

Neural Networks and PDEs in Modern Function Spaces: Sobolev and Barron Spaces

Introduction:

The rise of neural networks as powerful tools for managing high-dimensional data has prompted a reexamination of their mathematical foundations, particularly in terms of function spaces that govern their approximation properties. Traditionally, function spaces such as Sobolev spaces have played a central role in understanding the complexity and smoothness of functions, especially for partial differential equations (PDEs). However, when dealing with high-dimensional data, these classical spaces suffer from the “curse of dimensionality.” In contrast, modern function spaces like Barron spaces have emerged as important in analyzing neural network models, particularly two-layer networks, because they allow dimension-independent error rates.

Main Goal:

This project seeks to investigate whether data from PDEs and their solutions, when represented through neural networks, naturally reside in these modern function spaces, such as Barron spaces. The specific goals include:

1. Determining whether solutions of PDEs (e.g., Poisson, heat, or Hamilton-Jacobi equations) that are known to belong to classical spaces like Sobolev or Besov also reside in Barron spaces, as suggested by recent research [1,2].
2. Investigating the optimal function spaces (e.g., Barron or Sobolev spaces) for different machine learning models, particularly neural networks, to ensure precise approximation of PDE solutions.
3. Developing an understanding of how modern neural network spaces, such as the Barron space, compare to traditional spaces in approximating PDE solutions and managing the curse of dimensionality.

The project will integrate insights from courses on Sobolev spaces, linear analysis, and approximation theory, aiming to understand the mathematical framework underlying machine learning models for PDEs. As a potential extension, this project may explore the flow-induced function spaces related to deep neural networks, providing a more nuanced understanding of multilayer networks.

References:

[1] Weinan E, Chao Ma, and Lei Wu. The Barron Space and the Flow-Induced Function Spaces for Neural Network Models. *Constructive Approximation*, 2022 .

[2] Weinan E and Stephan Wojtowytsch. Some observations on high-dimensional partial differential equations with Barron data. *Proceedings of Machine Learning Research*, 2021 .

[3] Weinan E and Stephan Wojtowytsch. On the Banach spaces associated with multi-layer ReLU networks: Function representation, approximation theory, and gradient descent dynamics, *CSIAM Trans. Appl. Math.*, 2020.