

Homogenization theory for high-contrast time-dependent particles

Erik Orvehed Hiltunen, erikhilt@math.uio.no

September 9, 2024

1 Background

The field of *metamaterials* aims to develop materials with, in some sense, *exotic* wave scattering properties. Using a carefully designed, periodic, microstructure, it is possible to design materials with effective properties beyond classical physics, such as vanishing or negative refractive index, or to localize and guide waves along specified positions. Such materials have been studied, both theoretically and experimentally, for a range of wave systems such as acoustic, elastic, and electromagnetic waves [2, 3, 4, 5].

The current frontier of metamaterials lies in *spacetime*-periodic materials. This is a class of materials whose material parameters depend on both space and time. By breaking time-reversal symmetry, it is possible to achieve (for example) nonreciprocal wave propagation. In these materials, a wave is able to propagate freely in one direction, but is exponentially attenuated (decaying) in the opposite direction.

A revolutionary approach for studying materials with a small-scale structure is *effective-medium theory* (or *homogenization theory*). The main idea is to replace a complicated material with a simple, homogeneous material which captures the same wave scattering properties. There is a large field of mathematical literature on homogenization theory. The most interesting case, which is also substantially more challenging, is the homogenization in case of a resonant microstructure [1].

Please note: The next section will give you some flavour of the project. Everything is very briefly described, and there will most likely be things you cannot follow. This is okay! If you are interested in the project, the best thing to do is to reach out directly to me to discuss.

2 Problem formulation

The simplest model for wave propagation in metamaterials is the scalar wave equation, such as

$$\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{n^2(x, t)} \Delta \right) u(x, t) = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}. \quad (1)$$

where $n(x, t)$ describes the refractive index of the material. In this project, we will consider wave scattering from a finite collection of time-dependent particles in three dimensions: $d = 3$. Time-modulated materials means that $n(x, t)$ depends on t . The most important case is when n is periodic in t .

If a wave is scattered by a static particle, energy conservation guarantees the frequency of the scattered wave to agree with the frequency of the incident wave. For time-modulated particles, this basic physical principle is broken. Instead, the time-modulation of frequency Ω induces a coupling between waves of frequencies differing by Ω (c.f. Figure 1). As consequence, if the incident frequency is ω , there will be a family of scattered waves occurring at frequencies $\omega + n\Omega$, for integer n .

The goal of this project is to study wave scattering for a collection of *high-contrast point particles*. Here, “point particle” means that the each particle $D_i \subset \mathbb{R}^3$ is small, and can be written as

$$D_i = \epsilon B + z_i \quad (2)$$

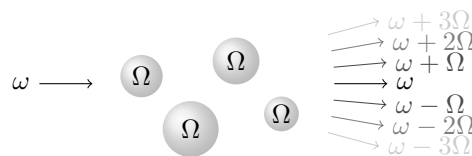


Figure 1: An incident field of a single frequency ω will give rise to a family of scattered harmonics. If the scatterers are modulated at a frequency Ω , the family of coupled harmonics will occur at frequencies $\omega + n\Omega$ for integer n .

for $\epsilon \ll 1$ and some origin $z_i \in \mathbb{R}^3$. Moreover, “high contrast” means that n is large inside D_i . In other words, for x inside D , we assume

$$n^2(x, t) = 1 + \eta(t) \quad (3)$$

for some periodic function $\eta(t)$ scaling as

$$\eta(t) = \frac{s(t)}{\epsilon^2}, \quad (4)$$

as $\epsilon \rightarrow 0$. The motivation for this scaling is that each particle exhibits resonances ω_i , for $i = 1, 2, \dots$. These resonances are frequencies at which the particle strongly interacts with the incident wave, producing a scattered field of large amplitude.

In the case of a single particle, and for a general incident field $u^{\text{in}}(x, t) = \sum_{j=-\infty}^{\infty} v_j^{\text{in}}(x) e^{-i(\omega + j\Omega)t}$, it was recently shown that the i^{th} mode of the scattered field can be approximated as [4]

$$v_i(x) \approx G^{k_i}(x - z_0) \sum_{j=-\infty}^{\infty} \Lambda_{ij} v_j^{\text{in}}(z_0), \quad (5)$$

where $G^k(x)$ is the (outgoing) Green’s function in three dimensions: $G^k(x) = \frac{e^{ik|x|}}{4\pi|x|}$. Here, the (infinite) matrix Λ are the *scattering coefficients* of the particle, and measures the amplitude of the scattered field of frequency $\omega + i\Omega$ given an incident field of frequency $\omega + j\Omega$. Equation (5) is a *point-particle* approximation of the scattered field, in the sense that it provides an approximation of the scattered field in the limit when $\epsilon \rightarrow 0$.

3 Goal of the project

The main goal of this project is to derive an effective medium theory for a large collection of time-dependent particles. To achieve this, the first step will be to generalise (5) to a point-scattering approximation for multiple particles using the Foldy-Lax approximation. Based on this formulation, the resonant homogenization theory of [1] will be applicable to derive the effective equations. In addition, an important goal of the project will be to study the physical phenomena of the effective model, for example, by proving the possibility of nonreciprocal wave propagation and the emergence of the non-Hermitian skin effect [6].

4 Required background

The project is suitable for any student with a solid foundation of PDE-theory and analysis, who wants to work in the intersection of theoretical mathematics, numerical analysis, and wave physics. The project can be tailored to the specific background of the student, e.g., a physics-oriented student might focus more on the physical implications, and a numerics-oriented student might focus more on the numerical analysis and implementation.

References

- [1] Habib Ammari and Hai Zhang. Effective medium theory for acoustic waves in bubbly fluids near minnaert resonant frequency. *SIAM Journal on Mathematical Analysis*, 49(4):3252–3276, 2017.
- [2] Christophe Caloz and Zoé-Lise Deck-Léger. Spacetime metamaterials—part i: general concepts. *IEEE Transactions on Antennas and Propagation*, 68(3):1569–1582, 2019.
- [3] Christophe Caloz and Zoe-Lise Deck-Leger. Spacetime metamaterials—part ii: Theory and applications. *IEEE Transactions on Antennas and Propagation*, 68(3):1583–1598, 2019.
- [4] Erik Orvehed Hiltunen and Bryn Davies. Coupled harmonics due to time-modulated point scatterers. *arXiv preprint arXiv:2404.03483*, 2024.
- [5] Peter Kuchment. An overview of periodic elliptic operators. *Bulletin of the American Mathematical Society*, 53(3):343–414, 2016.
- [6] Kei Matsushima and Takayuki Yamada. Non-bloch band theory for time-modulated discrete mechanical systems. *arXiv preprint arXiv:2407.09871*, 2024.