

Master project: Hitting times for diffusion processes

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1 Background and outline of the problem

The Itô stochastic differential equation (SDE)

$$dX(t) = a(X(t))dt + b(X(t))dW(t)$$

with $a : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $b : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times sd}$ and $W(t)$ denoting a d -dimensional standard Wiener process, is a diffusion process used to model the evolution of stock prices and the motion of heavy particles immersed in a heat bath of light particles, among other things. Well-posedness results have been established when the coefficients a and b have sufficient regularity, but it is challenging to develop efficient numerical methods for solving many SDE, for instance hitting-time problems.

For a given open, bounded and connected domain $D \subset \mathbb{R}^d$, the above SDE may be associated to the following linear parabolic partial differential equation (PDE) depending on the same coefficients a and b :

$$\begin{aligned} u_t(x, t) + a(x) \nabla u(x, t) + \frac{1}{2} \sum_{i,j,k=1}^d b_{ik} b_{kj} u_{x_i x_j}(x, t) &= 0 & (x, t) \in D \times [0, T) \\ u(x, T) &= g(x) & x \in D \\ u(x, t) &= g(x) & (x, t) \in \partial D \times [0, T]. \end{aligned}$$

Indeed, the Feynman–Kac formula tells us that the SDE paths can be viewed as stochastic characteristic functions of the PDE in the following sense

$$u(x, t) = \mathbb{E}[g(X(\min(\tau, T))) \mid X(t) = x],$$

where $\tau := \min\{s \geq t \mid X(s) \in \partial D\}$, denotes the hitting time: the first time $s \geq t$ that the SDE $X(s)$ hits the boundary of D .

Feynman–Kac presents a way to compute the solution of u at a point (x, t) by Monte Carlo methods drawing samples that are numerical approximations of the hitting time τ and the hitting state $X(\min(\tau, T))$. When $d \gg 1$, such

Monte Carlo methods can substantially outperform classic numerical methods for PDE, such as the finite element method. This promising feature is however challenging to fully leverage due to the low regularity of SDE paths, which makes it very challenging to develop higher order methods (essentially because classic Taylor expansions break down when higher order derivatives are not bounded).

2 Main goals of the project

The main goal of the project is to study if the recent high order, adaptive timestepping numerical method for hitting/exit times [1] can be extended to less regular domains and coefficients, and when the cut-off time $T \rightarrow \infty$. It also involves to read up on numerical methods for SDE, such as Euler–Maruyama and the Milstein method, on the Feynman-Kac formula, and on numerical software for solving PDE (see e.g., [3, 2])

3 Required background/required courses

Introductory course on numerical analysis, numerical methods for PDE, introductory course on SDE, and a good background in studying mathematical problems numerically on a computer.

References

- [1] Hoel, Håkon, and Sankarasubramanian Ragunathan. *Higher-order adaptive methods for exit times of Itô diffusions*. IMA Journal of Numerical Analysis (2023): drad077.
- [2] Higham, Desmond, and Peter Kloeden. *An introduction to the numerical simulation of stochastic differential equations*. Society for Industrial and Applied Mathematics, 2021.
- [3] Oksendal, Bernt. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.