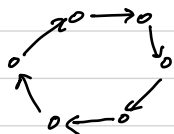


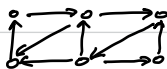
Exercises on quiver representations and cluster algebras

Let $k = \mathbb{C}$.

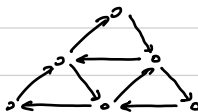
Ex. 0: What are the mutation equivalences among the following quivers?



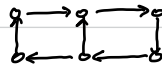
①



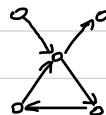
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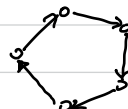
③



④



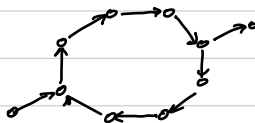
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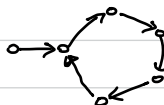
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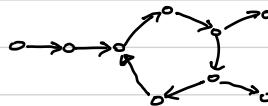
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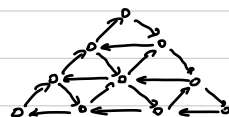
⑧



⑨



⑩



⑪

Ex. 1: Let Q be the quiver $\begin{array}{ccc} 1 & \searrow & \\ 2 & \rightarrow & 4 \\ 3 & \nearrow & \end{array}$ and V its representation $\begin{array}{ccc} k & \xrightarrow{[\cdot]} & k^2 \\ k & \xrightarrow{[\cdot]} & k^2 \end{array}$.

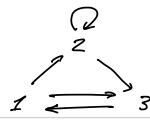
Recall the Caldero-Chapoton formula:

$$CC(V) = \frac{1}{x_1^{d_1} \dots x_n^{d_n}} \left(\sum_{0 \leq c \leq d} \chi(G_c(V)) \cdot \prod_{i=1}^n x_i^{\sum_{j \rightarrow i} c_j + \sum_{i \rightarrow j} (d_j - c_j)} \right)$$

Determine the dimension vectors $0 \leq c \leq d$ s.t. $G_c(V) = \emptyset$ and show that

$$CC(V) = \frac{1}{x_1 x_2 x_3 x_4^2} (1 + 2x_1 x_2 x_3 + x_1^2 x_2^2 x_3^2 + 3x_1 x_2 x_3 x_4 + 3x_4 + 3x_4^2 + x_4^3).$$

Ex-2: Let Q be an arbitrary finite quiver, e.g.



Let $V, W \in \text{rep}_k(Q)$. The **derived morphism complex** $R\text{Hom}(V, W)$ is the complex

$$\begin{array}{c}
 \text{degree 0} \qquad \qquad \qquad \text{degree 1} \\
 0 \rightarrow \prod_{i \in Q_0} \text{Hom}(V_i, W_i) \xrightarrow{\alpha: i \rightarrow j} \prod \text{Hom}(V_i, W_j) \rightarrow 0 \\
 (f_i) \longmapsto (W_\alpha \circ f_i - f_j \circ V_\alpha)_{\alpha \in Q},
 \end{array}
 \quad
 \begin{array}{ccc}
 V_i & \xrightarrow{V_\alpha} & V_j \\
 f_i \downarrow & \searrow g_\alpha & \downarrow f_j \\
 W_i & \xrightarrow{W_\alpha} & W_j
 \end{array}$$

a) Show that we have a natural isomorphism $\text{Hom}(V, W) \xrightarrow{\sim} H^0 R\text{Hom}(V, W)$.

b) Let $\text{Ext}^i(V, W) = H^i R\text{Hom}(V, W)$. Show that if $Q: 1 \rightrightarrows 2$ and

$$V(x) = k \xrightarrow{x_0} k, \quad (x_0: x_1) \in \mathbb{P}^1(k), \quad \text{then} \quad \dim \text{Ext}^1(V(x), V(x)) = 1.$$

c) Show that we have a natural bijection from $\text{Ext}^1(V, W)$ to the set of equivalence classes of short exact sequences $E: 0 \rightarrow W \xrightarrow{i} E \xrightarrow{p} V \rightarrow 0$, where $E \sim E'$ iff we have a com. diagram

$$\begin{array}{ccccccc}
 E: & 0 & \rightarrow & W & \xrightarrow{i} & E & \xrightarrow{p} V \rightarrow 0 \\
 & & & \parallel & & \downarrow p' & \parallel \\
 E': & 0 & \rightarrow & W & \xrightarrow{i'} & E' & \xrightarrow{p'} V \rightarrow 0
 \end{array}$$

Ex. 3: Let Q be an arbitrary finite quiver. Let $d = (d_i) \in \mathbb{N}^{Q_0}$.

Let $\text{rep}(Q, d) = \prod_{\alpha: i \rightarrow j} M_{d_j, d_i}(\mathbb{C})$ be the variety of repres. of dim d of Q .

Let $G(d) = \prod_{i \in Q_0} GL_{d_i}(\mathbb{C})$ act on $\text{rep}(Q, d)$ by "base change", i.e.

$g \cdot x$ is the family with components $(g \cdot x)_\alpha = g_j x_\alpha g_i^{-1}$ for $\alpha: i \rightarrow j$ in Q .

a) Show that we have a bijection

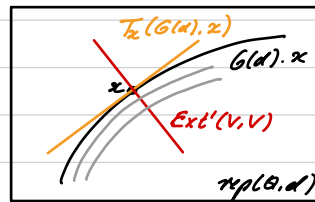
$$\{G(d)\text{-orbits in } \text{rep}(Q, d)\} \xrightarrow{\sim} \{\text{isoclasses of repres. } V \text{ with } \dim V = d\}$$

b) Let $x \in \text{rep}(Q, d)$ and V the representation with $V_i = k^{d_i}$, $V_\alpha = x_\alpha$ for $\alpha \in Q_1$.

Show that we have a natural bijection

$$\text{Ext}^1(V, V) \xrightarrow{\sim} T_x(\text{rep}(Q, d)) / T_x(G(d) \cdot x)$$

where $T_x(\dots)$ denotes the tangent space.



Ex. 4*: Let $Q: 1 \rightrightarrows 2$ be the Kronecker quiver and $\mathcal{C} = \overset{(*)}{\mathcal{D}_Q} / \tau^\mathbb{Z}$ the orbit category of $\mathcal{D}_Q = \mathcal{D}^b(\text{mod } kQ)$ under the action of the Auslander-Reiten translation τ .

Show that $\mathcal{C}$ cannot carry any structure of triangulated category.

Hint: Show that in a triangulated category, each monomorphism $i: X \rightarrow Y$ splits, i.e. there is $r: Y \rightarrow X$ s.th. $r \circ i = \text{id}_X$. Then construct a non split monomorphism $i: X \rightarrow Y$ in $\mathcal{C}$.

(*) Equivalently: $\mathcal{C} = \mathcal{D}^b(\text{loh } P^1(\mathcal{C})) / (1 \otimes U(2))^\mathbb{Z}$