Perturbation theory for dispersion relations of spacetime-periodic materials

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1 Background

The field of *metamaterials* aims to develop materials with, in some sense, *exotic* wave scattering properties. Using a carefully designed, periodic, microstructure, it is possible to design materials with effective properties beyond classical physics, such as vanishing or negative refractive index, or to localize and guide waves along specified positions. Such materials have been studied, both theoretically and experimentally, for a range of wave systems such as acoustic, elastic, and electromagnetic waves [1, 2, 3, 4].

The current frontier of metamaterials lies in *spacetime*-periodic materials. This is a class of materials whose material parameters depend on both space and time. By breaking time-reversal symmetry, it is possible to achieve (for example) nonreciprocal wave propagation. In these materials, a wave is able to propagate freely in one direction, but is exponentially attenuated (decaying) in the opposite direction.

Please note: The next section will give you some flavour of the project. Everything is very briefly described, and there will most likely be things you cannot follow. This is okay! If you are interested in the project, the best thing to do is to reach out directly to me to discuss.

2 Problem formulation

The simplest model for wave propagation in metamaterials is the scalar wave equation, such as

$$\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{n^2(x,t)}\Delta\right) u(x,t) = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}. \tag{1}$$

where $n(x,t)$ describes the refractive index of the material. For simplicity, we consider the one-dimensional wave equation $d = 1$. Time-modulated materials means that $n(x,t)$ depends on t. The most important case is when n is periodic in t.

In physics, the *dispersion relation*, or *band structure*, is a relationship between the frequency ω and the momentum k of waves. If $n(x,t) = 1$ for all x and t, describing a static, homogeneous material, the dispersion relation of (1) is $\omega = \pm c|k|$ (c.f. Figure 1(a)). For general n, characterising and computing the dispersion relation is a challenging but important task. If $n(x,t)$ is L-periodic in x, the momentum k is defined in the *Brillouin zone* $\mathcal{B}_x = \mathbb{R}/(\frac{2\pi}{L}\mathbb{Z})$. Observe that $\mathcal{B}_x$ has the topology of a (one-dimensional) torus. As consequence, the dispersion relation is "folded" into $\mathcal{B}_x$ (c.f. Figure 1(b)). If, additionally, $n(x,t)$ is T-periodic in t, the frequency ω is defined in the *temporal* Brillouin zone $\mathcal{B}_t = \mathbb{R}/(\frac{2\pi}{T}\mathbb{Z})$. The interplay between T and L will dictate if the folded bands coalesce (c.f. Figure 1(c)) or if they "miss" each other and fill the entire (k,ω)-space (c.f. Figure 1(d)).

The goal of this project is to study the band structure when n^2 is a shallow perturbation of a constant background:

$$n^2(x,t) = 1 + \epsilon\eta(x,t).$$

For small $\epsilon \ll 1$, the band structure of the material will be a small perturbation of the "free" band structure corresponding to $\epsilon = 0$, which is shown in Figure 1. In other words, if $\omega^{(0)}(k)$ is a point on the free band structure, we expect the perturbed band structure to satisfy

$$\omega(k) = \omega^{(0)}(k) + \epsilon\omega^{(1)}(k) + \epsilon^2\omega^{(2)}(k) + ... \tag{2}$$

If α is rational:

$$\alpha = \frac{p}{q}, \tag{3}$$

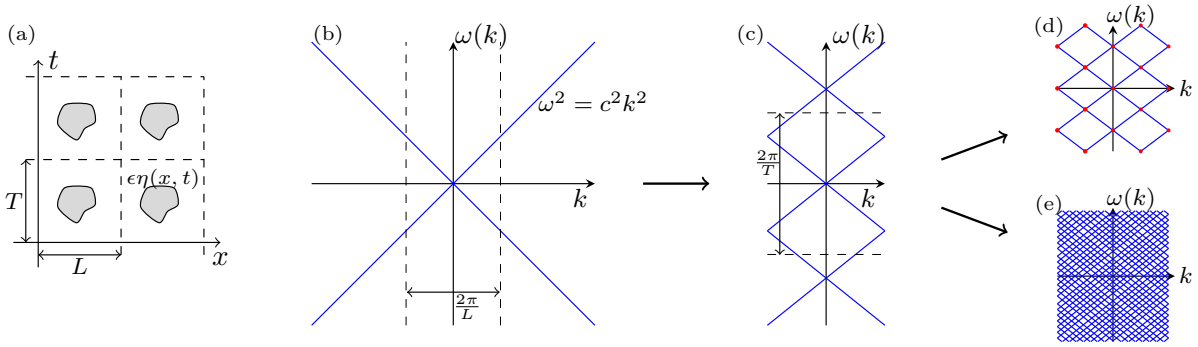

Figure 1: (a) We consider a (L,T)-spacetime-periodic material whose refractive index n satisfies $n^2(x,t) = 1+\epsilon\eta(x,t)$ for $\epsilon \ll 1$. (b) The dispersion of the free problem is given by $\omega^2 = c^2k^2$. (c)–(e) For spacetime-periodic media in the perturbation regime, the free dispersion is folded in both the momentum (c) and frequency (d),(e) variables. If $\alpha = L/cT$ is rational (shown in the case $\alpha = 3/2$ in (c)), the folded bands coalesce and each point (k,ω) on the band structure is infinitely degenerate. If, on the other hand, α is irrational, the folded bands fill the entire (k,ω)-space (d).

we can use (formal) eigenvalue perturbation theory to show that the perturbation $\omega^{(1)}$ can be described through the spectral (eigenvalue) problem for the *infinite* matrices $\mathbf{H}$ and $\mathbf{O}$ as follows:

$$-\frac{1}{2}\mathbf{HO}\mathbf{v} = \omega^{(1)}\mathbf{v}, \tag{4}$$

where $\mathbf{H}$ a (doubly infinite) Toeplitz matrix and $\mathbf{O}$ is a diagonal matrix $\mathbf{O}$ defined as

$$H_{mn} = \widetilde{\eta}\big(p(n-m), q(n-m)\big), \quad O_n = \omega^{(0)} + nq\frac{2\pi}{T}. \tag{5}$$

Here $\widetilde{\eta}\big(m,n\big)$ are the (two-dimensional) Fourier coefficients of the periodic function η. Thus, the spectrum of $-\frac{1}{2}\mathbf{HO}$ is of key importance in understanding the physical properties of the material.

3 Goal of the project

The main goal of this project is to study the spectrum of the infinite matrix $-\frac{1}{2}\mathbf{HO}$, viewed as an operator on the space ℓ^2 of square-summable sequences. Of interest are both qualitative properties, such as characterising the continuous and pure-point spectrum, and quantitative numerical calculations. In addition, the goal is to prove the existence of physical phenomena, such as the existence of k-gaps and nonreciprocal band gaps, and find conditions on η which guarantees these phenomena.

4 Required background

The project is suitable for any student with a solid foundation of PDE-theory and analysis, who wants to work in the intersection of theoretical mathematics, numerical analysis, and wave physics. The project can be tailored to the specific background of the student, e.g., a physics-oriented student might focus more on the physical implications, and a numerics-oriented student might focus more on the numerical analysis and implementation.

References

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[4] Peter Kuchment. An overview of periodic elliptic operators. *Bulletin of the American Mathematical Society*, 53(3):343–414, 2016.