

Resch's Apple and the Landau-Zener Problem

On the Spherical Clothoid

Alexandru Ionuț

Overview

1. Summary of prior work
2. Landau-Zener problem
3. New hypergeometric representation
4. Stereographic Projection

The curve that minimizes the L^2 norm of curvature

$$\kappa^2 \tau = C \tag{1}$$

$$[(\kappa^2)']^2 + \kappa^2 [(\kappa^2 - 2D)^2 - 4\psi^2] + 4C^2 = 0 \tag{2}$$

Spherical Clothoid

Assuming $\kappa^2 \ll 2D$ yields:

$$[(\kappa^2)']^2 - 4\alpha^2\kappa^2 + 4C^2 = 0 \quad (3)$$

$\alpha = \psi^2 - D^2$ is a constant. We have a simple solution:

$$\kappa^2 = \alpha^2 s^2 + \frac{C^2}{\alpha^2} \quad (4)$$

For $C \neq 0$, the curve is spherical with $R = \frac{\alpha}{C}$

$$\kappa_g = \alpha s \quad (5)$$

Landau-Zener problem

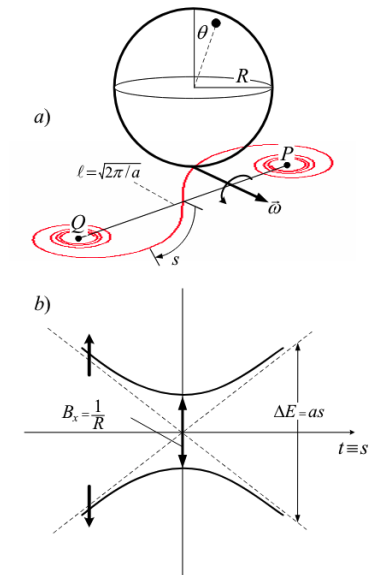
$$\frac{d}{ds} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\sin\left(\frac{\alpha s^2}{2}\right) \\ 0 & 0 & \cos\left(\frac{\alpha s^2}{2}\right) \\ \sin\left(\frac{\alpha s^2}{2}\right) & -\cos\left(\frac{\alpha s^2}{2}\right) & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \quad (6)$$

$$\tilde{x} = ab^* + ba^*, \tilde{y} = i(ab^* - ba^*), \tilde{z} = aa^* - bb^*. \quad (7)$$

a and b must satisfy the following system in order for $\tilde{x}, \tilde{y}$ and $\tilde{z}$ to satisfy (6):

$$i \frac{d}{ds} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & e^{-i\alpha s^2/2} \\ e^{i\alpha s^2/2} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad (8)$$

Rojo Bloch illustration



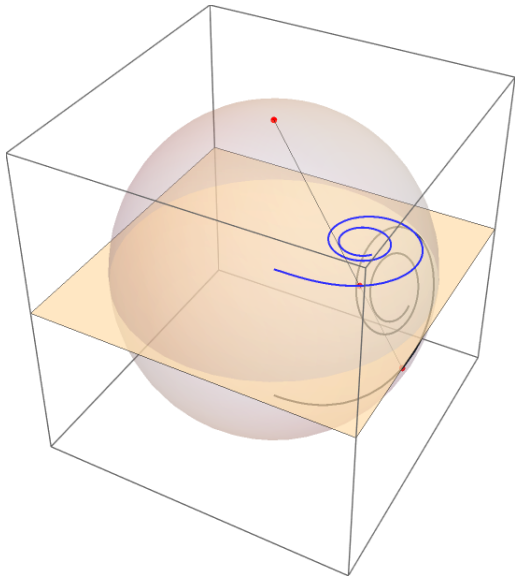
Hypergeometric representation

$$\begin{cases} x(s) = \Re \left(s {}_1F_1 \left(-\frac{i}{8\alpha}, \frac{1}{2}; -\frac{i\alpha s^2}{2} \right) {}_1F_1 \left(\frac{i}{8\alpha} + \frac{1}{2}, \frac{3}{2}; \frac{i\alpha s^2}{2} \right) \right) \\ y(s) = -2 \left| {}_1F_1 \left(\frac{i}{8\alpha}, \frac{1}{2}; \frac{i\alpha s^2}{2} \right) \right|^2 + 2 \\ z(s) = \Im \left(s {}_1F_1 \left(-\frac{i}{8\alpha}, \frac{1}{2}; -\frac{i\alpha s^2}{2} \right) {}_1F_1 \left(\frac{i}{8\alpha} + \frac{1}{2}, \frac{3}{2}; \frac{i\alpha s^2}{2} \right) \right) \end{cases} \quad (9)$$

Stereographic Projection

$$\begin{aligned}\zeta(\alpha; s) &= X(s) + iY(s) \\ &= \frac{x(s) + iz(s)}{1 - (y(s) - 1)} \\ &= \frac{s {}_1F_1\left(\frac{i}{8\alpha} + \frac{1}{2}, \frac{3}{2}; \frac{i\alpha s^2}{2}\right)}{{}_2F_1\left(\frac{i}{8\alpha}, \frac{1}{2}; \frac{i\alpha s^2}{2}\right)}\end{aligned}\tag{10}$$

Plot



The End