KMS states on the crossed product C^*-algebra of a homeomorphism

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joint with **Klaus Thomsen**

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Dynamical systems

Notation

In this talk a dynamical system is a homeomorphism on a (infinite) compact metric space. We denote throughout by X the compact metric space and by ϕ the homeomorphism $\phi : X \to X$.

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- We call $\phi : X \to X$ *minimal* if there exists no closed ϕ-invariant subset $Y \subseteq X$ different than $\emptyset$ and X.
- $\phi : X \to X$ is minimal if and only if the orbit of each $x \in X$ is dense in X.

C^*-algebras and dynamical systems

Definition

Assume X is a compact metric space and $\phi : X \to X$ is a homeomorphism. The crossed product C^*-algebra $C(X) \rtimes_\phi \mathbb{Z}$ of (X, ϕ) is the universal C^*-algebra generated by a copy of $C(X)$ and a unitary U satisfying

$$UfU^* = f \circ \phi^{-1} \quad \text{for all } f \in C(X).$$

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- *The set*

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  is dense in $C(X) \rtimes_\phi \mathbb{Z}$.
- *$C(X) \rtimes_\phi \mathbb{Z}$ is simple if and only if $\phi : X \to X$ is minimal.*

C^*-algebras and dynamical systems

Definition

Let X be a compact metric space and let $\phi : X \to X$ be a homeomorphism. For any continuous function $F : X \to \mathbb{R}$ we can define a one-parameter group $\{\alpha_t^F\}_{t\in\mathbb{R}}$ on $C(X) \rtimes_\phi \mathbb{Z}$ satisfying

$$\alpha_t^F(f) = f \text{ and } \alpha_t^F(U) = Ue^{-itF} = e^{-itF\circ\phi^{-1}}U$$

for all $f \in C(X)$ and $t \in \mathbb{R}$.

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What is the β-KMS states for α^F?

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Question

What is the β-KMS states for α^F?

Definition

Let $\mathcal{A}$ be a C^*-algebra, α a continuous one-parameter group on $\mathcal{A}$ and $\beta \in \mathbb{R}$. A state ω on $\mathcal{A}$ is a β-KMS state for α if :

$$\omega(AB) = \omega(B\alpha_{i\beta}(A))$$

for all A, B in a norm-dense, α-invariant $*$-subalgebra of the analytic elements of $\mathcal{A}$.

KMS states and Conformal measures

Definition (Patterson, Sullivan, Denker and Urbanski)

Let (X, ϕ) *be a dynamical system and let* $F : X \to \mathbb{R}$ *be continuous. An* $e^{\beta F}$ *conformal measure for* ϕ *is a Borel probability measure* m *on* X *satisfying that*

$$m(\phi(B)) = \int_B e^{\beta F} \, \mathrm{d}m$$

for all Borel sets B*.*

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for all Borel sets B.

Remark

m is an $e^{\beta F}$ conformal measure for ϕ if and only if

$$\int_X f \, \mathrm{d}m = \int_X f \circ \phi \; e^{\beta F} \, \mathrm{d}m$$

for all $f \in C(X)$.

KMS states and Conformal measures

Lemma

If ω is a β-KMS state for α^F on $C(X) \rtimes_\phi \mathbb{Z}$ then the restriction of ω to $C(X) \subseteq C(X) \rtimes_\phi \mathbb{Z}$ defines an $e^{\beta F}$-conformal measure m_ω. The map $\omega \mapsto m_\omega$ is surjective: For each $e^{\beta F}$-conformal measure m we can define a β-KMS states ω_m for α^F such that

$$\omega_m(a) = \int_X E(a)\, \mathrm{d}m \, ,$$

where $E : C(X) \rtimes_\phi \mathbb{Z} \to C(X)$ is the canonical conditional expectation.

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$\frac{1}{2}$proof.

Let ω be a β-KMS states for α^F. For any $f \in C(X)$ then

$$\begin{aligned}\omega(f) &= \omega(U(f \circ \phi)U^*) = \omega((f \circ \phi)U^* \alpha^F_{i\beta}(U)) \\ &= \omega((f \circ \phi)U^* U e^{\beta F}) = \omega((f \circ \phi)e^{\beta F}).\end{aligned}$$

□

KMS states and Conformal measures

Lemma

Let $\mathcal{O}$ be a finite ϕ-orbit with $\sum_{x \in \mathcal{O}} F(x) = 0$. The set $\mathcal{F}^{\mathcal{O}}$ of β-KMS states ω of α^F such that m_ω is concentrated on $\mathcal{O}$ is affinely homeomorphic to the simplex of Borel probability measures on $\mathbb{T}$.

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Theorem

Let ω be an extremal β-KMS state, $\beta \neq 0$. Then either

$$\omega(a) = \int_X E(a) \, \mathrm{d}m_\omega \quad \forall a \in C(X) \rtimes_\phi \mathbb{Z} \ ,$$

with m_ω concentrated on the set of points that are not ϕ-periodic or $\omega \in \mathcal{F}^{\mathcal{O}}$ for some finite ϕ-orbit $\mathcal{O}$ with $\sum_{x \in \mathcal{O}} F(x) = 0$.

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In particular $\omega \mapsto m_\omega$ is an affine homeomorphism when ϕ is minimal.

Conclusion

To describe the β-KMS states for α^F on $C(X) \rtimes_\phi \mathbb{Z}$ it suffices to describe the $e^{\beta F}$-conformal measures.

Illustrative Examples

Question

How complex and rich can the structure of the KMS states be?

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An $e^{\beta F}$-conformal measure is non-singular wrt. (X, ϕ).

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Theorem

Let m be an ergodic $e^{\beta F}$-conformal measure.

- *m is of type I_p if and only if it is atomic and concentrated on an p-periodic orbit $\mathcal{O}$ with $\sum_{x \in \mathcal{O}} F(x) = 0$.*
- *m is of type I_∞ if and only if it is atomic and concentrated on a single infinite orbit.*
- *m is of type II_1 if and only if it is equivalent to a ϕ-invariant non-atomic Borel probability measure.*
- *m is of type II_∞ if and only if it is equivalent to an infinite σ-finite non-atomic ϕ-invariant measure.*
- *m is of type III if and only if it is singular with respect to all σ-finite ϕ-invariant Borel measures.*

Illustrative Examples

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- *Type I_∞ done by Douady and Yoccoz for certain irrational rotations of $\mathbb{T}$.*
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- *Type II_∞ by Katznelson for certain minimal homeomorphisms of $\mathbb{T}$.*

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- *Type I_p easy.*
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- *Type II_1 easy.*
- *Type II_∞ by Katznelson for certain minimal homeomorphisms of $\mathbb{T}$.*
- *Type III by Katznelson for certain minimal homeomorphisms of $\mathbb{T}$.*

Illustrative Examples

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Lemma

Let $\phi : \mathbb{T} \to \mathbb{T}$ be an irrational rotation of the circle $\mathbb{T}$. There is a continuous real-valued function $F : \mathbb{T} \to \mathbb{R}$ such that

- *There are $e^{\beta F}$-conformal measures for all $\beta \in \mathbb{R}$.*
- *For $\beta \geq 0$ there is exactly one ergodic $e^{\beta F}$-conformal measure of type II_1.*
- *For $\beta < 0$ all ergodic $e^{\beta F}$-conformal measures are of type II_∞ or type III.*

Richness of set of KMS states

Theorem

Let X be a compact metric space and let $\phi : X \to X$ a homeomorphism. Let $\{x_1, x_2, \dots, x_q\} \subseteq X$ be a finite subset consisting of points x_p in X that are not periodic under ϕ and have disjoint orbits, and choose a $\beta_p \in]-\infty, 0[$ and an interval J_p of the form $]-\infty, \beta_p]$ or $]-\infty, \beta_p[$ for each $1 \leq p \leq q$. There exists a continuous function $F : X \to \mathbb{R}$ such that there is an $e^{\beta F}$-conformal measure concentrated on the ϕ-orbit of x_p if and only if

$$\beta \in J_p$$

for $p = 1, 2, \cdots, q$. The potential F can be chosen such that the flow α^F is approximately inner.

Existence of conformal measures

Existence of conformal measures

Theorem

Let (X, ϕ) be a dynamical system. Fix a $\beta \in \mathbb{R}$ and a continuous function $F : X \to \mathbb{R}$. TFAE

1. *There exists an $e^{\beta F}$-conformal measure for ϕ.*
2. *There exists a point $x \in X$ such that*
$$\limsup_k \frac{1}{k} \sum_{i=0}^{k-1} \beta F(\phi^i(x)) \leq 0 \quad \text{and}$$
$$\limsup_k \frac{1}{k} \sum_{i=1}^{k} -\beta F(\phi^{-i}(x)) \leq 0 \, .$$
3. *There exists a point $x \in X$ such that*
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From the formulas

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Corollary

Let $\phi : X \to X$ be a homeomorphism. Assume that $F(x) > 0$ for all $x \in X$ or that $F(x) < 0$ for all $x \in X$. There are no $e^{\beta F}$-conformal measure for ϕ when $\beta \neq 0$.

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Corollary

Let $\phi : X \to X$ be a homeomorphism. Let I be the KMS spectrum for α^F, i.e. the set of real numbers β such that there exists an $e^{\beta F}$-conformal measure for ϕ. Then I is one of following intervals:

- $I = \{0\}$,
- $I =]-\infty, 0]$,
- $I = [0, \infty[$, *or*
- $I = \mathbb{R}$.

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Corollary

Assume that there is an ergodic ϕ-invariant proability measure ν such that $\int_X F \, \mathrm{d}\nu = 0$. It follows that there is an $e^{\beta F}$-conformal measure for all $\beta \in \mathbb{R}$.

"proof".

Birkhoff's pointwise ergodic theorem. □

Existence of conformal measures

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we can derive some very interesting results:

Corollary

For a uniquely ergodic system (X, ϕ) with unique ϕ-invariant probability measure ν the only possibilities for a KMS spectrum is $\mathbb{R}$ of $\{0\}$. The spectrum is $\mathbb{R}$ iff $\int_X F \, \mathrm{d}\nu = 0$.

Example of KMS spectrums

Example

Let (X, ϕ) be a minimal dynamical system. If $F : X \to \mathbb{R}$ is the function $F = 1$ then the KMS spectrum of α^F is $\{0\}$.
If $F = 0$ then the KMS spectrum of α^F is $\mathbb{R}$.

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Example

Set $X = [0, 1]$ and set $\phi(x) = x^2$ for all $x \in X$. Let $F : [0, 1] \to \mathbb{R}$ be a continuous function strictly negative around 0 and strictly positive around 1, as an example $F(x) = -\frac{1}{2} + x$. Then the KMS spectrum is $[0, \infty[$.

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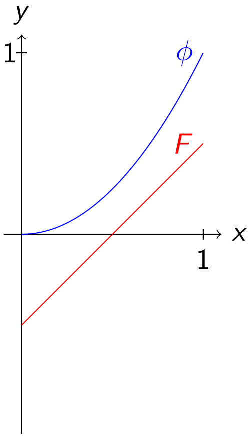

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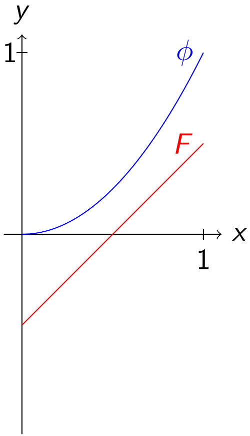

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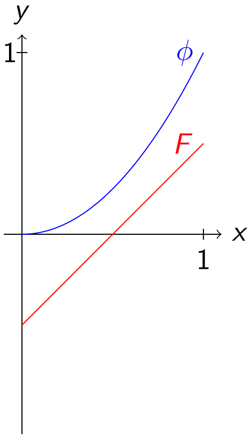

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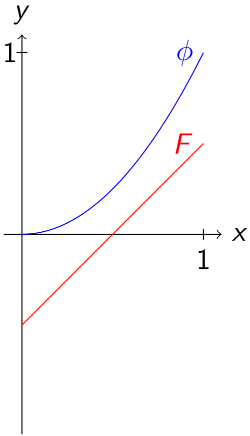

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KMS spectrums of minimal homeomorphisms

Question

Can we realize the KMS spectrums $[0, \infty[$ *and* $]-\infty, 0]$ *with minimal homeomorphisms?*

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Theorem

Let X *be a compact metric space and let* $\phi : X \to X$ *be a minimal homeomorphism. Assume* $F : X \to \mathbb{R}$ *is continuous. The only two possible KMS spectrums for* α^F *are* $\{0\}$ *and* $\mathbb{R}$.

Thank you for your attention!