

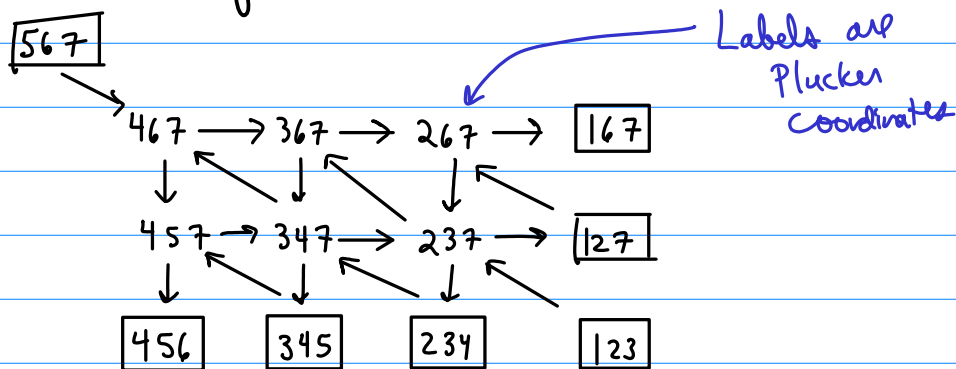
Exercises (for Lauren Williams' lecture)

- Verify that for a 2×4 matrix A , we have

$$p_{13}(A) p_{24}(A) = p_{12}(A) p_{34}(A) + p_{14}(A) p_{23}(A)$$
- Verify that if triangulations T and T' are connected by a flip at diagonal i , then the corresponding quivers Q_T and $Q_{T'}$ are related by $\hookrightarrow$ mutation at i , i.e.

$$\mu_i(Q_T) = Q_{T'}$$

- Recall the rectangles seed $\Sigma_{a,b}$:
 (example for $a=3, b=7$)



Show that if one starts from $\Sigma_{a,b}$ & mutates each mutable vertex exactly once, by mutating each row from L to R , starting from bottom & ending at top, then one gets $\Sigma_{a,b}^1$, the cyclic shift of $\Sigma_{a,b}$. (You can do it for $a=3, b=7$)

- Choose $i < j < k < d < e < f$

Let $\Omega = P_{ikf} \cdot P_{jde} - P_{jkd} \cdot P_{ief}$

Verify that

$$P_{ijk} \Omega = P_{ikf} P_{ijd} P_{jke} + P_{ikd} P_{ije} P_{jkf}$$

- The equation $P_{135} P_{246} - P_{134} P_{256} - P_{136} P_{245} - P_{123} P_{456} = 0$ holds for any 3×6 matrix (or element of $Gr_{3,6}$). However, it is not in the ideal generated by 3-term Plucker relations.

Show that we can multiply it by a monomial in Plucker coord's s.t. the result lies in the ideal gen by 3-term Plucker relations.

- Let $V = \mathbb{C}^{2k}$ be vect. space w/ coordinates $(x_1, \dots, x_{2k})$. Consider nondegenerate quadratic form Q on V given by
- $$Q(x_1, \dots, x_{2k}) = \sum_{i=1}^k (-1)^{i-1} x_i x_{2k+1-i}$$

Let $\mathcal{C} = \{v \in V \mid Q(v) = 0\}$ be isotropic cone and $\mathbb{P}(\mathcal{C})$ the corresp smooth quadric in $\mathbb{P}(V)$.

The homog. coord. ring of the quadric (or equiv, the coord ring of $\mathcal{C}$) is

$$A = \mathbb{C}[x_1, \dots, x_{2k}] / Q(x_1, \dots, x_{2k}).$$

Show that A is a cluster algebra.

Hint: For $1 \leq s \leq k-3$, define

$$p_s = \sum_{i=1}^{s+1} (-1)^{s+1-i} x_i x_{2k+1-i}$$

Then A has:

cluster variables $\{x_2, x_3, \dots, x_{k-1}\} \cup \{x_{k+2}, x_{k+3}, \dots, x_{2k-1}\}$

frozen variables $\{x_1, x_k, x_{k+1}, x_{2k}\} \cup \{p_s \mid 1 \leq s \leq k-3\}$.

Verify that there are 2^{k-2} clusters,
which are defined by choosing,

for each $2 \leq i \leq k-1$, exactly one of x_i and x_{2k+1-i} .

Exchange relations: (Fill in $\rightarrow$)

$$x_i x_{2k+1-i} = \begin{cases} \text{if } i=2 \\ \text{if } 3 \leq i \leq k-2 \\ \text{if } i=k-1 \end{cases}$$

What is the cluster type of A ?