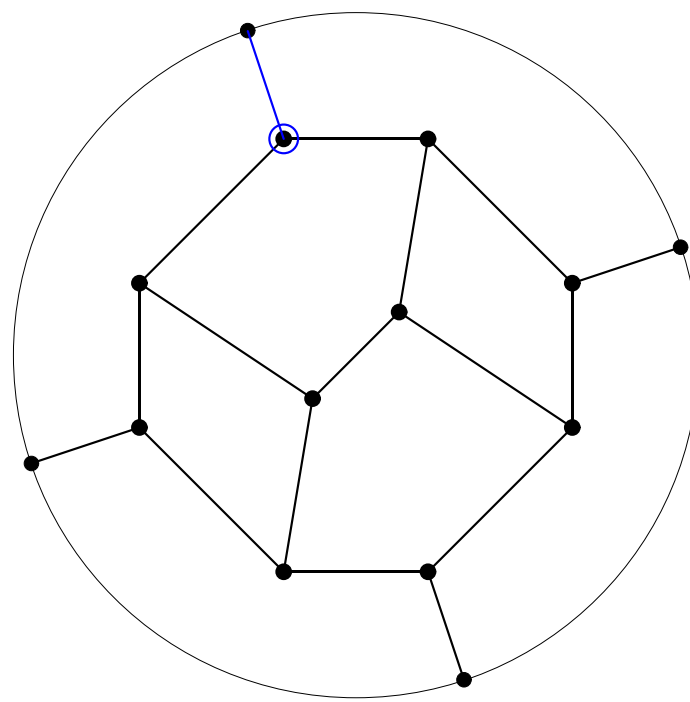
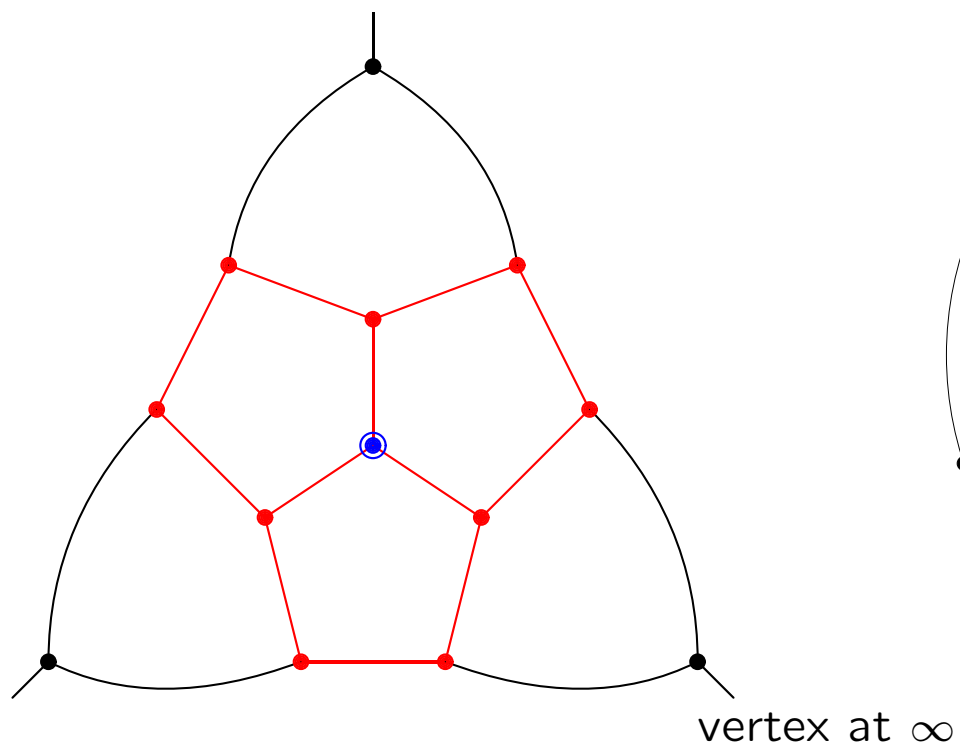
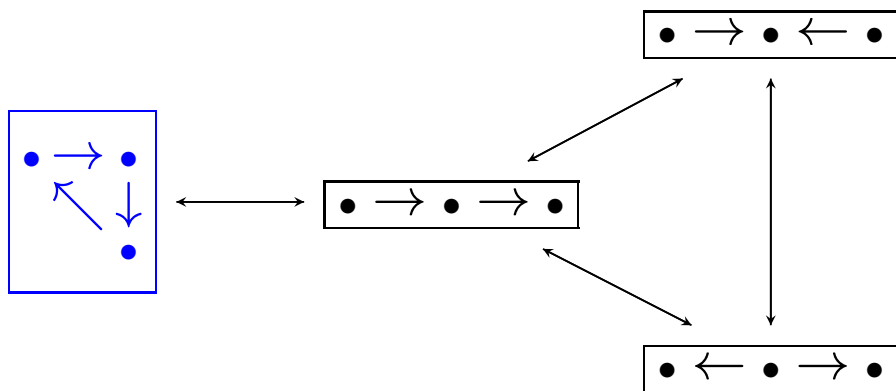
Lemma. Suppose that two mutable vertices u and v in a quiver Q are connected by a single arrow ($u \rightarrow v$ or $v \rightarrow u$). Then applying a sequence of 5 mutations

$$\mu_u \circ \mu_v \circ \mu_u \circ \mu_v \circ \mu_u \quad (\text{or } \mu_v \circ \mu_u \circ \mu_v \circ \mu_u \circ \mu_v)$$

to a seed $(Q, \mathbf{z})$ recovers the same seed, up to switching the cluster variables associated with u and v .



Mutation classes in type A_3



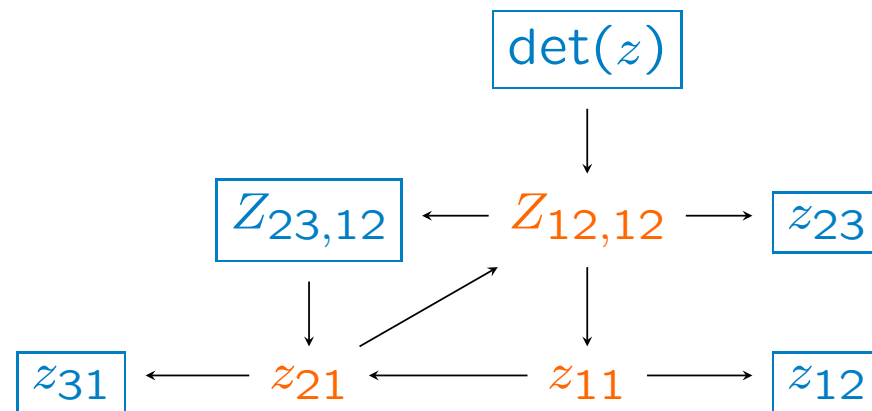
Exercise. The coordinate ring of the space of 3×3 matrices z of the form

$$z = \begin{bmatrix} z_{11} & z_{12} & 0 \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

is the polynomial ring

$$R = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, z_{23}, z_{31}, z_{32}, z_{33}].$$

Verify that the initial seed



defines a cluster structure (of type A_3) on R . Here we use the notation

$$Z_{ij,k\ell} = \det \begin{pmatrix} z_{ik} & z_{i\ell} \\ z_{jk} & z_{j\ell} \end{pmatrix}.$$

Compute all the seeds for this cluster structure.

Solution (sketch). We start by applying a single mutation to the initial seed, in each of the three possible ways.

1. Mutation at $Z_{12,12}$ replaces this cluster variable by $Z_{23,13}$, according to the exchange relation

$$Z_{12,12}Z_{23,13} = z_{11}z_{23}Z_{23,12} + \det(z)z_{21}.$$

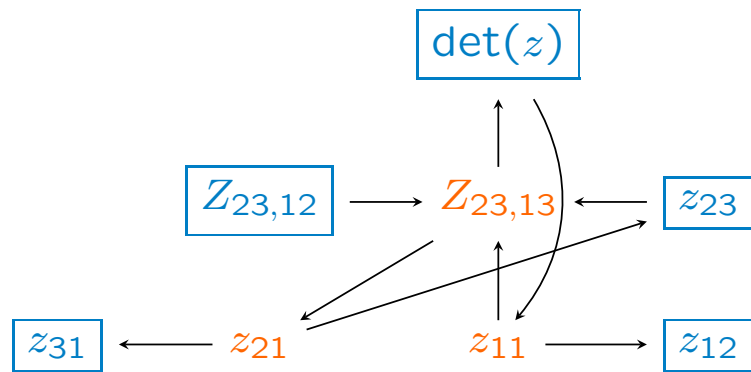
2. Mutation at z_{21} replaces this cluster variable by $Z_{13,12}$, according to the exchange relation

$$z_{21}Z_{13,12} = z_{11}Z_{23,12} + z_{31}Z_{12,12}.$$

3. Mutation at z_{11} replaces this cluster variable by z_{22} , according to the exchange relation

$$z_{11}z_{22} = z_{21}z_{12} + Z_{12,12}.$$

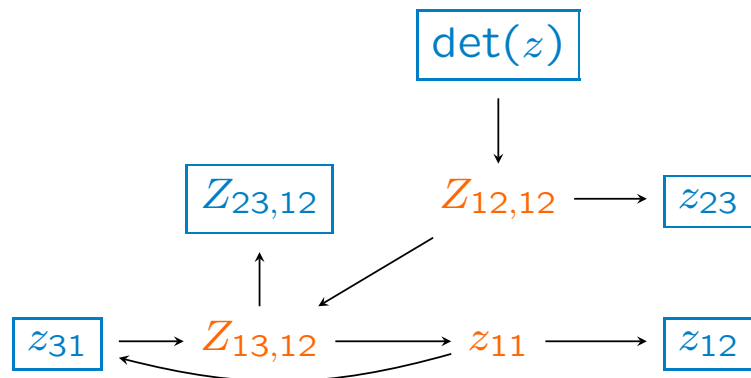
These three mutations produce the three seeds shown on the next slide. For each of the three seeds, we also show the exchange relations for it. The exchanges that return us back to the initial seed are boxed.



$$Z_{23,13}Z_{12,12} = z_{11}z_{23}Z_{23,12} + \det(z)z_{21}$$

$$z_{21}z_{33} = z_{23}z_{31} + Z_{23,13}$$

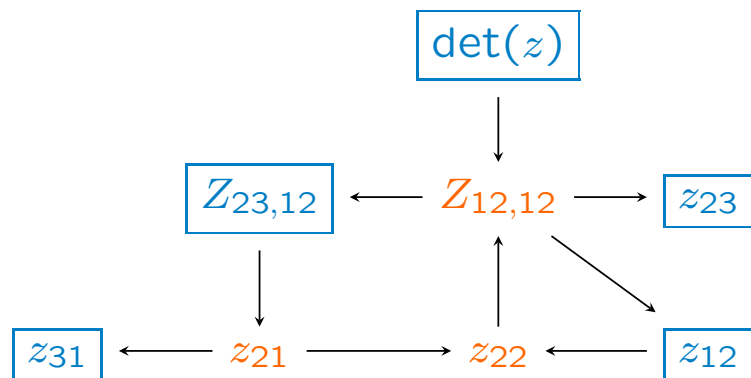
$$z_{11}Z_{23,23} = \det(z) + z_{12}Z_{23,13}$$



$$Z_{12,12}z_{33} = \det(z) + z_{23}Z_{13,12}$$

$$Z_{13,12}z_{21} = z_{11}Z_{23,12} + z_{31}Z_{12,12}$$

$$z_{11}z_{32} = z_{12}z_{31} + Z_{13,12}$$

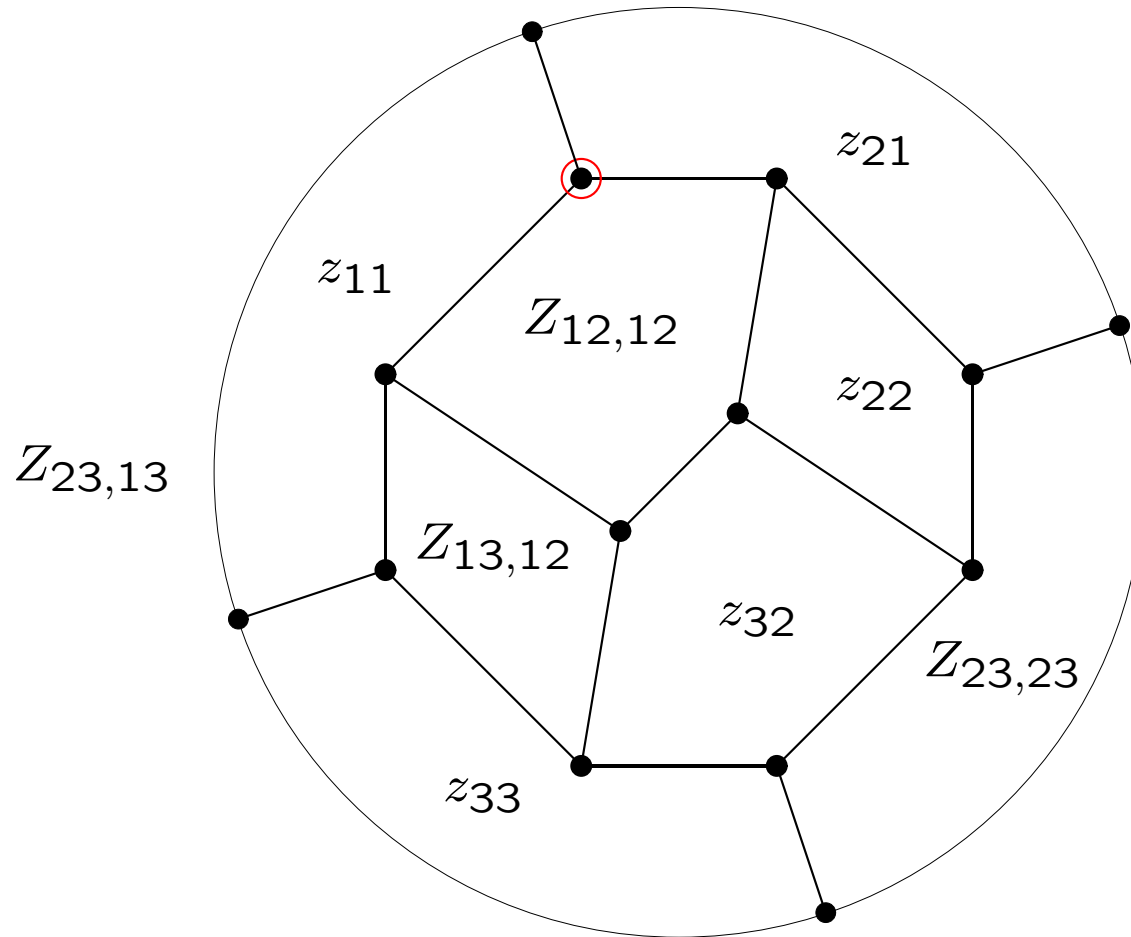


$$Z_{12,12}Z_{23,23} = z_{12}z_{23}Z_{23,12} + \det(z)z_{22}$$

$$z_{21}z_{32} = z_{22}z_{31} + Z_{23,12}$$

$$z_{22}z_{11} = z_{21}z_{12} + Z_{12,12}$$

The 9 cluster variables appearing in the seeds listed above comprise the complete list of cluster variables for this cluster algebra. These cluster variables are grouped into the following 14 clusters:



The initial cluster is marked by a red circle.

Exercise (S.-W. Yang–A. Zelevinsky [*Transform. Groups* **13** (2008)]).

Let R be the coordinate ring of the space of 4×4 matrices z of the form

$$z = \begin{bmatrix} v_1 & q_1 & 0 & 0 \\ 1 & v_2 & q_2 & 0 \\ 0 & 1 & v_3 & q_3 \\ 0 & 0 & 1 & v_4 \end{bmatrix} \in \mathrm{SL}_4(\mathbb{C}).$$

Thus R has 7 generators $q_1, q_2, q_3, v_1, v_2, v_3, v_4$ satisfying a single relation

$$\det(z) = v_1 v_2 v_3 v_4 - q_1 v_3 v_4 - q_2 v_1 v_4 - q_3 v_1 v_2 + q_1 q_3 = 1.$$

Verify that the initial seed

$$\begin{array}{ccccc} v_1 & \leftarrow & \det \begin{pmatrix} v_1 & q_1 \\ 1 & v_2 \end{pmatrix} & \leftarrow & \det \begin{pmatrix} v_1 & q_1 & 0 \\ 1 & v_2 & q_2 \\ 0 & 1 & v_3 \end{pmatrix} \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{q_1} & & \boxed{q_2} & & \boxed{q_3} \end{array}$$

defines a cluster structure on R . Compute all its seeds.

Remark. This is a cluster algebra of type A_3 with principal coefficients.