

1. Abelian categories

$\mathcal{A}$ eg. $\text{Rep}(Q)$, $\text{Mod}(R)$, $\text{Coh}(X)$

Have (co)kernels, exact sequences, complexes just as in $\text{Mod}(R)$. A s.e.s.

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

exhibits N as a "bound state" of M, P .

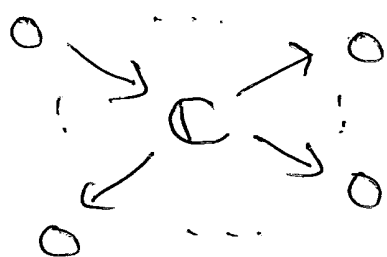
The "glue" is a class $\eta \in \text{Ext}_{\mathcal{A}}^1(P, M)$.

An object $S \in \mathcal{A}$ is simple if $\nexists 0 \subsetneq A \subsetneq S$.

Can think of it as a "fundamental particle".

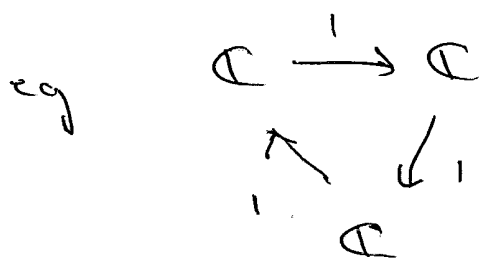
Example: $\mathcal{A} = \text{Rep}(Q)$

Vertex $i \in Q_0 \rightsquigarrow$ Simple $S_i \in \mathcal{A}$



$$\dim_{\mathbb{C}}(S_i) = 1$$

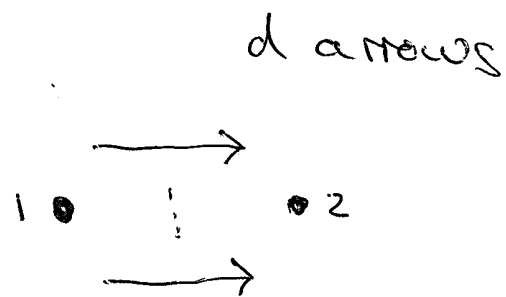
If Q has oriented cycles there are more simple



S_i is a vertex simple.

Example: $\mathcal{A} = \text{Rep}(Q)$

Q d -Kronecker quiver



$$0 \rightarrow \begin{array}{c} 0 \\ \downarrow \downarrow \\ \mathbb{C} \end{array} \rightarrow \begin{array}{c} \mathbb{C} \\ \downarrow \alpha_1 \end{array} \cdots \begin{array}{c} \mathbb{C} \\ \downarrow \alpha_d \end{array} \rightarrow \begin{array}{c} \mathbb{C} \\ \downarrow \downarrow \\ 0 \end{array} \rightarrow 0$$

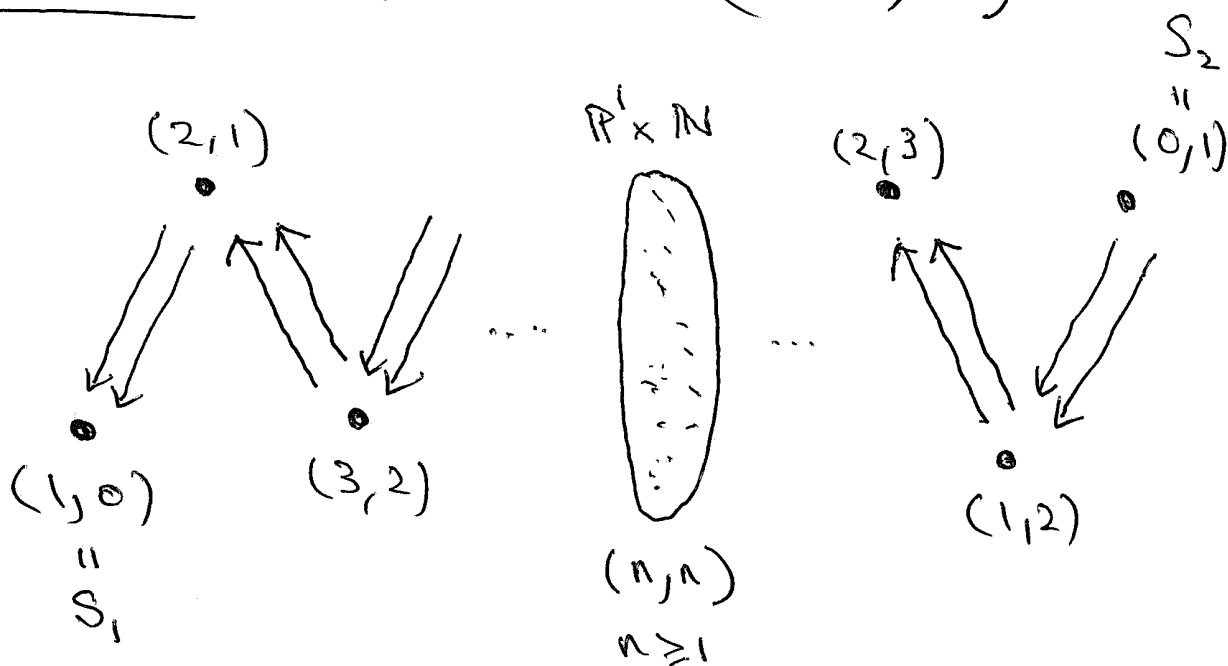
$$0 \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow 0$$

$$\Rightarrow \text{Ext}'_{\mathcal{A}}(S_1, S_2) = \mathbb{C}^d$$

For general Q , $i, j \in Q_0$

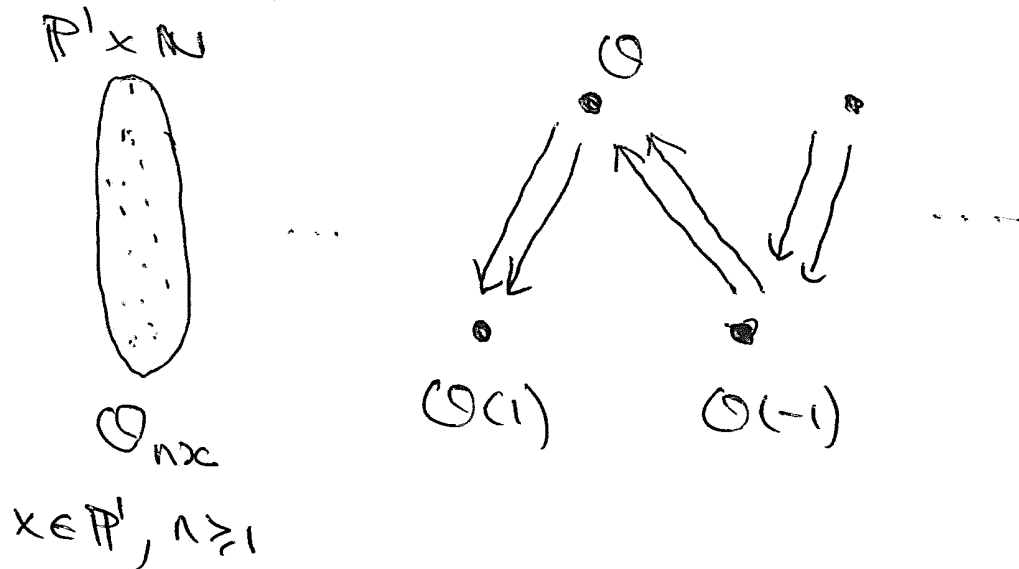
$$\text{Ext}'_{\mathcal{A}}(S_i, S_j) = \mathbb{C}^{\text{arrows } i \rightarrow j}$$

Example: $\mathcal{A} = \text{Rep}(\bullet \rightrightarrows \bullet)$



We drew indecomposable objects and irreducible morphisms.

Example : $\mathcal{A} = \text{Coh}(\mathbb{P}^1)$



2. Derived categories

$\mathcal{A}$ $\rightsquigarrow$ $D^b(\mathcal{A})$
 abelian category $\rightsquigarrow$ bounded derived category

Objects are bounded cochain complexes in $\mathcal{A}$.

$$0 \rightarrow \dots \rightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \rightarrow \dots \rightarrow 0$$

$$\forall i \quad d^i \circ d^{i-1} = 0$$

Morphisms are harder

to describe. We will give basic properties

$\exists$ functors

- $H^i : D^b(\mathcal{A}) \rightarrow \mathcal{A}$ "ith cohomology"

- $[1] : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A})$ "shift functor"

$$(M^\bullet[1])^i = M^{i+1}, \quad d_{M^\bullet[1]}^i = -d_{M^\bullet}^i$$

Lemma : $\mathcal{A} \hookrightarrow D^b(\mathcal{A})$

$$M \mapsto \cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$$

is fully faithful and induces an equivalence

$$\mathcal{A} \xrightarrow{\sim} \{E \in D^b(\mathcal{A}) : i \neq 0 \Rightarrow H^i(E) = 0\}$$

Objects concentrated in degree 0

Moreover $\operatorname{Hom}_{D^b(\mathcal{A})}^i(M, N[i]) = \operatorname{Ext}_{\mathcal{A}}^i(M, N)$

for $M, N \in \mathcal{A}$ ($= 0$ when $i < 0$)

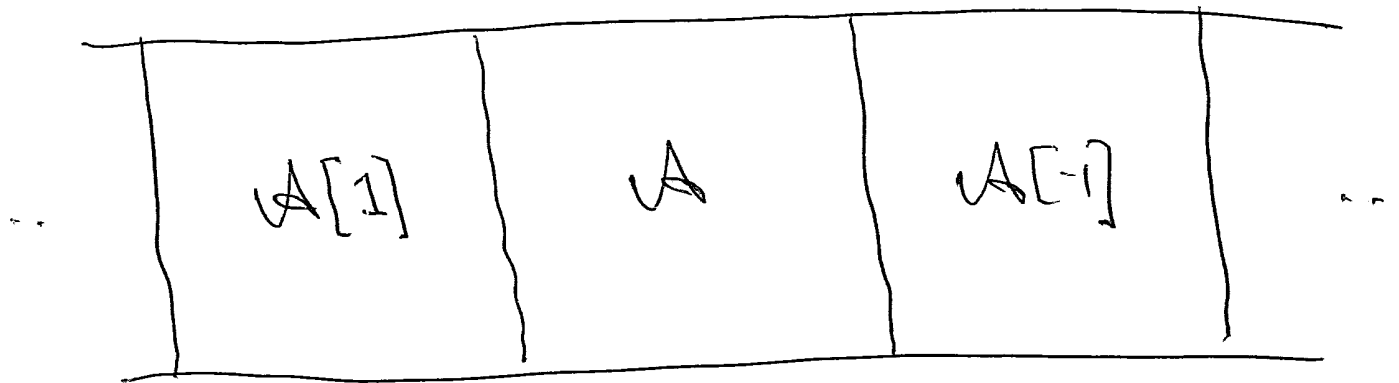
Lemma : Suppose $\operatorname{gl dim}(\mathcal{A}) \leq 1$

[i.e. $\operatorname{Ext}_{\mathcal{A}}^i(M, N) = 0 \quad \forall i \geq 1 \quad \forall M, N \in \mathcal{A}$]

eg. $\text{Rep}(\mathcal{Q})$ or $\text{Coh}(X)$ if $\dim_{\mathbb{C}}(X) \leq 1$]

Then $E \cong \bigoplus_{i \in \mathbb{Z}} H^i(E)[-i] \quad \forall E \in D^b(\mathcal{A})$

So in this case the picture is



Note: $D^b \text{Rep}(\bullet \rightrightarrows \bullet) \cong D^b \text{Coh}(\mathbb{P}^1)$ $D^b(\mathcal{A})$

3. Triangulated categories

$D^b(\mathcal{A})$ is not abelian but triangulated
(extra structure)

• $[1]: D^b(\mathcal{A}) \hookrightarrow$ "shift"

• distinguished triangles

$$M \xrightarrow{f} N$$

$$\begin{array}{ccc} & \nearrow h & \nwarrow g \\ & P & \end{array}$$

$$\begin{array}{c} \dots \rightarrow P[-1] \xrightarrow{h[-1]} M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} \dots \\ \hookrightarrow M[-1] \xrightarrow{f[-1]} N[-1] \rightarrow \dots \end{array}$$

In $D^b(\mathcal{A})$ these come from mapping cones
 $\Rightarrow$ long exact seq. in cohomology

$$\begin{array}{c} \cdots \longrightarrow H^{i-1}(P) \longrightarrow \\ \hookrightarrow H^i(M) \longrightarrow H^i(N) \longrightarrow H^i(P) \longrightarrow \\ \hookrightarrow H^{i+1}(M) \longrightarrow \cdots \end{array}$$

A s.e.s. in $\mathcal{A}$

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

gives a triangle

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \nearrow h & & \searrow g \\ & P & \end{array}$$

$$h \in \text{Hom}_{D^b(\mathcal{A})}(P, M[i]) = \text{Ext}'_{\mathcal{A}}(P, M)$$

So triangles generalise s.e.s. and should be thought of as bound states. But

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \nearrow h & & \searrow g \\ & P & \end{array}$$

distinguished

$\Leftrightarrow$

$$\begin{array}{ccc} N & \xrightarrow{g} & P \\ \nearrow f[i] & & \searrow h \\ & M[i] & \end{array}$$

distinguished

"anti-particles"!

4. Filtrations

In an abelian category $\mathcal{A}$ have filtrations

$$0 = E_0 \subset E_1 \subset \dots \subset E_{n-1} \subset E_n = E$$

Setting $F_i = E_i / E_{i-1}$ have s.e.s.

$$\begin{array}{ccccccc} 0 = E_0 & \hookrightarrow & E_1 & \hookrightarrow & \dots & \hookrightarrow & E_{n-1} \hookrightarrow E_n = E \\ & & \searrow & & & & \searrow \\ & & F_1 & & & & F_n \end{array}$$

Can make longer filtrations by gluing in filtrations of the F_i , or shorter ones by forgetting some of the E_i

$$\begin{array}{ccccc} E_1 & \rightarrow & E_2 & \rightarrow & F_2 \\ \parallel & & \downarrow & & \downarrow \\ E_1 & \rightarrow & E_3 & \rightarrow & F \\ & & \downarrow & & \downarrow \\ & & F_3 & = & F_3 \end{array} \quad \begin{array}{l} E_3 / E_2 \cong \frac{(E_3 / E_1)}{(E_2 / E_1)} \\ \text{3rd iso. theorem} \end{array}$$

(*)

We can play the same games in triangulated categories.

$$0 = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_{n-1} \rightarrow E_n = E$$

$\nwarrow \searrow$
 F_1

$\nwarrow \searrow$
 F_n

The analogue of $(*)$ is the octahedral axiom,

Standard ~~structures~~ truncations

$\mathcal{A}$ abelian, $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$.

Truncation $T_{\leq i} : \mathcal{D} \rightarrow \mathcal{D}$
functors

$$T_{\leq i}(M^\bullet) = (\dots \rightarrow M^{i-1} \xrightarrow{d^{i-1}} \ker(d^i) \rightarrow 0 \rightarrow \dots)$$

$$H^j(T_{\leq i}(M^\bullet)) = \begin{cases} H^j(M^\bullet) & j \leq i \\ 0 & j > i \end{cases}$$

Have triangles in $\mathcal{D}$

$$T_{\leq i-1}(M^\bullet) \xrightarrow{\eta_{i-1}} T_{\leq i}(M^\bullet)$$

$\nwarrow$

$\searrow$

$$H^i(M^\bullet)[-i]$$

Proof: Take the cone on η_{i-1} . The long exact sequence in cohomology $\Rightarrow$ cone concentrated in degree i .

So every object $M \in D^b(\mathcal{A})$ has a canonical filtration by the objects $H^i(M)[-i]$.

6. Hearts

$\mathcal{D}$ a triangulated category

Defⁿ: A full subcat $\mathcal{A} \subset \mathcal{D}$ is a heart if

$$(a) \ A_1, A_2 \in \mathcal{A}, \ i_1 > i_2 \in \mathbb{Z}$$

$$\Rightarrow \operatorname{Hom}_{\mathcal{D}}(A_1[i_1], A_2[i_2]) = 0$$

$$(b) \ \forall E \in \mathcal{D} \ \exists k_1 > \dots > k_n \in \mathbb{Z}$$

and triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \rightarrow & E_1 & \rightarrow & \dots & \rightarrow & E_{n-1} \rightarrow E_n = E \\
 & \nearrow & \searrow & & & & \nearrow & \searrow \\
 & & A_1[k_1] & & & & A_n[k_n] &
 \end{array}$$

with $A_1, \dots, A_n \in \mathcal{A}$.

Example : Standard heart $\mathcal{A} \subset \mathcal{D}^b(\mathcal{A})$

$$(a) \operatorname{Ext}_{\mathcal{A}}^k(A_1, A_2) = 0 \quad k < 0$$

(b) ~~standard~~ truncation functors.

Prop : $\mathcal{A}$ heart is an abelian category.

(a) $\Rightarrow$ filtration in (b) unique.

$$\text{Set } H_{\mathcal{A}}^j(E) = \begin{cases} A_i & j = -k_i \\ 0 & \text{o/wise} \end{cases}$$

Get functors $H_{\mathcal{A}}^i : \mathcal{D} \rightarrow \mathcal{A}$ and

triangles give long exact seq. as before.

7. Grothendieck groups

$$K_0(\mathcal{D}) = \left(\bigoplus_{M \in \mathcal{D}/\cong} \mathbb{Z} \cdot [M] \right) / \left[\begin{array}{ccc} & & [N] = [M] + [P] \\ & M \rightarrow N & \\ \nearrow X & & \searrow \\ & P & \end{array} \right]$$

Rotating the triangle

$$\Rightarrow [M[1]] = -[M]$$

$$\mathcal{A} \subset \mathcal{D} \quad \leadsto \quad K_0(\mathcal{A}) \xrightarrow{\cong} K_0(\mathcal{D})$$

heart

Inverse map

$$[E] \in K_0(\mathcal{D}) \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [H_{\mathcal{A}}^i(E)] \in K_0(\mathcal{A})$$

8. Tilting

Defⁿ: A torsion pair in an abelian cat. $\mathcal{A}$ is a pair of full subcats $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ s.t.

$$(a) \quad \text{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0 \quad \mathcal{T} \in \mathcal{T}, \mathcal{F} \in \mathcal{F}$$

$$(b) \quad \forall E \in \mathcal{A} \quad \exists \text{ s.e.s.}$$

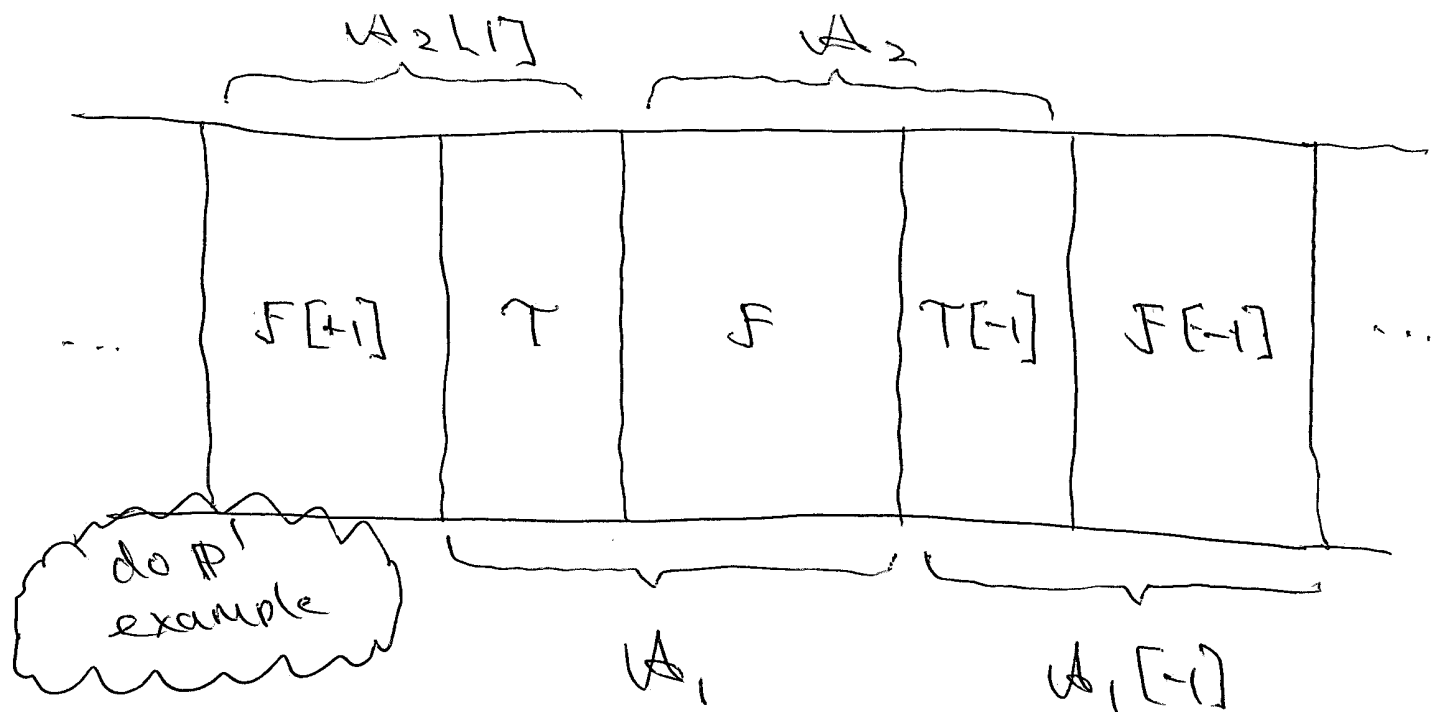
$$0 \rightarrow \mathcal{T} \rightarrow E \rightarrow \mathcal{F} \rightarrow 0 \quad \mathcal{T} \in \mathcal{T}, \mathcal{F} \in \mathcal{F}$$

Lemma: $\mathcal{A}_1 \subset \mathcal{D}$ heart, $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}_1$,
torsion pair.

Then

$$\mathcal{A}_2 = \left\{ E \in \mathcal{D} : H_{\mathcal{A}}^i(E) \in \begin{cases} \mathcal{F} & i=0 \\ \mathcal{T} & i=1 \\ 0 & \text{o/wise} \end{cases} \right\}$$

is a heart in $\mathcal{D}$.



Lemma : Given hearts $(\mathcal{A}_1, \mathcal{A}_2) \subset \mathcal{D}$

TRUE :

(i) $E \in \mathcal{A}_1 \Rightarrow H_{\mathcal{A}_2}^i(E) = 0$ unless $i = -1, 0$.

(ii) $E \in \mathcal{A}_2 \Rightarrow H_{\mathcal{A}_1}^i(E) = 0$ unless $i = 0, 1$.

(iii) $\mathcal{A}_2$ arises from above construction.

We call $(\mathcal{A}_1, \mathcal{A}_2)$ a tilting pair, $\mathcal{A}_2$ a right tilt of $\mathcal{A}_1$, $\mathcal{A}_1$ a left tilt of $\mathcal{A}_2$.

Note : $(\mathcal{A}_2[1], \mathcal{A}_1)$ also a tilting pair.

Hearts most basic structure in Δ -category,
tilting the basic adjacency relation.

9. Finite hearts

$\mathcal{D}$ Hom-finite triangulated cat. / $\mathbb{C}$

$$\dim_{\mathbb{C}} \left(\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(E, F[i]) \right) < \infty$$

Can define Euler form

$$X(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}(E, F[i])$$

$$X(-, -) : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \rightarrow \mathbb{Z}$$

A heart $\mathcal{A} \subset \mathcal{D}$ is finite-length if it is Artinian and Noetherian. This gives Jordan-Hölder filtrations

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = E \quad F_i = E_i / E_{i-1} \text{ simple}$$

We call $\mathcal{A} \subset \mathcal{D}$ a finite heart if also $\exists$ only finitely many simples (up to iso)

$$K_0(\mathcal{D}) = K_0(\mathcal{A}) = \bigoplus_{i=1}^n \mathbb{Z} \cdot [S_i]$$

Get basis for $K_0(\mathcal{D})$.

Define quiver $Q(A)$: Vertices are iso. classes of simples, and with

$$e_{jk} = \dim_{\mathbb{Q}} \text{Ext}_{\mathcal{A}}^1(S_j, S_k) \text{ arrows } j \rightarrow k.$$

Example : $\mathcal{Q}$ a finite quiver without oriented cycles, then $\mathcal{A} = \text{Rep}(\mathcal{Q})$ is a finite heart in $D^b(\mathcal{A})$ and $Q(\mathcal{A}) = \mathcal{Q}$. Same holds for any finite quiver if we consider completed path algebra.

Simple tilts

$\mathcal{A} \subset \mathcal{D}$ finite heart, $S \in \mathcal{A}$ simple

$\langle S \rangle \subset \mathcal{A}$ full-subcat of objects whose JH factors are all $\cong S$.

$${}^{\perp}S = \{ E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(E, S) = 0 \}$$

$$S^{\perp} = \{ E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(S, E) = 0 \}$$

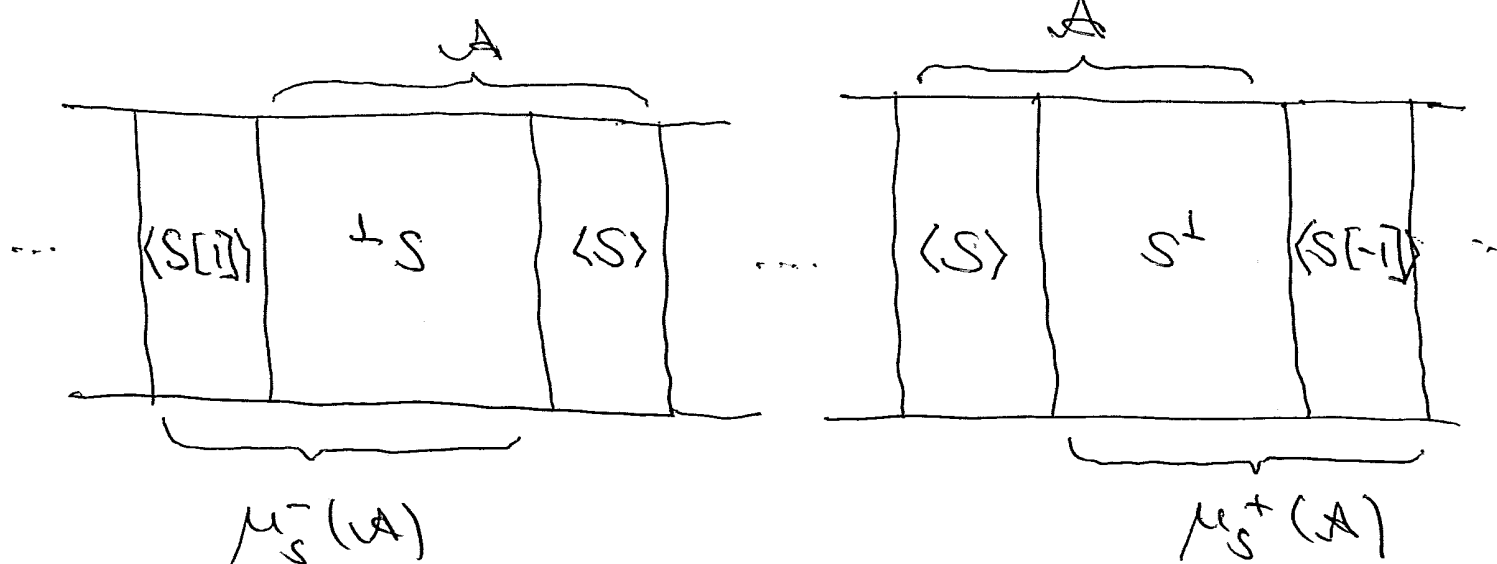
Get torsion pairs $({}^{\perp}S, \langle S \rangle), (\langle S \rangle, S^{\perp})$

and corresponding tilts

$$\mu_S^-(A) = \{ E \in \mathcal{D} : H^i(E) \in \begin{cases} \langle S \rangle & i = -1 \\ \perp S & i = 0 \\ 0 & \text{o/w} \end{cases} \}$$

$$\mu_S^+(A) = \{ E \in \mathcal{D} : H^i(E) \in \begin{cases} S^\perp & i = 0 \\ \langle S \rangle & i = 1 \\ 0 & \text{o/w} \end{cases} \}$$

~~But a finite heart may not be finite~~



Prop: $A \in \mathcal{D}$ finite heart, $\mathcal{Q}(A)$ has no loops. Then $\mu_{S_i}^+(A)$ is a finite heart with simples $\hat{S}_i = S_i[-1]$ and universal extensions

$$0 \rightarrow S_j \rightarrow \hat{S}_j \rightarrow S_i^{\oplus e_{ij}} \rightarrow 0$$

(*)

for $j \neq i$

1. Proof: Write $S = S_i$. Applying $\text{Hom}_D(S, -)$ to $*$ gives

$$0 \rightarrow \text{Hom}_A(S, \hat{S}_j) \rightarrow \mathbb{C}^{e_{ij}} \rightarrow \mathbb{C}^{e_{ij}} \rightarrow 0$$

$$\hookrightarrow \text{Ext}_A^1(S, \hat{S}_j) \rightarrow 0$$

$$\Rightarrow \text{Hom}_A(S, \hat{S}_j) = \text{Ext}_A^1(S, \hat{S}_j) = 0$$

Consider s.e.s. in $\mu_S^+(A)$

$$0 \rightarrow A \rightarrow \hat{S}_j \rightarrow B \rightarrow 0$$

Taking l.e.s. in A -cohomology gives

$$0 \rightarrow H_A^0(A) \rightarrow \hat{S}_j \rightarrow H_A^0(B) \rightarrow H_A^1(A) \rightarrow 0$$

$$\begin{array}{ccccccc} \cap & & \searrow & \nearrow & \cap & & \cap \\ S^\perp & & P & & S^\perp & & \langle S \rangle \end{array}$$

$$P \in \langle S \rangle \Rightarrow P=0 \Rightarrow H_A^0(B) = H_A^1(A)$$

$$\Rightarrow B=0$$

$$H_A^0(A) \in \langle S \rangle \Rightarrow P = \hat{S}_j \Rightarrow \begin{array}{l} \text{2nd s.e.s} \\ \text{splits} \end{array} \Rightarrow A=0$$

A similar argument shows $\hat{S}_i$ is simple.
 Rotating $\oplus$ gives s.e.s. in $\mu_{S_i}^+(\mathcal{A})$

$$S_i^{\oplus e_{ij}}[-1] \rightarrow S_j \rightarrow \hat{S}_j$$

Thus all objects of $\mu_{S_i}^+(\mathcal{A})$ have filtrations by the $\hat{S}_j$. The cone in $K_0(\mathcal{D})$ generated by the $[\hat{S}_j]$ does not contain 0. So $\exists \ell: K_0(\mathcal{D}) \rightarrow \mathbb{Z}$ s.t. $\ell(E) > 0 \quad \forall_{0 \neq E \in \mu_{S_i}^+(\mathcal{A})}$. This gives finite-length. $\square$

Defn: The tilting graph $\text{Tilt}(\mathcal{D})$ is the oriented graph with vertices finite hearts, and arrows $\mathcal{A} \rightarrow \mu_S^+(\mathcal{A})$ given by simple tilts.

12. CY₃ setting + mutations

Abstracting Serre duality on $D^b\text{Coh}(X)$

with X a smooth proj. CY 3-fold gives

Defⁿ: A Hom-finite $\mathbb{K}$ triangulated cat.
is CY_3 if $\exists$ functorial isos

$$\mathrm{Hom}_{\mathcal{D}}(E, F[i]) \cong \mathrm{Hom}_{\mathcal{D}}(F, E[3-i])^*$$

$\Rightarrow \chi(-, -)$ is skew-symmetric.

$\mathcal{A} \subset \mathcal{D}$ finite heart, simples $S_1, \dots, S_n$

~~Hom~~ ~~$\mathcal{A}$~~

$$\mathrm{Hom}_{\mathcal{D}}^i(S_j, S_k) = 0 \quad i \leq 0 \quad j \neq k$$

$CY_3 \Rightarrow$ also for $i \geq 3$

$$v_{jk} := -\chi(S_j, S_k) = \dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{D}}^1(S_j, S_k)$$

$$- \dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{D}}^2(S_j, S_k) = e_{jk} - e_{kj}$$

If $Q(\mathcal{A})$ is 2-acyclic, then

$$e_{jk} = \max(0, v_{jk}).$$

Lemma: $\mathcal{D} \subset \mathcal{Y}_3$, $\mathcal{A}_1, \mathcal{A}_2$ finite hearts related by a simple tilt. Then $Q(\mathcal{A}_1)$, $Q(\mathcal{A}_2)$ related by a mutation.

Proof: Can assume $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{A}_2 = \mu_{s_i}^+(\mathcal{A})$.

$$[\hat{s}_j] = \begin{cases} [s_j] + v_{ij} [s_i] & j \neq i, v_{ij} > 0 \\ [s_j] & j \neq i, v_{ij} \leq 0 \\ -[s_i] & j = i \end{cases}$$

This gives

$$\hat{v}_{jk} = -\chi(\hat{s}_j, \hat{s}_k) =$$

$$\begin{cases} v_{jk} + v_{ij} v_{ik} & i \notin \{j, k\} \begin{matrix} v_{ij} > 0 \\ v_{ik} \leq 0 \end{matrix} \\ v_{jk} + v_{ji} v_{ik} & i \notin \{j, k\} \begin{matrix} v_{ij} \leq 0 \\ v_{ik} > 0 \end{matrix} \\ v_{jk} & i \notin \{j, k\} \\ -v_{jk} & i \in \{j, k\} \end{cases}$$

which gives usual mutation rule

13. Ginzburg algebra

Theorem Q 2-acyclic, W generic potential

$$\mathcal{D} = \mathcal{D}^b \Pi_3(Q, W).$$

(i) $\mathcal{D}$ is $\mathcal{Y}_3$ Δ -category

(ii) $\exists$ dist. finite heart $\mathcal{A}_0 \subset \mathcal{D}$ s.t.

$$Q(\mathcal{A}_0) = Q.$$

(iii) Repeatedly applying simple tilts always gives finite hearts with 2-acyclic quivers

$\Rightarrow$ connected cmpt $\text{Tilt}_0(\mathcal{D}) \subset \text{Tilt}(\mathcal{D})$
 containing $\mathcal{A}_0$ is 2n-regular. Vertices
 are called reachable hearts.

$\mathcal{A} \subset \mathcal{D}$ reachable $S_1, \dots, S_n$ simples

$$\text{Hom}_{\mathcal{D}}^d(S_i, S_i) = \begin{cases} \mathbb{Q} & i=0,3 \\ 0 & \text{o/w} \end{cases}$$

$\Rightarrow S_i$ spherical

$\leadsto$ auto-equivalences $\text{Tw}_{S_i} \in \text{Aut}(\mathcal{D})$

$$\begin{array}{ccc} \text{Hom}^0(S, E) \otimes S & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & \text{Tw}_S(E) & \end{array} \quad \forall E \in \mathcal{D}$$

In partic. simples of $\mu_{S_i}^+(\mathcal{A})$ are

$$\hat{S}_j = \begin{cases} Tw_{S_i}(S_j) & i \neq j \quad v_{ij} > 0 \\ S_j & i \neq j \quad v_{ij} \leq 0 \\ S_i[-1] & i = j \end{cases}$$

$$0 \rightarrow S_j \rightarrow Tw_{S_i}(S_j) \rightarrow \text{Ext}_{\mathcal{A}}^1(S_i, S_j) \otimes S_i \rightarrow$$

Using $\mu_{S_i}^{\pm}$ are inverse operation, simples of $\mu_{S_i}^{-}(\mathcal{A})$ are

$$\hat{S}_j = \begin{cases} Tw_{S_i}^{-1}(S_j) & i \neq j \quad v_{ij} < 0 \\ S_j & i \neq j \quad v_{ij} \geq 0 \\ S_i[1] & i = j \end{cases}$$

We find $\boxed{\mu_{S_i}^{+}(\mathcal{A}) = Tw_{S_i}(\mu_{S_i}^{-}(\mathcal{A}))}$

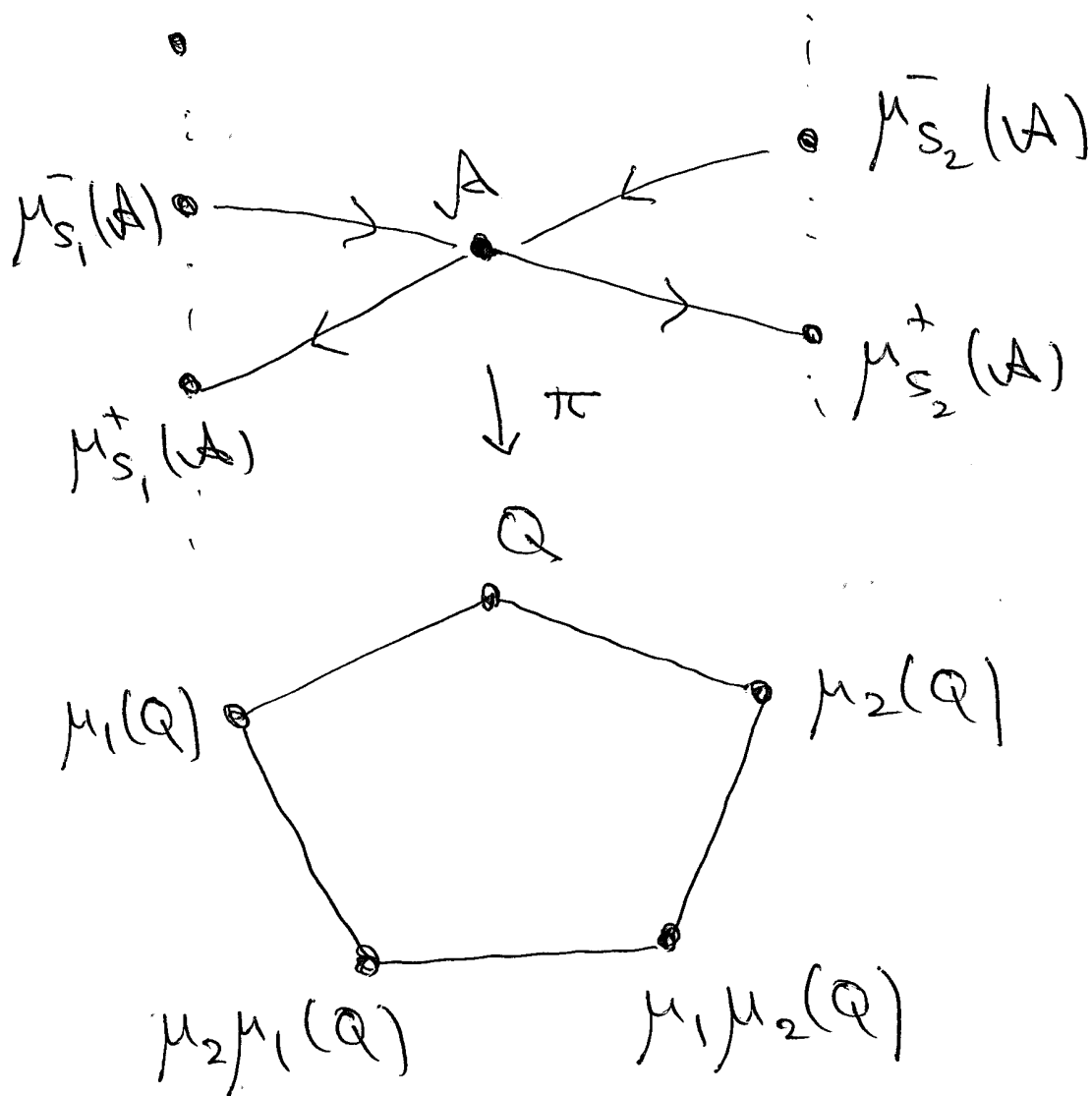
Moreover $Tw_{Tw_{S_i}(S_j)} = Tw_{S_i}^{-1} Tw_{S_j} Tw_{S_i}$

and $Tw_{S_i[n]} = Tw_{S_i}$, so

$\langle Tw_{S_1}, \dots, Tw_{S_n} \rangle \in \text{Aut}(\mathcal{D})$ independent of $\mathcal{A}$.

$Br_0(\mathcal{D})$

Define $\text{Exch}_0(\mathcal{D}) = \frac{\text{Tilt}_0(\mathcal{D})}{\text{Br}_0(\mathcal{D})}$



Amazing fact : $\text{Exch}_0(\mathcal{D})$ coincides with the exchange graph of Q defined using cluster combinatorics

15. Nagao-Plamondon section

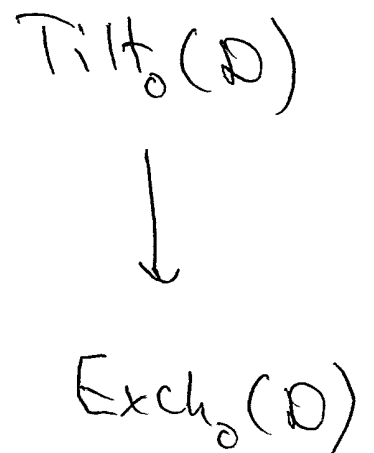
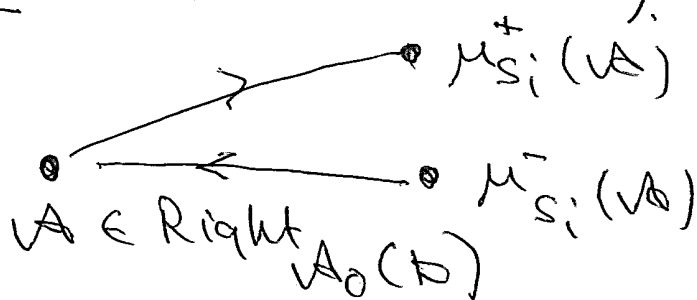
Choose a reachable heart $\mathcal{A}_0 \subset \mathcal{D}$.

$\exists$ subgraph $\text{Right}_{\mathcal{A}_0} \subset \text{Tilt}_0(\mathcal{D})$
consisting of right tilts of $\mathcal{A}_0$.

Theorem : $\text{Right}_{\mathcal{A}_0}(\mathcal{D}) \hookrightarrow \text{Tilt}_0(\mathcal{D}) \twoheadrightarrow \text{Exch}_0(\mathcal{D})$
is. every orbit of

$\text{Br}_0(\mathcal{D})$ on $\text{Tilt}_0(\mathcal{D})$ contains a unique heart which is a right tilt of $\mathcal{A}_0$.

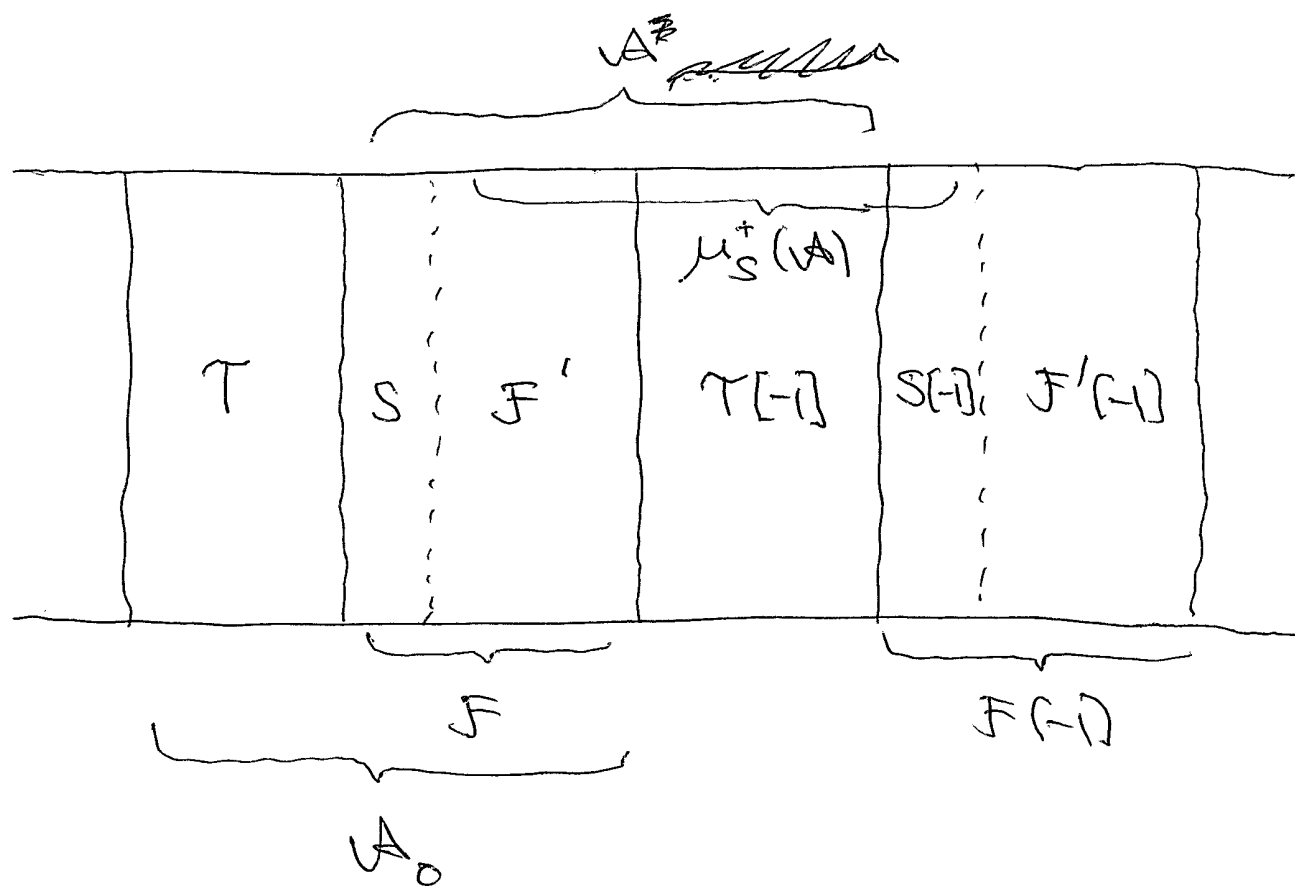
Proof : (Surjectivity)



Either $S \in \mathcal{F}$ or $S \in \mathcal{T}[-1]$

In first case $\mu_S^+(A)$ is a right tilt

In second $\mu_S^-(A)$ is a right tilt.



Consequences : • $\text{Exch}_0(\mathcal{D})$ becomes oriented.

• Each $A \in \text{Right}_{A_0}(\mathcal{D})$ gives a basis of $K_0(\mathcal{D})$ $[S_1], \dots, [S_n]$.

~~$\mathbb{Z}^n$~~ $\mathbb{Z}^{\oplus n}$ $S_i \in A_0$ or $A_0[-1]$
(via A_0) $\Rightarrow \pm[S_i] \in \mathbb{Z}_{\geq 0}^n$

Sign coherence.