

Master project: Numerical methods for the stochastic Allen–Cahn equation

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September 9, 2024

1 Background and outline of the problem

The Allen–Cahn equation is a reaction-diffusion equation that is used to model phase separations and the dynamics between different phases in materials science. To account for small-scale variations and dynamical fluctuations in materials, one often introduces a stochastic term in the Allen–Cahn, leading to the following stochastic partial differential equation (SPDE)

$$\begin{aligned}u_t(x, t) &= \Delta u(x, t) + f(u(x, t)) + dW(x, t) & (x, t) \in D \times (0, T] \\ u(x, 0) &= u_0(x),\end{aligned}$$

where D is an open, bounded and connected set $D \subset \mathbb{R}^d$, Δ denotes the Laplace operator, $W(x, t)$ is a space-time Gaussian process, and one typically imposes periodic or Neumann zero boundary conditions. The nonlinear reaction term $f : \mathbb{R} \rightarrow \mathbb{R}$ classically takes the form $f(u) = u - u^3$, which is a function with two stable stationary points ± 1 . One views ± 1 as material phases, and it turns out that the solution $u(x, t)$ will tend to take values near ± 1 , and one may view the values as meta-stable states.

The Allen–Cahn SPDE must typically be solved by numerical methods, and even this is challenging due to the low regularity of $W(x, t)$ and possibly also the low regularity of the initial condition. Recently, a very promising spectral numerical method with a splitting step solving the reaction part of the SPDE exactly was shown theoretically to have very good performance, cf. [1, Remark 2.5].

2 Main goals of the project

The goal of the project is to read up on numerical methods for stochastic partial differential equations with particular focus on the Allen–Cahn equation (see e.g., [2]), and to implement and test numerically the performance of the splitting method in [1] for problems on simple domains in 1d and 2d, for different reaction terms $f(u)$, initial conditions, and Gaussian processes $W(x, t)$. If time

permits, combine the numerical method with a multilevel Monte Carlo method, for approximations of quantities of interest of the SPDE solution, cf. [3].

3 Required background/required courses

Introductory course on numerical analysis (like MAT3110), numerical methods for PDE (the spectral methods covered in MAT3360 provides a good background), introductory course in stochastic analysis (like MAT4720), and a good background in studying mathematical problems numerically on a computer.

References

- [1] Djurdjevac, Ana, Máté Gerencsér, and Helena Kremp. *Higher order approximation of nonlinear SPDEs with additive space-time white noise*. arXiv preprint arXiv:2406.03058 (2024).
- [2] Lord, Gabriel J., Catherine E. Powell, and Tony Shardlow. *An introduction to computational stochastic PDEs*. Vol. 50. Cambridge University Press, 2014.
- [3] Chada, Neil K., et al. Improved efficiency of multilevel Monte Carlo for stochastic PDE through strong pairwise coupling. *Journal of Scientific Computing* 93.3 (2022): 62.